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Sliding motion and global dynamics of a Filippov fire-blight model with economic thresholds

Can Chen[a,b], Yanmei Kang[b], Robert Smith?[c,*]

[a] *Department of Mathematics and Physics, College of Science, Zhengzhou University of Aeronautics, Zhengzhou 450015, PR China*
[b] *Department of Applied Mathematics, Xi'an Jiaotong University, Xi'an 710049, PR China*
[c] *Department of Mathematics and Faculty of Medicine, The University of Ottawa, 585 King Edward Ave, Ottawa, ON K1N 6N5, Canada*

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ABSTRACT

Cutting off infected branches has always been an effective method for removing fire-blight infection in an orchard. We introduce a Filippov fire-blight model with a threshold policy: cutting off infected branches and replanting susceptible trees. The dynamics of the proposed piecewise smooth model are described by differential equations with discontinuous right-hand sides. For each susceptible threshold value S_T, we investigate the global dynamical behaviour of the Filippov system, including the existence of all the possible equilibria, their stability and sliding-mode dynamics, as we vary the infected threshold level I_T. Our results show that model solutions ultimately approach the equilibrium that lies in the region above I_T or below I_T or on $I = I_T$, or the equilibrium $E_T = (S_T, I_T)$ on the surface of discontinuity. Furthermore, control strategies should be taken when the solution of this system approaches the equilibrium that lies in the region above I_T. The findings indicate that proper choice of susceptible and infected threshold levels can either preclude an outbreak of fire blight or lead the number of infected trees to a desired level.

1. Introduction

Fire blight is one of the major threats to fruit-bearing trees, primarily apple, pear and other members of the Rosaceae family, due to the fact that it can destroy an entire orchard in a single growing season [1–3]. The infection is transmitted by gram-negative bacteria, *Erwinia amylovora*, which is capable of infecting blossoms, vegetative shoots, woody tissues, rootstock crowns and fruits of the trees [4,5]. The total economic loss of fire blight is not always easy to appreciate, as it is an erratic disease, but severe outbreaks can lead to millions of dollars of production and tree losses [6]. In the USA alone, it has been reported that the annual economic loss is approximately \$100 million [7,8]. Similarly, in Europe, significant economic losses have been

* Corresponding author.
E-mail addresses: can3111054023@stu.xjtu.edu.cn (C. Chen), ymkang@mail.xjtu.edu.cn (Y. Kang), rsmith43@uottawa.ca (R. Smith?).

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reported; e.g., in Switzerland, a major outbreak of fire blight occurred between 1997 and 2000, resulting in the loss of $9 million within this period [9]. Furthermore, the worldwide economic importance due to this disease is likely to increase [10].

Currently, even though there is no cure for fire blight, preventative strategies have been implemented to reduce the spread of fire-blight infection, such as pruning and removal of diseased plant parts. Furthermore, it has been recognized that cutting off infected branches is an effective method of removing fire-blight infection, because it can disrupt the equilibrium between vegetative and reproductive growth [11,12]. Two sets of experiments were conducted during 1999 to 2001 in Israel to evaluate the efficacy of pruning infected pear tissues to combat fire blight. They found that if pruning was carried out when the trees were dormant (in December), then none of these plants had a severely infected canopy the following spring [13]. Nevertheless, the loss of fruit production can be economically devastating for growers, even if the disease does not kill the tree. In reality, complete eradication of the infected trees is generally not possible, nor is it economically desirable. Therefore an efficient control strategy is needed to avoid overpruning and reduce economic losses.

Mathematical models can be a useful tool for designing strategies to control the spread of plant diseases and determining their efficacy, especially in the absence of an effective treatment [14]. Many different types of mathematical models on plant diseases have been proposed [15,16], including ordinary differential equation models [17,18] and impulsive differential equation models [19,20]. A combination of an epidemiological model, together with the analysis of evolutionary stable strategies, was used to analyse the effectiveness of continuous control measures for combating vegetatively propagated plant diseases [17]. Meng et al. [19] constructed plant-disease models with continuous and impulsive cultural control strategies to investigate how to control plant-disease transmission by replanting of healthy plants and removal of infected trees. Tang et al. [20] first developed a plant-disease model with pulse replanting and roguing strategies at fixed moments, then formulated a state-dependent impulsive model by implementing a cultural control strategy only when the number of infected plants reaches an economic threshold value.

However, in these plant-disease models, there exist some disadvantages. On the one hand, if control strategies occur continuously or impulsively at fixed moments, regardless of whether the number of infected trees reaches the economic threshold or not, this will consume a huge amount of economic damage and labour costs, because it is not necessary to implement the control strategy when the number of infected plants is not relatively high. On the other hand, in the state-dependent impulsive plant disease models, once the number of infected trees reaches the economic threshold, the growers would theoretically implement the control strategy instantaneously and reduce it below the economic threshold at that precise moment, which seems unrealistic.

Consequently, a more realistic threshold policy is required to provide useful information in fire-blight management strategies, so that the economic damage can be reduced to a minimum level. Therefore, by incorporating non-instantaneous control with the threshold policy, the spread of fire blight can be described by nonlinear ordinary differential equations with discontinuous right-hand sides, called Filippov systems [21,22]. Although Filippov systems have been used to investigate many infectious diseases [23–26], very little is known about the effects of the discontinuous control functions on the dynamics of fire-blight. Thus our main purpose is to construct a Filippov fire-blight model by considering cutting off infected branches and replanting susceptible trees. Then, by applying the theory of Filippov systems to the proposed model, we aim to establish conditions under which the growers can achieve minimize economic losses and maximize returns.

The rest of this paper is structured as follows. In Section 2, we propose a Filippov fire-blight model incorporating cutting off infected branches and replanting susceptible trees. The dynamical behaviour of the proposed Filippov system, including the existence of all the possible equilibria, their stability and sliding-mode dynamics, is investigated by varying the infected and susceptible threshold values in Sections 3–6. Finally, we present a discussion and biological conclusion on the results of this work in Section 7.

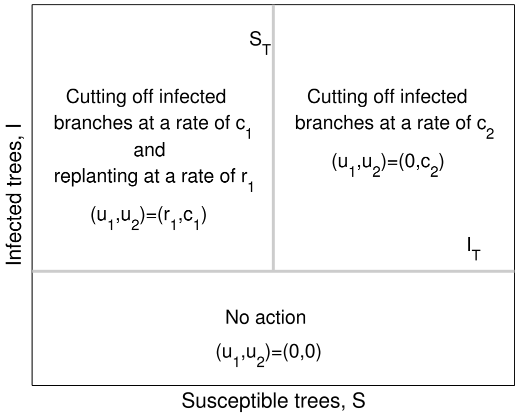

Fig. 1. Schematic diagram of the threshold policy.

2. Filippov fire-blight model and preliminaries

We consider a threshold policy in a fire-blight model consisting of cutting off infected branches and replanting susceptible trees. In order to better control the spread of fire blight and achieve maximal economic benefits, control measures should be taken depending on whether the numbers of infected and susceptible trees exceed the economic threshold values or not. This threshold policy is defined as follows: if the number of infected trees is less than the infected threshold value I_T, then the control strategy is not necessary; above I_T, we remove infected branches at a rate of c_1 and replant susceptible trees at a rate of r_1 simultaneously if the number of susceptible trees is less than the susceptible threshold level S_T, and we only remove infected branches at a rate of c_2 if $S > S_T$. Here we choose the replanting proportional to the number of susceptible trees. The value of the replanting rate r_1 might dependent on the number of available workers, so we use the number of susceptible trees as a proxy for the availability of disposable funds for the orchard. A schematic diagram of the threshold policy is illustrated in Fig. 1. Furthermore, it is natural to assume that the removal rate of infected branches when there are few susceptible trees is larger than the removal rate of infected branches when there are many susceptible trees; that is, $c_1 > c_2$. We not only consider pruning infected limbs with a higher rate c_1 when $S < S_T$ but also increase the number of susceptible trees by replanting at a rate r_1. The reason is that we need to maintain enough susceptible trees for fruit production when the number of infected trees exceeds I_T. Additionally, we assume that the replanting rate of susceptible trees r_1 is less than the natural death rate of trees μ; that is, $\mu > r_1$.

Let Λ be the total rate at which the susceptible trees enter the system, β be the infection rate through the environment and α be the disease death rate. We consider the dynamics of susceptible trees $S(t)$ and infected state $I(t)$, where $S(t)$, $I(t)$ denote the number of susceptible and infected trees at time t, respectively. Therefore a Filippov fire-blight model with cultural control strategy, consisting of cutting off infected branches and replanting susceptible trees, is described by differential equations with discontinuous right-hand sides as follows:

$$\begin{pmatrix} S' \\ I' \end{pmatrix} = f(S,I) = \begin{pmatrix} \Lambda - \beta SI - \mu S + u_1 S + u_2 I \\ \beta SI - \mu I - \alpha I - u_2 I \end{pmatrix}, \tag{2.1}$$

with

$$(u_1,u_2) = \begin{cases} (0,0) & \text{for } I < I_T, \\ (r_1,c_1) & \text{for } S < S_T \text{ and } I > I_T, \\ (0,c_2) & \text{for } S > S_T \text{ and } I > I_T. \end{cases} \tag{2.2}$$

Furthermore the S, I space $\mathbb{R}_+^2$ can be divided into the following five regions:

$$\begin{aligned}
G_1 &= \Big\{(S,I) \in \mathbb{R}_+^2 : I < I_T\Big\},\\
G_2 &= \Big\{(S,I) \in \mathbb{R}_+^2 : S < S_T \text{ and } I > I_T\Big\},\\
G_3 &= \Big\{(S,I) \in \mathbb{R}_+^2 : S > S_T \text{ and } I > I_T\Big\},\\
M_1 &= \Big\{(S,I) \in \mathbb{R}_+^2 : I = I_T\Big\},\\
M_2 &= \Big\{(S,I) \in \mathbb{R}_+^2 : S = S_T \text{ and } I > I_T\Big\}.
\end{aligned}$$

Additionally, the normal vectors that are perpendicular to M_1 and M_2 are defined as $n_1 = (0,1)^T$ and $n_2 = (1,0)^T$, respectively.

Since the right-hand side of system (2.1) with (2.2) is piecewise continuous, we consider its solutions in Filippov's sense. The theory of existence and uniqueness of solutions of such systems can be found in [21]. The following definitions of equilibria and sliding regions on M_j $(j = 1, 2)$ of the Filippov system (2.1) with (2.2) are necessary throughout the paper (we refer the interested reader to [27–29] for further details). Denote the right-hand side of system (2.1) with (2.2) in region G_i by F_i, $i = 1, 2, 3$.

Definition 2.1. A point E^R is called a real equilibrium of system (2.1) with (2.2) if $F_i(E^R) = 0$ and $E^R \in G_i$, $i = 1, 2, 3$.

Definition 2.2. A point E^V is called a virtual equilibrium of system (2.1) with (2.2) if $F_i(E^V) = 0$ and $E^V \notin G_i$, $i = 1, 2, 3$.

Definition 2.3. A 'sliding mode' exists if there are subsets Σ of the manifold M_j such that the flows of f (outside of M_j) are directed towards each other on them, $j = 1, 2$.

Denote the sliding-mode equations that describe the motion in the sliding region $\Sigma \subset M_j$ by $H_j(S, I)$, $j = 1, 2$.

Definition 2.4. A point E^P is called a pseudoequilibrium if E^P is an equilibrium on the sliding mode Σ; that is, $H_j(E^P) = 0$ and $E^P \in \Sigma \subset M_j$, $j = 1, 2$.

2.1. Dynamics in region G_1

In this section, we investigate the dynamics in region G_1, which are described by

$$\begin{pmatrix} S' \\ I' \end{pmatrix} = F_1(S,I) = \begin{pmatrix} \Lambda - \beta SI - \mu S \\ \beta SI - \mu I - \alpha I \end{pmatrix}. \tag{2.3}$$

Even though system (2.3) has been studied in some papers [19,20], for clarity we present the main results and give a brief proof here. The basic reproduction number is $R_{01} = \frac{\Lambda\beta}{\mu(\mu+\alpha)}$. System (2.3) always has a disease-free equilibrium, $E_{10} = (\frac{\Lambda}{\mu}, 0)$. If $R_{01} > 1$, there is a unique endemic equilibrium, $E_1 = (S_1^*, I_1^*) = \left(\frac{\mu+\alpha}{\beta}, \frac{\Lambda\beta - \mu(\mu+\alpha)}{\beta(\mu+\alpha)}\right)$.

Lemma 2.1. *The set $D_1 = \Big\{(S,I) \in \mathbb{R}_+^2 : S + I \le \frac{\Lambda}{\mu}\Big\}$ is a positively invariant and attracting region for model (2.3) with any given initial values in $\mathbb{R}_+^2$.*

Proof. Let $N = S + I$. Since $N' = \Lambda - \mu N - \alpha I \leq \Lambda - \mu N$, we have

$$N(t) \leq \frac{\Lambda}{\mu} + \Big(N(0) - \frac{\Lambda}{\mu}\Big)e^{-\mu t}.$$

Therefore, we obtain $N(t) \leq \frac{\Lambda}{\mu}$ if $N(0) \leq \frac{\Lambda}{\mu}$; that is, the set D_1 is positively invariant.

Suppose that $N > \frac{\Lambda}{\mu}$, then we get $N' < -\alpha I < 0$, which implies that the set D_1 is an attracting region. □

Theorem 2.1. *The disease-free equilibrium, E_{10}, is globally asymptotically stable if $R_{01} < 1$, while the unique endemic equilibrium, E_1, is globally asymptotically stable if $R_{01} > 1$.*

Proof. If $R_{01} < 1$, taking a Lyapunov function $V(t) = I(t)$ and applying LaSalle's invariance principle, one finds that E_{10} is globally asymptotically stable. If $R_{01} > 1$, we choose a Dulac function $D = \frac{1}{SI}$ and using the Bendixson–Dulac criteria, we can exclude the existence of limit cycles, and hence E_1 is globally asymptotically stable.

Furthermore, E_1 could either be a stable spiral if $\Delta < 0$; or be a stable node if $\Delta \geq 0$, where $\Delta = \left(\frac{\Lambda\beta}{\mu+\alpha}\right)^2 - 4(\Lambda\beta - \mu(\mu+\alpha))$. □

2.2. Dynamics in region G_2

In region G_2, the dynamics are governed by

$$\begin{pmatrix} S' \\ I' \end{pmatrix} = F_2(S, I) = \begin{pmatrix} \Lambda - \beta SI - \mu S + r_1 S + c_1 I \\ \beta SI - \mu I - \alpha I - c_1 I \end{pmatrix}. \tag{2.4}$$

The basic reproduction number is $R_{02} = \frac{\Lambda\beta}{(\mu-r_1)(\mu+\alpha+c_1)}$. System (2.4) always has a disease-free equilibrium, $E_{20} = (\frac{\Lambda}{\mu-r_1}, 0)$. If $R_{02} > 1$, there exists a unique endemic equilibrium, $E_2 = (S_2^*, I_2^*) = \left(\frac{\mu+\alpha+c_1}{\beta}, \frac{\Lambda\beta-(\mu-r_1)(\mu+\alpha+c_1)}{\beta(\mu+\alpha)}\right)$.

Lemma 2.2. *The set $D_2 = \left\{(S, I) \in \mathbb{R}_+^2 : S + I \leq \frac{\Lambda}{\mu-r_1}\right\}$ is a positively invariant and attracting region for model (2.4) with any given initial values in $\mathbb{R}_+^2$.*

Theorem 2.2. *The disease-free equilibrium, E_{20}, is globally asymptotically stable if $R_{02} < 1$, while the unique endemic equilibrium, E_2, is globally asymptotically stable if $R_{02} > 1$.*

2.3. Dynamics in region G_3

The dynamics in region G_3 are described by

$$\begin{pmatrix} S' \\ I' \end{pmatrix} = F_3(S, I) = \begin{pmatrix} \Lambda - \beta SI - \mu S + c_2 I \\ \beta SI - \mu I - \alpha I - c_2 I \end{pmatrix}. \tag{2.5}$$

The basic reproduction number is $R_{03} = \frac{\Lambda\beta}{\mu(\mu+\alpha+c_2)}$. System (2.5) always has a disease-free equilibrium, $E_{30} = (\frac{\Lambda}{\mu}, 0)$. If $R_{03} > 1$, there is a unique endemic equilibrium, $E_3 = (S_3^*, I_3^*) = \left(\frac{\mu+\alpha+c_2}{\beta}, \frac{\Lambda\beta-\mu(\mu+\alpha+c_2)}{\beta(\mu+\alpha)}\right)$.

Lemma 2.3. *The set $D_3 = \left\{(S, I) \in \mathbb{R}_+^2 : S + I \leq \frac{\Lambda}{\mu}\right\}$ is a positively invariant and attracting region for model (2.5) with any given initial values in $\mathbb{R}_+^2$.*

Theorem 2.3. *The disease-free equilibrium, E_{30}, is globally asymptotically stable if $R_{03} < 1$, while the unique endemic equilibrium, E_3, is globally asymptotically stable if $R_{03} > 1$.*

3. Case 1: $S_T < S_1^* < S_3^* < S_2^*$

In this and the following three sections, we aim to address the richness of the possible equilibria and sliding modes on M_1 and M_2 that the system (2.1) with (2.2) can exhibit. Note that we only consider $R_{01} > 1$, $R_{02} > 1$, $R_{03} > 1$ to guarantee the existence of endemic equilibrium in each region; otherwise the system will stabilize to its disease-free equilibrium. We have $S_1^* < S_3^* < S_2^*$ and $I_1^* > I_3^*$. Then we consider the following cases generated by $S_T < S_1^*$, $S_1^* < S_T < S_3^*$, $S_3^* < S_T < S_2^*$ and $S_T > S_2^*$, with varied infected threshold value I_T. Furthermore, the existence of all the possible equilibria, their stability and sliding-mode dynamics will be examined from one case to the other. According to the dynamics in each case, the biological phenomena will be described. In addition, we will summarize the main results and describe the biological implication of all these cases at the end of this paper.

3.1. Sliding mode on M_1 and its dynamics

We initially examine the existence of the sliding mode on M_1. According to Definition 2.3, if $\langle n_1, F_1\rangle > 0$ and $\langle n_1, F_3\rangle < 0$ on $\Sigma_1 \subset M_1$, then Σ_1 is a sliding region. In Case 1, the manifold Σ_1 is a discontinuity surface between the two different structures F_1 and F_3. Therefore, we can verify that the sliding domain $\Sigma_1 \subset M_1$ is defined as:

$$\Sigma_1 = \{(S, I) \in M_1 : S_1^* < S < S_3^*\}. \tag{3.1}$$

We utilize the Filippov convex method [21,30] as follows:

$$\begin{pmatrix} S' \\ I' \end{pmatrix} = \sigma_1 F_1 + (1 - \sigma_1)F_3, \text{ where } \sigma_1 = \frac{\langle n_1, F_3\rangle}{\langle n_1, F_3 - F_1\rangle}.$$

Therefore we can obtain differential equations describing the sliding-mode dynamics along the manifold Σ_1 for system (2.1) with (2.2):

$$\begin{pmatrix} S' \\ I' \end{pmatrix} = \begin{pmatrix} \Lambda - \mu S - (\mu + \alpha)I_T \\ 0 \end{pmatrix}. \tag{3.2}$$

System (3.2) has a unique equilibrium $E_{s1} = (S_{s1}^*, I_T)$, where $S_{s1}^* = \frac{\Lambda - (\mu+\alpha)I_T}{\mu}$. Hence $E_{s1} \in \Sigma_1 \subset M_1$ is a pseudoequilibrium for system (3.2) if and only if $S_1^* < S_{s1}^* < S_3^*$; that is, $I_3^* < I_T < I_1^*$. Furthermore, it is stable on $\Sigma_1 \subset M_1$.

Theorem 3.1. *E_{s1} is a stable pseudoequilibrium on $\Sigma_1 \subset M_1$.*

Proof. We have

$$\frac{\partial}{\partial S}\Big(\Lambda - \mu S - (\mu + \alpha)I_T\Big)\Big|_{E_{s1}} = -\mu < 0.$$

Hence solutions are attracting. □

3.2. Existence of a sliding mode on M_2 and its dynamics

In order to better prevent fire-blight infection and to increase fruit production, we first assume that the removal rates of infected branches are larger than the total death rates. Since the average durations are inverses of these rates, this corresponds to an assumption that the removals occur before the tree dies.

Assumption 1. Assume that $c_1 > \mu + \alpha$ and $c_2 > \mu + \alpha$.

Next we examine the existence of the sliding mode on M_2. Since $\langle n_2, F_2\rangle > 0$ and $\langle n_2, F_3\rangle < 0$, we have $I(\beta S_T - c_1) < \Lambda - (\mu - r_1)S_T$ and $I(\beta S_T - c_2) > \Lambda - \mu S_T$. However, since $S_T < S_1^*$, then $\beta S_T < \mu + \alpha$; i.e., $\beta S_T - c_2 < \mu + \alpha - c_2 < 0$. Therefore there is no sliding mode on M_2.

3.3. Global behaviour

In this section, we investigate the asymptotical behaviour of system (2.1) and (2.2) with a fixed susceptible threshold $S_T < S_1^*$ while the infected threshold I_T varies. In Case 1, for a fixed threshold level S_T such that $S_T < S_1^* < S_3^* < S_2^*$, E_2 is a virtual equilibrium for any values of the infected threshold I_T, denoted by E_2^V, so there is no real equilibrium in region G_2. Nevertheless, E_1 and E_3 may be real equilibria depending on the values of I_T.

3.3.1. Case 1.1: $I_T < I_3^*$

In this case, E_1 is a virtual equilibrium, whereas E_3 is a real equilibrium, denoted by E_1^V and E_3^R, respectively. Furthermore, E_{s1} is not a pseudoequilibrium on $\Sigma_1 \subset M_1$. Then we claim that E_3^R is globally asymptotically stable if $I_T < I_3^*$.

Theorem 3.2. *E_3^R is globally asymptotically stable if $S_T < S_1^* < S_3^* < S_2^*$ and $I_T < I_3^*$.*

Proof. Suppose there exists a closed trajectory Γ (shown in Fig. 2) that surrounds the real equilibrium E_3^R and the sliding mode Σ_1. Denote $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$, where $\Gamma_i = \Gamma \cap G_i$, $i = 1, 2, 3$. Let U be the bounded region delimited by Γ and $U_i = U \cap G_i$ for $i = 1, 2, 3$. Let the Dulac function be $D = \frac{1}{SI}$. Then

$$\iint_U \Big(\frac{\partial(Df_1)}{\partial S} + \frac{\partial(Df_2)}{\partial I}\Big) dSdI = \sum_{i=1}^{3} \iint_{U_i} \Big(\frac{\partial(DF_{i1})}{\partial S} + \frac{\partial(DF_{i2})}{\partial I}\Big) dSdI = -\frac{3\Lambda}{S^2 I} - \frac{c_1 + c_2}{S^2} < 0,$$

where f_1 is the first component of f and f_2 is the second component of f, F_{i1} is the first component of F_i and F_{i2} is the second component of F_i, $i = 1, 2, 3$. Let $\tilde{U}_i$ be the region bounded by $\tilde{\Gamma}_i$, $\tilde{P}_i$ and $\tilde{Q}_i$, where $\tilde{U}_i$ and $\tilde{\Gamma}_i$ depend on ε and converge to U_i and Γ_i as ε approaches 0. We can obtain that

$$\iint_{U_i} \Big(\frac{\partial(DF_{i1})}{\partial S} + \frac{\partial(DF_{i2})}{\partial I}\Big) dSdI = \lim_{\varepsilon \to 0} \iint_{\tilde{U}_i} \Big(\frac{\partial(DF_{i1})}{\partial S} + \frac{\partial(DF_{i2})}{\partial I}\Big) dSdI.$$

Since $dS = F_{11}dt$ and $dI = F_{12}dt$ along $\tilde{\Gamma}_1$ and $dI = 0$ along $\tilde{P}_1$, then, applying Green's theorem to region $\tilde{U}_1$, we have

$$\begin{aligned}
\iint_{\tilde{U}_1} \Big(\frac{\partial(DF_{11})}{\partial S} + \frac{\partial(DF_{12})}{\partial I}\Big) dSdI &= \oint_{\partial \tilde{U}_1} DF_{11}dI - DF_{12}dS \\
&= \int_{\tilde{\Gamma}_1} DF_{11}dI - DF_{12}dS + \int_{\tilde{P}_1} DF_{11}dI - DF_{12}dS \\
&= -\int_{\tilde{P}_1} DF_{12}dS.
\end{aligned} \tag{3.3}$$

Similarly, we can get

$$\iint_{\tilde{U}_2} \Big(\frac{\partial(DF_{21})}{\partial S} + \frac{\partial(DF_{22})}{\partial I}\Big) dSdI = -\int_{\tilde{P}_2} DF_{22}dS + \int_{\tilde{Q}_2} DF_{21}dI \tag{3.4}$$

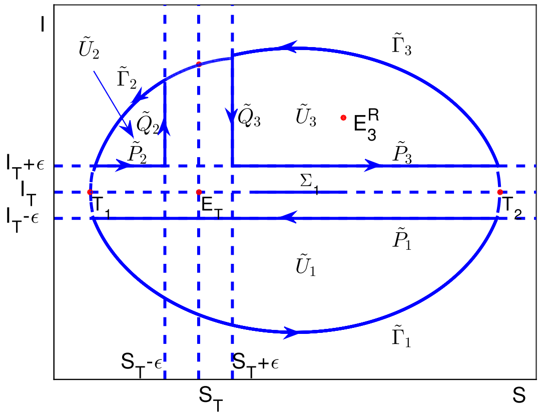

Fig. 2. Schematic diagram illustrating the nonexistence of a closed trajectory in system (2.1) with (2.2) in Case 1.1 when E_3^R is a real equilibrium.

and

$$\iint_{\tilde{U}_3}\Big(\frac{\partial(DF_{31})}{\partial S}+\frac{\partial(DF_{32})}{\partial I}\Big)dSdI=-\int_{\tilde{P}_3}DF_{32}dS+\int_{\tilde{Q}_3}DF_{31}dI. \tag{3.5}$$

Following (3.3)–(3.5), we have

$$\begin{aligned}0&>\sum_{i=1}^{3}\iint_{U_i}\Big(\frac{\partial(DF_{i1})}{\partial S}+\frac{\partial(DF_{i2})}{\partial I}\Big)dSdI=\lim_{\varepsilon\to 0}\sum_{i=1}^{3}\iint_{\tilde{U}_i}\Big(\frac{\partial DF_{i1}}{\partial S}+\frac{\partial DF_{i2}}{\partial I}\Big)dSdI\\&=\lim_{\varepsilon\to 0}\Big(-\int_{\tilde{P}_1}DF_{12}dS-\int_{\tilde{P}_2}DF_{22}dS+\int_{\tilde{Q}_2}DF_{21}dI-\int_{\tilde{P}_3}DF_{32}dS+\int_{\tilde{Q}_3}DF_{31}dI\Big).\end{aligned} \tag{3.6}$$

Denote the intersection points of the closed trajectory Γ and the line $I=I_T$ by T_1 and T_2, and the intersection point of Γ and the line $S=S_T$ in the region of $I>I_T$ by T_3. In addition, denote the intersection point of the line $I=I_T$ and the line $S=S_T$ by E_T. Note that $T_{11}<S_T<T_{21}$ and $T_{32}>I_T$. Then the inequality (3.6) becomes

$$\begin{aligned}0&>-\int_{T_{21}}^{T_{11}}\Big(\beta-\frac{\mu+\alpha}{S}\Big)dS-\int_{T_{11}}^{S_T}\Big(\beta-\frac{\mu+\alpha+c_1}{S}\Big)dS+\int_{I_T}^{T_{32}}\Big(\frac{\Lambda}{SI}-\beta-\frac{\mu-r_1}{I}+\frac{c_1}{S}\Big)dI\\&\quad-\int_{S_T}^{T_{21}}\Big(\beta-\frac{\mu+\alpha+c_2}{S}\Big)dS+\int_{T_{32}}^{I_T}\Big(\frac{\Lambda}{SI}-\beta-\frac{\mu}{I}+\frac{c_2}{S}\Big)dI\\&=c_1\ln\Big(\frac{S_T}{T_{11}}\Big)+c_2\ln\Big(\frac{T_{21}}{S_T}\Big)+r_1\ln\Big(\frac{T_{32}}{I_T}\Big)+\int_{I_T}^{T_{32}}\Big(\frac{c_1-c_2}{S}\Big)dI>0.\end{aligned}$$

This is a contradiction. Consequently, this rules out the existence of such a closed trajectory Γ surrounding the sliding mode and the real equilibrium E_3^R. □

Throughout this paper, the S-nullclines and I-nullclines of the system (2.1) with (2.2) are represented by the blue dashed curves and black dash–dot lines, respectively. Note that $S=S_1^*,S_2^*$ and S_3^* are the I-nullclines of F_1, F_2 and F_3, denoted by L_{12}, L_{22} and L_{32}, respectively. Moreover, the curve $\{(S,I)\in G_1:\Lambda-\beta SI-\mu S=0\}$ is the S-nullcline of system F_1, denoted by L_{11}, whereas the curves $\{(S,I)\in G_2:\Lambda-\beta SI-\mu S+r_1S+c_1I=0\}$ and $\{(S,I)\in G_3:\Lambda-\beta SI-\mu S+c_2I=0\}$ are the S-nullclines of system F_2 and F_3, denoted by L_{21} and L_{31}, respectively.

Fig. 3 indicates that all solutions of system (2.1) with (2.2) with any initial values in $\mathbb{R}_+^2$ will converge to E_3^R as $t\to\infty$, as stated in Theorem 3.2. Note that, in this case, the eventual number of infected trees is larger than the infected threshold level I_T.

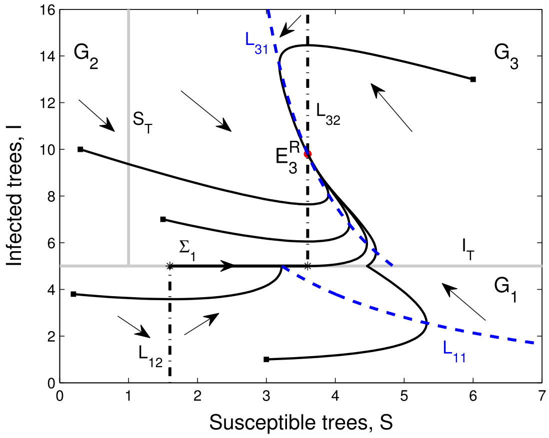

Fig. 3. E_3^R is globally asymptotically stable in Case 1.1. Parameter values are chosen as follows: $\Lambda = 5$, $\beta = 0.25$, $\mu = 0.3$, $\alpha = 0.1$, $r_1 = 0.15$, $c_1 = 0.8$, $c_2 = 0.5$, $S_T = 1$ and $I_T = 5$.

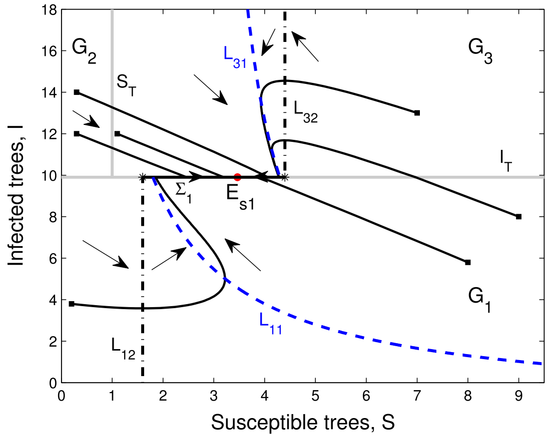

Fig. 4. E_{s1} is globally asymptotically stable in Case 1.2. Parameter values are fixed as follows: $\Lambda = 5$, $\beta = 0.25$, $\mu = 0.3$, $\alpha = 0.1$, $r_1 = 0.15$, $c_1 = 0.8$, $c_2 = 0.7$, $S_T = 1$ and $I_T = 9.9$.

3.3.2. Case 1.2: $I_3^* < I_T < I_1^*$

In this case, both E_1 and E_3 are virtual equilibria, denoted by E_1^V and E_3^V, respectively. Nevertheless, E_{s1} is a stable pseudoequilibrium on $\Sigma_1 \subset M_1$. Employing a similar method as in the proof of Theorem 3.2 to exclude the existence of limit cycles, we can derive the following result.

Theorem 3.3. *E_{s1} is globally asymptotically stable if $S_T < S_1^* < S_3^* < S_2^*$ and $I_3^* < I_T < I_1^*$.*

All trajectories of system (2.1) with (2.2) with any initial conditions in $\mathbb{R}_+^2$ will eventually converge to E_{s1} as t increases. Thus in this case, the eventual number of infected trees is equal to the infected threshold level I_T; that is, there is no risk of an outbreak, as shown in Fig. 4.

3.3.3. Case 1.3: $I_T > I_1^*$

In this case, E_1 is a real equilibrium, whereas E_3 is a virtual equilibrium, denoted by E_1^R and E_3^V, respectively. Moreover, E_{s1} is not a pseudoequilibrium on $\Sigma_1 \subset M_1$. Then we can get the following result.

Theorem 3.4. *E_1^R is globally asymptotically stable if $S_T < S_1^* < S_3^* < S_2^*$ and $I_T > I_1^*$.*

The proof of this theorem is identical to the proof of Theorem 3.2. All orbits of system (2.1) with (2.2) with any initial values in $\mathbb{R}^2_+$ will finally converge to E_1^R as $t \to \infty$. Therefore, in this case, the eventual number of infected trees is below the infected threshold value I_T. The phase portrait for this case is represented in Fig. 5.

4. Case 2: $S_1^* < S_T < S_3^* < S_2^*$

4.1. Sliding mode on M_1 and its dynamics

For Case 2, there are two sliding domains on M_1:

$$\Sigma_2 = \{(S, I) \in M_1 : S_1^* < S < S_T\} \text{ and } \Sigma_3 = \{(S, I) \in M_1 : S_T < S < S_3^*\}. \tag{4.1}$$

The dynamics on $\Sigma_2 \subset M_1$ are described by

$$\begin{pmatrix} S' \\ I' \end{pmatrix} = \begin{pmatrix} \Lambda - \mu S + \frac{r_1}{c_1} S(\beta S - \mu - \alpha) - (\mu + \alpha) I_T \\ 0 \end{pmatrix}. \tag{4.2}$$

First, we investigate the existence of a positive equilibrium on $\Sigma_2 \subset M_1$ of system (4.2). Define

$$\Delta = (r_1(\mu + \alpha) + \mu c_1)^2 - 4\beta r_1 c_1(\Lambda - (\mu + \alpha) I_T), \; I_T^* = \frac{\Lambda}{\mu + \alpha} - \frac{(r_1(\mu + \alpha) + \mu c_1)^2}{4\beta r_1 c_1(\mu + \alpha)}.$$

Proposition 4.1. *According to the values of the infected threshold I_T, we have*

(i) *if $I_T < I_T^*$; that is, $\Delta < 0$, then there is no equilibrium for system (4.2);*
(ii) *if $I_T^* < I_T < \frac{\Lambda}{\mu+\alpha}$; that is, $0 < \Delta < (r_1(\mu + \alpha) + \mu c_1)^2$, then system (4.2) has two positive equilibria $E_{s2}^{\pm} = (S_{s2}^{\pm}, I_T)$, where $S_{s2}^{\pm} = \frac{r_1(\mu+\alpha)+\mu c_1 \pm \sqrt{\Delta}}{2\beta r_1}$;*
(iii) *if $I_T > \frac{\Lambda}{\mu+\alpha}$; that is, $\Delta > (r_1(\mu + \alpha) + \mu c_1)^2$, then system (4.2) has a unique positive equilibrium $E_{s2} = (S_{s2}^*, I_T)$, where $S_{s2}^* = \frac{r_1(\mu+\alpha)+\mu c_1 + \sqrt{\Delta}}{2\beta r_1}$.*

Next we seek conditions under which the equilibrium becomes a pseudoequilibrium on the sliding mode $\Sigma_2 \subset M_1$. Denote

$$S_T^* = \frac{r_1(\mu + \alpha) + \mu c_1}{2\beta r_1}, \; H_1 = \frac{\beta r_1 S_T^2 - S_T(r_1(\mu + \alpha) + \mu c_1) + \Lambda c_1}{c_1(\mu + \alpha)}, \tag{4.3}$$

where H_1 takes its minimum value I_T^* at $S_T = S_T^*$, $S_1^* < S_T^*$ and $H_1 > I_T^*$.

Proposition 4.2. *Suppose that $S_T^* < S_3^*$.*

(i) *When $S_1^* < S_T < S_T^*$, which implies $H_1 < I_1^*$. Then we have $E_{s2}^+ \notin \Sigma_2 \subset M_1$ and*
   - *if $H_1 < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_2 \subset M_1$.*

(ii) *When $S_T^* < S_T < S_3^*$, we have the following.*
   - *Suppose $H_1 < I_1^*$, we have*
     - *if $I_T^* < I_T < H_1$, then $E_{s2}^+ \in \Sigma_2 \subset M_1$, $E_{s2}^- \in \Sigma_2 \subset M_1$;*
     - *if $H_1 < I_T < I_1^*$, then $E_{s2}^+ \notin \Sigma_2 \subset M_1$, $E_{s2}^- \in \Sigma_2 \subset M_1$.*

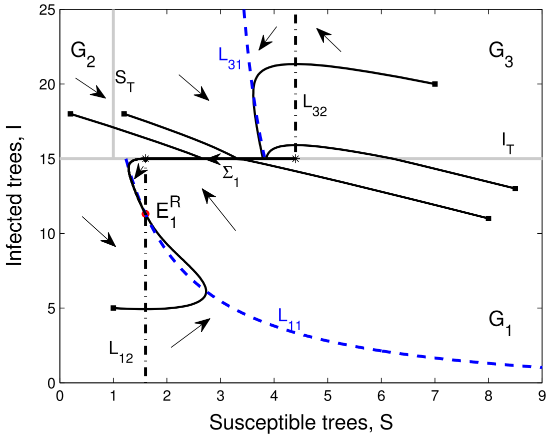

Fig. 5. E_1^R is globally asymptotically stable in Case 1.3. Parameter values are fixed as follows: $\Lambda = 5$, $\beta = 0.25$, $\mu = 0.3$, $\alpha = 0.1$, $r_1 = 0.15$, $c_1 = 0.8$, $c_2 = 0.7$, $S_T = 1$ and $I_T = 15$.

- *Suppose $H_1 > I_1^*$, we have*
  - *if $I_T^* < I_T < I_1^*$, then $E_{s2}^+ \in \Sigma_2 \subset M_1$, $E_{s2}^- \in \Sigma_2 \subset M_1$;*
  - *if $I_1^* < I_T < H_1$, then $E_{s2}^+ \in \Sigma_2 \subset M_1$, $E_{s2}^- \notin \Sigma_2 \subset M_1$.*

Proposition 4.3. *Suppose that $S_T^* > S_3^*$, which implies $H_1 < I_1^*$. Then we have $E_{s2}^+ \notin \Sigma_2 \subset M_1$ and*

- *if $H_1 < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_2 \subset M_1$.*

Theorem 4.1. *E_{s2}^- is a stable pseudoequilibrium on $\Sigma_2 \subset M_1$ if it is feasible, whereas E_{s2}^+ is an unstable pseudoequilibrium on $\Sigma_2 \subset M_1$ if it is feasible.*

Proof. We have

$$\frac{\partial}{\partial S}\Big(\Lambda - \mu S + \frac{r_1}{c_1}S(\beta S - \mu - \alpha) - (\mu + \alpha)I_T\Big)\Big|_{E_{s2}^{\pm}} = \pm\frac{\sqrt{\Delta}}{c_1}.$$

Hence the positive root is repelling and the negative root is attracting. □

Proposition 4.4. *Since $S_{s2}^* \geq 2S_T^* > S_2^* > S_T$, then E_{s2} is never a pseudoequilibrium on $\Sigma_2 \subset M_1$.*

The dynamics on $\Sigma_3 \subset M_1$ are governed by (3.2), and $E_{s1} = (S_{s1}^*, I_T) \in \Sigma_3 \subset M_1$ is a pseudoequilibrium if and only if $S_T < S_{s1}^* < S_3^*$; i.e., $I_3^* < I_T < H_2$, where

$$H_2 = \frac{\Lambda - \mu S_T}{\mu + \alpha}, \quad \text{with} \quad I_3^* < H_2 < I_1^* \text{ and } \; H_1 > H_2 > I_3^*.$$

Furthermore, E_{s1} is stable if it is feasible, as shown in Theorem 3.1.

4.2. Sliding mode on M_2 and its dynamics

From $\langle n_2, F_2\rangle > 0$ and $\langle n_2, F_3\rangle < 0$, we have $I(\beta S_T - c_1) < \Lambda - (\mu - r_1)S_T$ and $I(\beta S_T - c_2) > \Lambda - \mu S_T$. In Case 2, since $S_1^* < S_T < S_3^* < S_2^*$, we can get

$$\mu + \alpha - c_2 < \beta S_T - c_2 < \mu + \alpha \;\text{ and }\; \mu + \alpha - c_1 < \beta S_T - c_1 < \mu + \alpha.$$

According to Assumption 1, on the one hand, if $\mu+\alpha-c_2<\beta S_T-c_2\le 0$, similar to Case 1, there is no sliding mode on M_2; if $0<\beta S_T-c_2<\mu+\alpha$, from $\langle n_2,F_3\rangle<0$, we have $I>\frac{\Lambda-\mu S_T}{\beta S_T-c_2}$. On the other hand, if $\mu+\alpha-c_1<\beta S_T-c_1\le 0$, then $I(\beta S_T-c_1)<\Lambda-(\mu-r_1)S_T$ holds for any $I>I_T$; if $0<\beta S_T-c_1<\mu+\alpha$, from $\langle n_2,F_2\rangle>0$, we have $I<\frac{\Lambda-(\mu-r_1)S_T}{\beta S_T-c_1}$. Denote

$$B_1=\frac{\Lambda-\mu S_T}{\beta S_T-c_2},\ B_2=\frac{\Lambda-(\mu-r_1)S_T}{\beta S_T-c_1}.$$

Based on the above discussions, we derive the following results.

Theorem 4.2. *Let Assumption 1 hold.*

(i) *If $\beta S_T-c_2\le 0$, then there is no sliding mode on M_2.*
(ii) *If $\beta S_T-c_2>0$ and $\beta S_T-c_1\le 0$, then the sliding domain on M_2 is*

$$\Sigma_4=\{(S,I)\in M_2: I>\max\{I_T,B_1\}\}. \tag{4.4}$$

(iii) *If $\beta S_T-c_1>0$, then the sliding domain on M_2 is*

$$\Sigma_5=\{(S,I)\in M_2: \max\{I_T,B_1\}<I<B_2\}. \tag{4.5}$$

Therefore, in Case 2, there are three situations on M_2: no sliding mode, sliding domain Σ_4 or sliding domain Σ_5. First, if there is no sliding mode on M_2, the analysis is similar to Case 1. Next we consider the dynamics on $\Sigma_4\subset M_2$ or $\Sigma_5\subset M_2$. Again, by the Filippov convex method [21,30], the sliding-mode equations along the manifold $\Sigma_4\subset M_2$ or $\Sigma_5\subset M_2$ are the same, which can be described by

$$\begin{pmatrix}S'\\ I'\end{pmatrix}=\begin{pmatrix}0\\ I\left(\beta S_T-\mu-\alpha-c_2+(c_1-c_2)\frac{\Lambda-\beta S_T I-\mu S_T+c_2 I}{r_1 S_T+(c_1-c_2)I}\right)\end{pmatrix}. \tag{4.6}$$

System (4.6) has an equilibrium $E_{s3}=(S_T,I_{s3}^*)$, where

$$I_{s3}^*=\frac{r_1S_T(\beta S_T-\mu-\alpha-c_2)+(\Lambda-\mu S_T)(c_1-c_2)}{(\mu+\alpha)(c_1-c_2)}.$$

Since $I_{s3}^*<B_1$, so E_{s3} is never a pseudoequilibrium on $\Sigma_4\subset M_2$ or $\Sigma_5\subset M_2$. This implies that, even if there exists a sliding domain (Σ_4 or Σ_5) on M_2, there is no pseudoequilibrium.

4.3. Global behaviour

For Case 2, E_2 is a virtual equilibrium, denoted by E_2^V, so it is not present in region G_2. However, E_1 and E_3 may be present, depending on the values of the infected threshold I_T.

4.3.1. Case 2.1: $I_T<I_3^*$

In this case, E_1 is a virtual equilibrium, whereas E_3 is a real equilibrium, denoted by E_1^V and E_3^R, respectively. Furthermore, E_{s1} is not a pseudoequilibrium on $\Sigma_3\subset M_1$. After a simple calculation, we first give the following equivalent relation.

Proposition 4.5. *If $I_T^*<I_3^*$, we have $S_T^*>S_3^*$.*

Note that the equivalent relations between I_3^* and I_T^* cannot be determined. Then, in this case, we consider the two situations: $I_3^*>I_T^*$ and $I_3^*<I_T^*$. According to Propositions 4.2 and 4.3, we can obtain the following results.

Proposition 4.6. *Suppose that $I_T^* < I_3^*$.*

(i) *If $I_T < I_T^*$, then there is no equilibrium on $\Sigma_2 \subset M_1$.*
(ii) *If $I_T^* < I_T < I_3^*$, then $E_{s2}^- \notin \Sigma_2 \subset M_1$, $E_{s2}^+ \notin \Sigma_2 \subset M_1$.*

Proposition 4.7. *Suppose that $I_T^* > I_3^*$. Then there is no equilibrium on $\Sigma_2 \subset M_1$.*

Therefore, E_3^R is the unique equilibrium of system (2.1) with (2.2). Again, using the approach similar to the one used in the proof of Theorem 3.2, we can get the following theorem.

Theorem 4.3. *E_3^R is globally asymptotically stable if $S_1^* < S_T < S_3^* < S_2^*$ and $I_T < I_3^*$.*

In this case, we choose parameters as follows: $\Lambda = 5$, $\beta = 0.25$, $\mu = 0.3$, $\alpha = 0.2$, $r_1 = 0.2$ and $I_T = 5$. According to the values of c_1, c_2 and S_T, we can determine the existence of the sliding mode on M_2.

I. First, we choose $c_1 = 0.8$, $c_2 = 0.7$ and $S_T = 2.6$ such that $\beta S_T - c_2 < 0$. From Theorem 4.2(i), there is no sliding mode on M_2. The phase portrait is given in Fig. 6(A).
II. Next we choose $c_1 = 0.8$, $c_2 = 0.55$ and $S_T = 3.2$ such that $\beta S_T - c_2 > 0$ and $\beta S_T - c_1 < 0$. From Theorem 4.2(ii), so there is a sliding domain Σ_4 on M_2. Fig. 6(B) displays the phase portrait for this situation.
III. Finally, we choose $c_1 = 0.6$, $c_2 = 0.55$ and $S_T = 3.6$ such that $\beta S_T - c_1 > 0$. From Theorem 4.2(iii), there is a sliding domain Σ_5 on M_2. The phase portrait is represented in Fig. 6(C).

From Fig. 6, we see that whether there is a sliding mode on M_2 or not, all solutions of system (2.1) with (2.2) will approach E_3^R as t increases, as stated in Theorem 4.3.

4.3.2. Case 2.2: $I_3^* < I_T < I_1^*$

In this case, both E_1 and E_3 are virtual equilibria, denoted by E_1^V and E_3^V, respectively. According to Propositions 4.2 and 4.3, we discuss the two situations: $I_3^* > I_T^*$ and $I_3^* < I_T^*$.

Proposition 4.8. *Suppose that $I_T^* < I_3^*$, which implies $S_T^* > S_3^*$ and $H_1 < I_1^*$. Then we have $E_{s2}^+ \notin \Sigma_2 \subset M_1$ and*

(i) *if $I_3^* < I_T < H_2$, then $E_{s1} \in \Sigma_3 \subset M_1$, $E_{s2}^- \notin \Sigma_2 \subset M_1$;*
(ii) *if $H_2 < I_T < H_1$, then $E_{s1} \notin \Sigma_3 \subset M_1$, $E_{s2}^- \notin \Sigma_2 \subset M_1$;*
(iii) *if $H_1 < I_T < I_1^*$, then $E_{s1} \notin \Sigma_3 \subset M_1$, $E_{s2}^- \in \Sigma_2 \subset M_1$.*

Proposition 4.9. *Suppose that $I_T^* > I_3^*$.*

(i) *When $I_3^* < I_T < I_T^*$, then there is no equilibrium on $\Sigma_2 \subset M_1$.*
  - *Suppose that $I_T^* > H_2$. Then*
    - *if $I_3^* < I_T < H_2$, then $E_{s1} \in \Sigma_3 \subset M_1$;*
    - *if $H_2 < I_T < I_T^*$, then $E_{s1} \notin \Sigma_3 \subset M_1$.*
  - *Suppose that $I_T^* < H_2$. Then we have $E_{s1} \in \Sigma_3 \subset M_1$.*

(ii) *When $I_T^* < I_T < I_1^*$, we have the following.*
  - *Suppose that $S_T^* < S_3^*$.*
    - *If $S_1^* < S_T < S_T^*$, which implies $H_1 < I_1^*$. Then we have $E_{s2}^+ \notin \Sigma_2 \subset M_1$.*

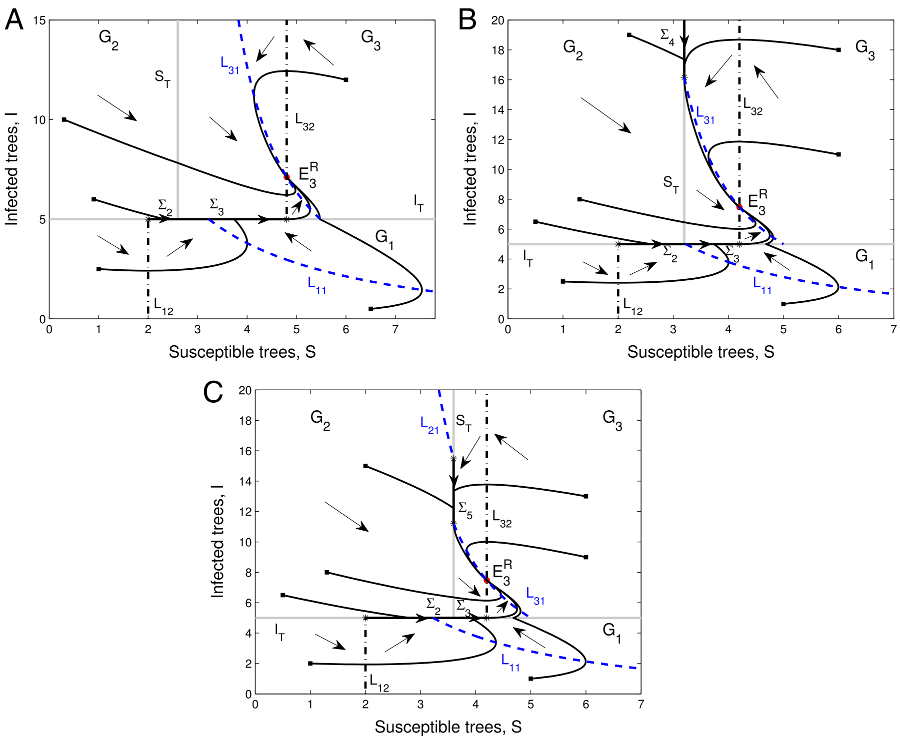

Fig. 6. E_3^R is globally asymptotically stable in Case 2.1. For (A), there is no sliding domain on M_2. For (B), there is a sliding domain Σ_4 on M_2. For (C), there is a sliding domain Σ_5 on M_2.

- *Suppose that $I_T^* > H_2$. Then we have $E_{s1} \notin \Sigma_3 \subset M_1$.*
  - *If $I_T^* < I_T < H_1$, then $E_{s2}^- \notin \Sigma_2 \subset M_1$.*
  - *If $H_1 < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_2 \subset M_1$.*
- *Suppose that $I_T^* < H_2$.*
  - *If $I_T^* < I_T < H_2$, then $E_{s1} \in \Sigma_3 \subset M_1$, $E_{s2}^- \notin \Sigma_2 \subset M_1$.*
  - *If $H_2 < I_T < H_1$, then $E_{s1} \notin \Sigma_3 \subset M_1$, $E_{s2}^- \notin \Sigma_2 \subset M_1$.*
  - *If $H_1 < I_T < I_1^*$, then $E_{s1} \notin \Sigma_3 \subset M_1$, $E_{s2}^- \in \Sigma_2 \subset M_1$.*

– *If $S_T^* < S_T < S_3^*$, we have $I_T^* > H_2$. Then $E_{s1} \notin \Sigma_3 \subset M_1$, $E_{s2}^- \in \Sigma_2 \subset M_1$.*

- *Suppose $H_1 > I_1^*$. Then we have $E_{s2}^+ \in \Sigma_2 \subset M_1$.*
- *Suppose $H_1 < I_1^*$. Then we have*
  - *if $I_T^* < I_T < H_1$, then $E_{s2}^+ \in \Sigma_2 \subset M_1$;*
  - *if $H_1 < I_T < I_1^*$, then $E_{s2}^+ \notin \Sigma_2 \subset M_1$.*

• *Suppose that $S_T^* > S_3^*$, which implies $H_1 < I_1^*$. Then we have $E_{s2}^+ \notin \Sigma_2 \subset M_1$.*

– *Suppose that $I_T^* > H_2$. Then we have $E_{s1} \notin \Sigma_3 \subset M_1$.*

- *If $I_T^* < I_T < H_1$, then $E_{s2}^- \notin \Sigma_2 \subset M_1$.*
- *If $H_1 < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_2 \subset M_1$.*

– *Suppose that $I_T^* < H_2$.*

- *If $I_T^* < I_T < H_2$, then $E_{s1} \in \Sigma_3 \subset M_1$, $E_{s2}^- \notin \Sigma_2 \subset M_1$.*

* *If $H_2 < I_T < H_1$, then $E_{s1} \notin \Sigma_3 \subset M_1$, $E_{s2}^- \notin \Sigma_2 \subset M_1$.*
* *If $H_1 < I_T < I_1^*$, then $E_{s1} \notin \Sigma_3 \subset M_1$, $E_{s2}^- \in \Sigma_2 \subset M_1$.*

All these conditions can be condensed into four situations.

I. $E_{s1} \in \Sigma_3 \subset M_1$, $E_{s2}^- \notin \Sigma_2 \subset M_1$, $E_{s2}^+ \notin \Sigma_2 \subset M_1$, when (S_T, I_T) belongs to $\Omega_1 \cup \Omega_2$:

$$\Omega_1 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_1^* < S_T < \min\{S_3^*, S_T^*\},\ \ \max\{I_3^*, I_T^*\} < I_T < H_2,\ \text{if}\ I_T^* < H_2\},$$
$$\Omega_2 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*,\ I_3^* < I_T < \min\{I_T^*, H_2\},\ \text{if}\ I_T^* > I_3^*\}.$$

$E_{s1} \in \Sigma_3 \subset M_1$ is a globally asymptotically stable pseudoequilibrium. All solutions with any initial conditions in $\mathbb{R}_+^2$ will approach E_{s1} as t increases, as shown in Fig. 7(A).

II. $E_{s1} \notin \Sigma_3 \subset M_1$, $E_{s2}^- \notin \Sigma_2 \subset M_1$, $E_{s2}^+ \notin \Sigma_2 \subset M_1$, when (S_T, I_T) belongs to $\Omega_3 \cup \Omega_4 \cup \Omega_5$:

$$\Omega_3 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*,\ H_2 < I_T < H_1,\ \text{if}\ I_T^* < I_3^*\},$$
$$\Omega_4 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*,\ H_2 < I_T < I_T^*,\ \text{if}\ I_T^* > H_2\},$$
$$\Omega_5 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_1^* < S_T < \min\{S_3^*, S_T^*\},\ \ \max\{I_T^*, H_2\} < I_T < H_1,\ \text{if}\ I_T^* > I_3^*\}.$$

No equilibrium exists for system (2.1) with (2.2), and all trajectories will converge in finite time to the equilibrium point $E_T = (S_T, I_T)$. The phase portrait is displayed in Fig. 7(B).

III. $E_{s1} \notin \Sigma_3 \subset M_1$, $E_{s2}^- \in \Sigma_2 \subset M_1$, $E_{s2}^+ \notin \Sigma_2 \subset M_1$, when (S_T, I_T) belongs to Ω_6:

$$\Omega_6 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*,\ H_1 < I_T < I_1^*,\ \text{if}\ H_1 < I_1^*\}.$$

$E_{s2}^- \in \Sigma_2 \subset M_1$ is a globally asymptotically stable pseudoequilibrium. All orbits with arbitrary initial values in $\mathbb{R}_+^2$ will converge to E_{s2}^- as $t \to \infty$, as represented in Fig. 7(C).

IV. $E_{s1} \notin \Sigma_3 \subset M_1$, $E_{s2}^- \in \Sigma_2 \subset M_1$, $E_{s2}^+ \in \Sigma_2 \subset M_1$, when (S_T, I_T) belongs to Ω_7:

$$\Omega_7 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_T^* < S_T < S_3^*,\ I_T^* < I_T < \min\{H_1, I_1^*\},\ \text{if}\ S_T^* < S_3^*, I_T^* > I_3^*\}.$$

$E_{s2}^- \in \Sigma_2 \subset M_1$ is stable, whereas $E_{s2}^+ \in \Sigma_2 \subset M_1$ is unstable. All solutions of system (2.1) with (2.2) will eventually approach E_{s2}^- or E_T. The phase portrait is given in Fig. 7(D).

4.3.3. Case 2.3: $I_T > I_1^*$

In this case, E_1 is a real equilibrium, whereas E_3 is a virtual equilibrium, denoted by E_1^R and E_3^V, respectively. Furthermore, E_{s1} is not a pseudoequilibrium on $\Sigma_3 \subset M_1$. According to Propositions 4.2 and 4.3, E_{s2}^- is not a pseudoequilibrium on $\Sigma_2 \subset M_1$. Then we can get the following results.

Proposition 4.10. *In Case 2.3, $I_T > I_1^*$, E_1^R is a real equilibrium.*

(i) *Suppose $H_1 > I_1^*$. Then we have:*

- *if $I_1^* < I_T < H_1$, then $E_{s2}^+ \in \Sigma_2 \subset M_1$;*
- *if $I_T > H_1$, then $E_{s2}^+ \notin \Sigma_2 \subset M_1$.*

(ii) *Suppose $H_1 < I_1^*$. Then we have $E_{s2}^+ \notin \Sigma_2 \subset M_1$.*

Therefore we summarize these into two situations.

I. $E_{s2}^+ \notin \Sigma_2 \subset M_1$, when (S_T, I_T) belongs to $\tilde{\Omega}_1$:

$$\tilde{\Omega}_1 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*,\ I_T > \max\{I_1^*, H_1\}\}.$$

E_1^R is globally asymptotically stable. All solutions with any initial conditions in $\mathbb{R}_+^2$ will eventually approach E_1^R as t increases. The phase portrait is given in Fig. 8(A).

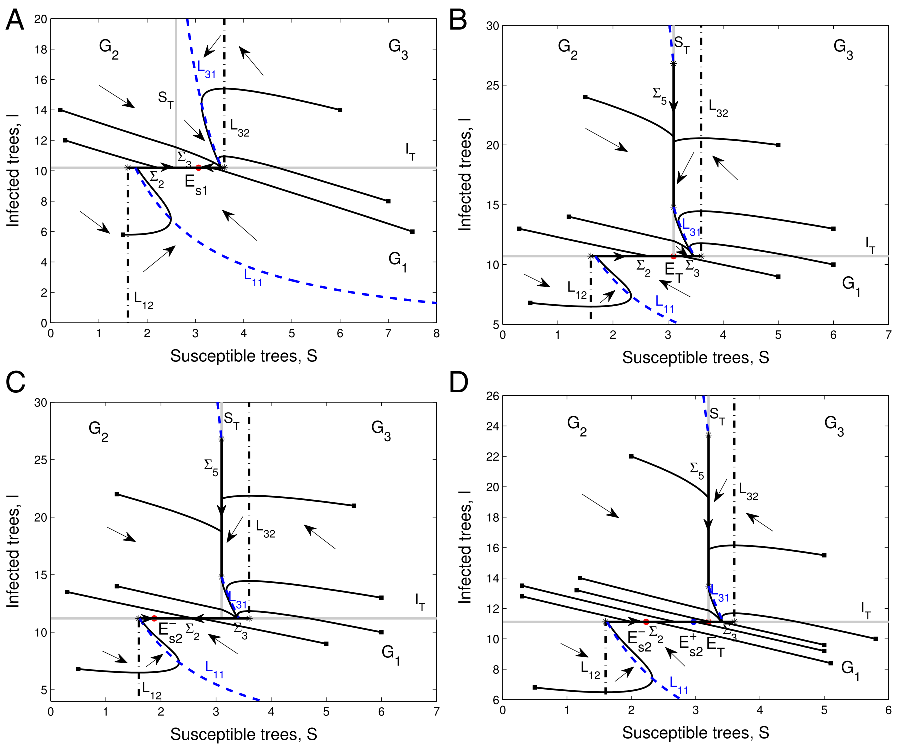

Fig. 7. Basic behaviour of solutions of system (2.1) with (2.2) in Case 2.2. Parameters are fixed as follows: $\Lambda = 5$, $\beta = 0.25$, $\mu = 0.3$, $\alpha = 0.1$, $r_1 = 0.2$, $c_1 = 0.6$, $c_2 = 0.5$. For (A), we choose $S_T = 2.6$ and $I_T = 10.2$ such that situation I is satisfied. For (B), we choose $S_T = 3.1$ and $I_T = 10.7$ such that situation II is satisfied. For (C), we choose $S_T = 3.1$ and $I_T = 11.2$ such that situation III is satisfied. For (D), we choose $S_T = 3.2$ and $I_T = 11.12$ such that situation IV is satisfied.

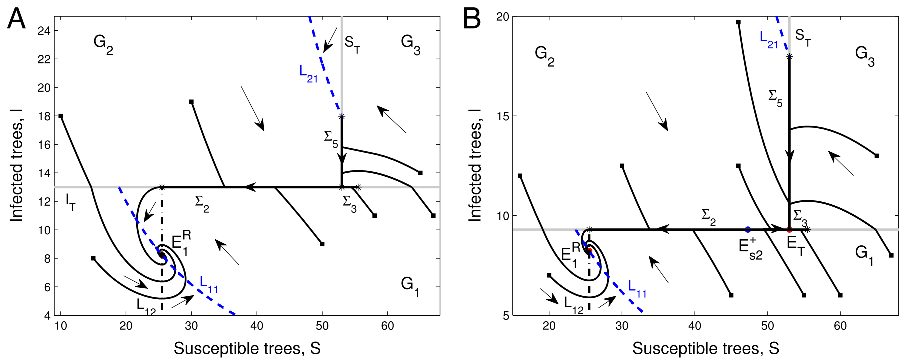

Fig. 8. Basic behaviour of solutions of system (2.1) with (2.2) in Case 2.3. Parameters are fixed as follows: $\Lambda = 7$, $\beta = 0.02$, $\mu = 0.11$, $\alpha = 0.4$, $r_1 = 0.1$, $c_1 = 0.7$, $c_2 = 0.6$, $S_T = 53$. For (A), we choose $I_T = 9.3$ such that situation I is satisfied. For (B), we choose $I_T = 13$ such that situation II is satisfied.

II. $E_{s2}^{+} \in \Sigma_2 \subset M_1$, when (S_T, I_T) belongs to $\tilde{\Omega}_2$:

$$\tilde{\Omega}_2 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_1^* < S_T < S_3^*,\ I_1^* < I_T < H_1, \text{if } H_1 > I_1^*\}.$$

E_{s2}^{+} is an unstable pseudoequilibrium on $\Sigma_2 \subset M_1$. Thus all trajectories will converge to E_1^R or $E_T = (S_T, I_T)$, as shown in Fig. 8(B).

5. Case 3: $S_1^* < S_3^* < S_T < S_2^*$

5.1. Sliding mode on M_1 and its dynamics

For Case 3, the sliding domain on M_1 is

$$\Sigma_6 = \{(S, I) \in M_1 : S_1^* < S < S_T\}. \tag{5.1}$$

The dynamics on $\Sigma_6 \subset M_1$ are described by (4.2). Then we discuss conditions under which there exists a pseudoequilibrium on the sliding mode $\Sigma_6 \subset M_1$.

Proposition 5.1. *Suppose that $S_T^* < S_3^*$.*

(i) *Suppose $H_1 < I_1^*$. Then we have:*

- *if $I_T^* < I_T < H_1$, then $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$;*
- *if $H_1 < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \notin \Sigma_6 \subset M_1$.*

(ii) *Suppose $H_1 > I_1^*$. Then we have:*

- *if $I_T^* < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$;*
- *if $I_1^* < I_T < H_1$, then $E_{s2}^- \notin \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$.*

Proposition 5.2. *Suppose that $S_3^* < S_T^* < S_2^*$.*

(i) *When $S_3^* < S_T < S_T^*$, which implies $H_1 < I_1^*$. Then we have $E_{s2}^+ \notin \Sigma_6 \subset M_1$ and*

- *if $H_1 < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_6 \subset M_1$.*

(ii) *When $S_T^* < S_T < S_2^*$, we have the following.*

- *Suppose $H_1 < I_1^*$. Then we have*
  - *if $I_T^* < I_T < H_1$, then $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$;*
  - *if $H_1 < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \notin \Sigma_6 \subset M_1$.*
- *Suppose $H_1 > I_1^*$. Then we have*
  - *if $I_T^* < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$;*
  - *if $I_1^* < I_T < H_1$, then $E_{s2}^- \notin \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$.*

Proposition 5.3. *Suppose that $S_T^* > S_2^*$, which implies $H_1 < I_1^*$. Then we have $E_{s2}^+ \notin \Sigma_6 \subset M_1$ and*

- *if $H_1 < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_6 \subset M_1$.*

From Proposition 4.4, E_{s2} is never a pseudoequilibrium on $\Sigma_6 \subset M_1$.

5.2. Sliding mode on M_2 and its dynamics

In Case 3, since $S_1^* < S_3^* < S_T < S_2^*$, then $\beta S_T - c_2 > \mu + \alpha > 0$ and $\mu + \alpha + c_2 - c_1 < \beta S_T - c_1 < \mu + \alpha$. Therefore if $\beta S_T - c_1 \leq 0$, the sliding domain on M_2 is

$$\tilde{\Sigma}_7 = \{(S, I) \in M_2 : I > \max\{I_T, B_1\}\}. \tag{5.2}$$

If $\beta S_T - c_1 > 0$, the sliding domain on M_2 is

$$\Sigma_7 = \{(S, I) \in M_2 : \max\{I_T, B_1\} < I < B_2\}. \tag{5.3}$$

The dynamics of the sliding domain $\tilde{\Sigma}_7$ or Σ_7 on M_2 are described by (4.6). System (4.6) has a unique equilibrium $E_{s3} = (S_T, I_{s3}^*)$. After a simple calculation, we can derive the following inequalities.

Proposition 5.4. *In Case 3, since $S_1^* < S_3^* < S_T < S_2^*$, if $\beta S_T - c_1 \le 0$, we have $I_{s3}^* > B_1$; if $\beta S_T - c_1 > 0$, we have $B_1 < I_3^*$, $B_2 > I_2^*$ and $B_1 < I_{s3}^* < B_2$.*

Theorem 5.1. *E_{s3} is a stable pseudoequilibrium if it is feasible.*

Proof. We have

$$\frac{\partial}{\partial I}\Big(I\big(\beta S_T - \mu - \alpha - c_2 + (c_1 - c_2)\frac{\Lambda - \beta S_T I - \mu S_T + c_2 I}{r_1 S_T + (c_1 - c_2)I}\big)\Big)\Big|_{E_{s3}}$$
$$= -\frac{(c_1 - c_2)I_{s3}^*\big(r_1 S_T(\beta S_T - c_2) + (\Lambda - \mu S_T)(c_1 - c_2)\big)}{(r_1 S_T + (c_1 - c_2)I_{s3}^*)^2} < 0.$$

Hence solutions are attracting. □

This implies that, E_{s3} is a stable pseudoequilibrium on $\Sigma_7 \subset M_2$ or $\tilde{\Sigma}_7 \subset M_2$ if $I_{s3}^* > I_T$. Since the global dynamics of the Filippov system cannot be affected by the sliding domain $\tilde{\Sigma}_7$ or Σ_7, we mainly consider the sliding mode Σ_7 in this section.

5.3. Global behaviour

In Case 3, E_2 and E_3 are virtual equilibria, denoted by E_2^V and E_3^V, respectively. However, E_1 may be present, depending on the values of the infected threshold I_T.

5.3.1. Case 3.1: $I_T < I_1^*$

In this case, E_1 is a virtual equilibrium, denoted by E_1^V. After a simple calculation, we have $I_{s3}^* < H_1$. According to Propositions 5.1–5.3, we can obtain the following results.

Proposition 5.5. *When $I_T < I_T^*$, then there is no equilibrium on $\Sigma_6 \subset M_1$.*

(i) *Suppose that $I_T^* < I_{s3}^*$. Then we have $E_{s3} \in \Sigma_7 \subset M_2$.*
(ii) *Suppose that $I_T^* > I_{s3}^*$.*

- *If $I_T < I_{s3}^*$, then $E_{s3} \in \Sigma_7 \subset M_2$.*
- *If $I_{s3}^* < I_T < I_T^*$, then $E_{s3} \notin \Sigma_7 \subset M_2$.*

Proposition 5.6. *Assume $I_T^* < I_T < I_1^*$.*

(i) *Suppose that $S_T^* < S_3^*$.*

- *Suppose that $I_T^* < I_{s3}^*$.*
  - *Suppose that $H_1 < I_1^*$. Then we have*
    - *if $I_T^* < I_T < I_{s3}^*$, then $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \in \Sigma_7 \subset M_2$;*
    - *if $I_{s3}^* < I_T < H_1$, then $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$;*
    - *if $H_1 < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.*

- *Suppose that* $H_1 > I_1^*$. *Then*
  - *if* $I_T^* < I_T < I_{s3}^*$, *then* $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \in \Sigma_7 \subset M_2$;
  - *if* $I_{s3}^* < I_T < I_1^*$, *then* $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.

- *Suppose that* $I_T^* > I_{s3}^*$.
  - *Suppose that* $H_1 < I_1^*$. *Then*
    - *if* $I_T^* < I_T < H_1$, *then* $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$;
    - *if* $H_1 < I_T < I_1^*$, *then* $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.
  - *Suppose that* $H_1 > I_1^*$, *then* $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.

(ii) *Suppose that* $S_3^* < S_T^* < S_2^*$.

- *Assume* $S_3^* < S_T < S_T^*$, *which implies* $H_1 < I_1^*$, *we have* $E_{s2}^+ \notin \Sigma_6 \subset M_1$.
  - *Suppose that* $I_T^* < I_{s3}^*$. *Then*
    - *if* $I_T^* < I_T < I_{s3}^*$, *then* $E_{s2}^- \notin \Sigma_6 \subset M_1$, $E_{s3} \in \Sigma_7 \subset M_2$;
    - *if* $I_{s3}^* < I_T < H_1$, *then* $E_{s2}^- \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$;
    - *if* $H_1 < I_T < I_1^*$, *then* $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.
  - *Suppose that* $I_T^* > I_{s3}^*$. *Then*
    - *if* $I_T^* < I_T < H_1$, *then* $E_{s2}^- \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$;
    - *if* $H_1 < I_T < I_1^*$, *then* $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.
- *Assume* $S_T^* < S_T < S_2^*$.
  - *Suppose that* $I_T^* < I_{s3}^*$.
    - *Suppose that* $H_1 < I_1^*$. *Then*
      - *if* $I_T^* < I_T < I_{s3}^*$, *then* $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \in \Sigma_7 \subset M_2$;
      - *if* $I_{s3}^* < I_T < H_1$, *then* $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$;
      - *if* $H_1 < I_T < I_1^*$, *then* $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.
    - *Suppose that* $H_1 > I_1^*$. *Then*
      - *if* $I_T^* < I_T < I_{s3}^*$, *then* $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \in \Sigma_7 \subset M_2$;
      - *if* $I_{s3}^* < I_T < I_1^*$, *then* $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.
  - *Suppose that* $I_T^* > I_{s3}^*$.
    - *Suppose that* $H_1 < I_1^*$. *Then*
      - *if* $I_T^* < I_T < H_1$, *then* $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$;
      - *if* $H_1 < I_T < I_1^*$, *then* $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.
    - *Suppose that* $H_1 > I_1^*$. *Then* $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.

(iii) *Suppose that* $S_T^* > S_2^*$, *which implies* $H_1 < I_1^*$. *Then we have* $E_{s2}^+ \notin \Sigma_6 \subset M_1$.

- *Suppose that* $I_T^* < I_{s3}^*$. *Then*
  - *if* $I_T^* < I_T < I_{s3}^*$, *then* $E_{s2}^- \notin \Sigma_6 \subset M_1$, $E_{s3} \in \Sigma_7 \subset M_2$;
  - *if* $I_{s3}^* < I_T < H_1$, *then* $E_{s2}^- \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$;
  - *if* $H_1 < I_T < I_1^*$, *then* $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.

- *Suppose that $I_T^* > I_{s3}^*$. Then*
  - *if $I_T^* < I_T < H_1$, then $E_{s2}^- \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$;*
  - *if $H_1 < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.*

All above discussions can be condensed into five situations.

I. $E_{s2}^- \notin \Sigma_6 \subset M_1$, $E_{s2}^+ \notin \Sigma_6 \subset M_1$, $E_{s3} \in \Sigma_7 \subset M_2$, when (S_T, I_T) belongs to $\Phi_1 \cup \Phi_2$:

$$\Phi_1 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_3^* < S_T < S_2^*,\ I_T < \min\{I_T^*, I_{s3}^*\}\},$$
$$\Phi_2 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_3^* < S_T < \min\{S_2^*, S_T^*\},\ I_T^* < I_T < I_{s3}^*,\ \text{if } I_T^* < I_{s3}^*\}.$$

E_{s3} is a globally asymptotically stable pseudoequilibrium on $\Sigma_7 \subset M_2$. All solutions with any initial conditions in $\mathbb{R}_+^2$ will approach E_{s3} as t increases, as shown in Fig. 9(A).

II. $E_{s2}^- \notin \Sigma_6 \subset M_1$, $E_{s2}^+ \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$, when (S_T, I_T) belongs to $\Phi_3 \cup \Phi_4$:

$$\Phi_3 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_3^* < S_T < S_2^*,\ I_{s3}^* < I_T < I_T^*,\ \text{if } I_T^* > I_{s3}^*\},$$
$$\Phi_4 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_3^* < S_T < \min\{S_2^*, S_T^*\},\ \max\{I_T^*, I_{s3}^*\} < I_T < H_1,\ \text{if } S_T^* > S_3^*\}.$$

No equilibrium exists for system (2.1) with (2.2). All trajectories will converge in finite time to $E_T = (S_T, I_T)$, as displayed in Fig. 9(B).

III. $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$, when (S_T, I_T) belongs to Φ_5:

$$\Phi_5 = \{(S_T, I_T) \in \mathbb{R}_+^2 : \max\{S_3^*, S_T^*\} < S_T < S_2^*,\ \max\{I_T^*, I_{s3}^*\} < I_T < \min\{H_1, I_1^*\}\}.$$

E_{s2}^- is a stable pseudoequilibrium, whereas E_{s2}^+ is unstable on $\Sigma_6 \subset M_1$. All solutions of system (2.1) with (2.2) will eventually approach E_{s2}^- or E_T, as represented in Fig. 9(C).

IV. $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$, when (S_T, I_T) belongs to Φ_6:

$$\Phi_6 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_3^* < S_T < S_2^*,\ H_1 < I_T < I_1^*,\ \text{if } H_1 < I_1^*\}.$$

E_{s2}^- is a globally asymptotically stable pseudoequilibrium on $\Sigma_6 \subset M_1$. All orbits with any initial values in $\mathbb{R}_+^2$ will approach E_{s2}^- as $t \to \infty$. The phase portrait is displayed in Fig. 9(D).

V. $E_{s2}^- \in \Sigma_6 \subset M_1$, $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \in \Sigma_7 \subset M_2$, when (S_T, I_T) belongs to Φ_7:

$$\Phi_7 = \{(S_T, I_T) \in \mathbb{R}_+^2 : \max\{S_3^*, S_T^*\} < S_T < S_2^*,\ I_T^* < I_T < I_{s3}^*, \text{if } S_T^* < S_2^*, I_T^* < I_{s3}^*\}.$$

E_{s2}^- and E_{s3} are stable pseudoequilibria, whereas E_{s2}^+ is unstable. All solutions of system (2.1) with (2.2) will approach E_{s2}^- or E_{s3}. The phase portrait is given in Fig. 9(E).

5.3.2. *Case 3.2: $I_T > I_1^*$*

In this case, E_1 is a real equilibrium, denoted by E_1^R. Furthermore, E_{s2}^- is not a pseudoequilibrium on $\Sigma_6 \subset M_1$. From Propositions 5.1–5.3, we can establish conditions under which E_{s2}^+ and E_{s3} are pseudoequilibria on $\Sigma_6 \subset M_1$ and $\Sigma_7 \subset M_2$, respectively.

Proposition 5.7. *Assume $I_T > I_1^*$.*

(i) *Suppose that $S_T^* < S_3^*$.*

- *Suppose that $H_1 < I_1^*$. Then $E_{s2}^+ \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.*
- *Suppose that $H_1 > I_1^*$.*

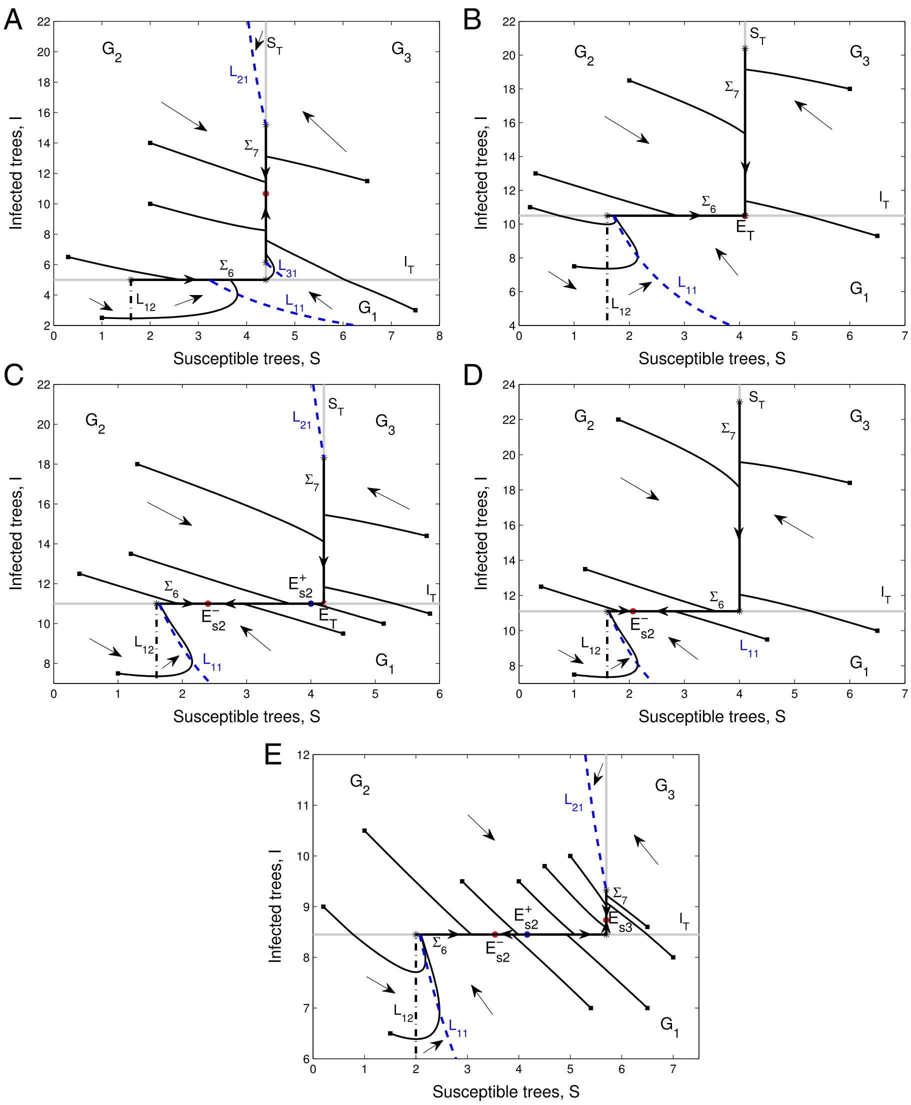

Fig. 9. Basic behaviour of solutions of system (2.1) with (2.2) in Case 3.1. Parameters are fixed as follows: $\Lambda = 5$, $\beta = 0.25$, $\mu = 0.3$, $\alpha = 0.1$, $r_1 = 0.2$, $c_1 = 0.8$, $c_2 = 0.5$. For (A), we choose $S_T = 4.4$ and $I_T = 5$ such that situation I is satisfied. For (B), we choose $S_T = 4.1$ and $I_T = 10.5$ such that situation II is satisfied. For (C), we choose $S_T = 4.2$ and $I_T = 11$ such that situation III is satisfied. For (D), we choose $S_T = 4$ and $I_T = 11.1$ such that situation IV is satisfied. For (E), we choose $\Lambda = 5$, $\beta = 0.25$, $\mu = 0.3$, $\alpha = 0.2$, $r_1 = 0.2$, $c_1 = 0.95$, $c_2 = 0.5$, $S_T = 5.7$ and $I_T = 8.45$ such that situation V is satisfied.

– *Suppose that* $I_{s3}^* < I_1^*$. *Then* $E_{s3} \notin \Sigma_7 \subset M_2$ *and*

  ∗ *if* $I_1^* < I_T < H_1$, *then* $E_{s2}^+ \in \Sigma_6 \subset M_1$;

  ∗ *if* $I_T > H_1$, *then* $E_{s2}^+ \notin \Sigma_6 \subset M_1$.

– *Suppose that* $I_{s3}^* > I_1^*$. *Then*

  ∗ *if* $I_1^* < I_T < I_{s3}^*$, *then* $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \in \Sigma_7 \subset M_2$;

  ∗ *if* $I_{s3}^* < I_T < H_1$, *then* $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$;

  ∗ *if* $I_T > H_1$, *then* $E_{s2}^+ \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.

(ii) *Suppose that* $S_3^* < S_T^* < S_2^*$.

- *Assume* $S_3^* < S_T < S_T^*$, *which implies* $H_1 < I_1^*$. *Then* $E_{s2}^+ \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.
- *Assume* $S_T^* < S_T < S_2^*$.
  - *Suppose that* $H_1 < I_1^*$. *Then we have* $E_{s2}^+ \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.
  - *Suppose that* $H_1 > I_1^*$.
    - *Suppose that* $I_{s3}^* < I_1^*$. *Then* $E_{s3} \notin \Sigma_7 \subset M_2$ *and*
      - *if* $I_1^* < I_T < H_1$, *then* $E_{s2}^+ \in \Sigma_6 \subset M_1$;
      - *if* $I_T > H_1$, *then* $E_{s2}^+ \notin \Sigma_6 \subset M_1$.
    - *Suppose that* $I_{s3}^* > I_1^*$. *Then*
      - *if* $I_1^* < I_T < I_{s3}^*$, *then* $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \in \Sigma_7 \subset M_2$;
      - *if* $I_{s3}^* < I_T < H_1$, *then* $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$;
      - *if* $I_T > H_1$, *then* $E_{s2}^+ \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.

(iii) *Suppose that* $S_T^* > S_2^*$, *which implies* $H_1 < I_1^*$. *Then we have* $E_{s2}^+ \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$.

Thus we summarize above discussions into three situations.

I. $E_{s2}^+ \notin \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$, when (S_T, I_T) belongs to $\tilde{\Phi}_1$:

$$\tilde{\Phi}_1 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_3^* < S_T < S_2^*,\ I_T > \max\{H_1, I_1^*\}\}.$$

E_1^R is globally asymptotically stable. All solutions with any initial conditions in $\mathbb{R}_+^2$ will approach E_1^R as t increases, as shown in Fig. 10(A).

II. $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \notin \Sigma_7 \subset M_2$, when (S_T, I_T) belongs to $\tilde{\Phi}_2$:

$$\tilde{\Phi}_2 = \{(S_T, I_T) \in \mathbb{R}_+^2 : \max\{S_3^*, S_T^*\} < S_T < S_2^*,\ \max\{I_1^*, I_{s3}^*\} < I_T < H_1,\ \text{if } S_T^* < S_2^*,\ I_1^* < H_1\}.$$

E_{s2}^+ is an unstable pseudoequilibrium on $\Sigma_6 \subset M_1$. All trajectories will converge to E_1^R or $E_T = (S_T, I_T)$, as shown in Fig. 10(B).

III. $E_{s2}^+ \in \Sigma_6 \subset M_1$, $E_{s3} \in \Sigma_7 \subset M_2$, when (S_T, I_T) belongs to $\tilde{\Phi}_3$:

$$\tilde{\Phi}_3 = \{(S_T, I_T) \in \mathbb{R}_+^2 : \max\{S_3^*, S_T^*\} < S_T < S_2^*,\ I_1^* < I_T < I_{s3}^*,\ \text{if } S_T^* < S_2^*,\ I_1^* < I_{s3}^*\}.$$

E_{s3} is a stable pseudoequilibrium on $\Sigma_7 \subset M_2$, whereas E_{s2}^+ is unstable on $\Sigma_6 \subset M_1$. All solutions of system (2.1) with (2.2) will eventually approach E_1^R or E_{s3}, as given in Fig. 10(C).

6. Case 4: $S_T > S_2^* > S_3^* > S_1^*$

6.1. Existence of a sliding mode on M_1 and its dynamics

For Case 4, the sliding mode on M_1 is

$$\Sigma_8 = \{(S, I) \in M_1 : S_1^* < S < S_2^*\}. \tag{6.1}$$

The dynamics on $\Sigma_8 \subset M_1$ are governed by (4.2). Next we seek conditions under which the equilibrium becomes a pseudoequilibrium on the sliding mode $\Sigma_8 \subset M_1$. Since H_1 takes its minimum value I_T^* at $S_T = S_T^*$, then $H_1|_{S_T = S_2^*} = I_2^* > I_T^*$.

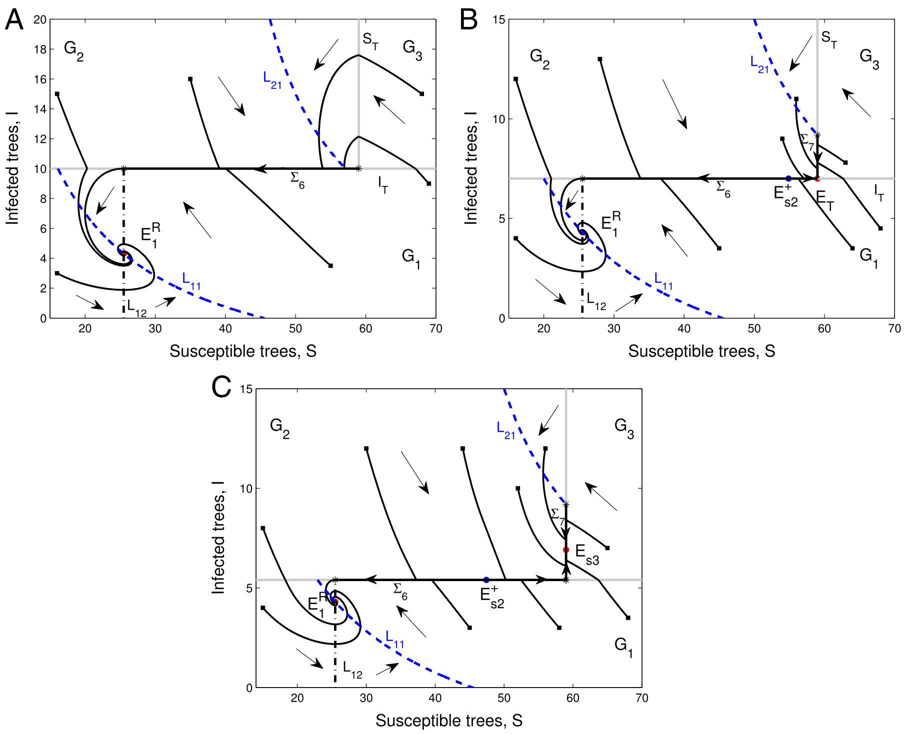

Fig. 10. Basic behaviour of solutions of system (2.1) with (2.2) in Case 3.2. Parameters are fixed as follows: $\Lambda = 5$, $\beta = 0.02$, $\mu = 0.11$, $\alpha = 0.4$, $r_1 = 0.1$, $c_1 = 0.7$, $c_2 = 0.5$ and $S_T = 59$. For (A), we choose $I_T = 10$ such that situation I is satisfied. For (B), we choose $I_T = 7$ such that situation II is satisfied. For (C), we choose $I_T = 5.4$ such that situation III is satisfied.

Proposition 6.1. *Suppose that $S_T^* > S_2^*$, which implies $I_2^* < I_1^*$. Then we have $E_{s2}^+ \notin \Sigma_8 \subset M_1$ and*

- *if $I_T^* < I_T < I_2^*$, then $E_{s2}^- \notin \Sigma_8 \subset M_1$;*
- *if $I_2^* < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_8 \subset M_1$.*

Proposition 6.2. *Suppose that $S_T^* < S_2^*$.*

(i) *Suppose that $I_2^* > I_1^*$. Then*

- *if $I_T^* < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_8 \subset M_1$, $E_{s2}^+ \in \Sigma_8 \subset M_1$;*
- *if $I_1^* < I_T < I_2^*$, then $E_{s2}^- \notin \Sigma_8 \subset M_1$, $E_{s2}^+ \in \Sigma_8 \subset M_1$.*

(ii) *Suppose that $I_2^* < I_1^*$. Then*

- *if $I_T^* < I_T < I_2^*$, then $E_{s2}^- \in \Sigma_8 \subset M_1$, $E_{s2}^+ \in \Sigma_8 \subset M_1$;*
- *if $I_2^* < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_8 \subset M_1$, $E_{s2}^+ \notin \Sigma_8 \subset M_1$.*

Again, from Proposition 4.4, E_{s2} is never a pseudoequilibrium on $\Sigma_8 \subset M_1$.

6.2. Sliding mode on M_2 and its dynamics

Since $S_1^* < S_3^* < S_2^* < S_T$, then $\beta S_T - c_2 > \mu + \alpha > 0$ and $\beta S_T - c_1 > \mu + \alpha > 0$. Therefore the sliding domain on M_2 exists if there is a nonempty set:

$$\Sigma_9 = \{(S, I) \in M_2 : \max\{I_T, B_1\} < I < B_2\}. \tag{6.2}$$

The dynamics of the sliding domain Σ_9 on M_2 are described by (4.6). System (4.6) has a unique equilibrium $E_{s3} = (S_T, I_{s3}^*)$. Since $I_{s3}^* > B_2$, so E_{s3} is never a pseudoequilibrium on $\Sigma_9 \subset M_2$.

6.3. Global behaviour

For a fixed threshold level S_T such that $S_T > S_2^* > S_3^* > S_1^*$, E_3 is a virtual equilibrium, denoted by E_3^V. However, E_1 and E_2 may be real equilibria depending on the values of the infected threshold I_T. From Propositions 6.1 and 6.2, we can get the following results.

Proposition 6.3. *Suppose $S_T^* > S_2^*$, which implies $I_2^* < I_1^*$. Then we have*

- *if $I_T < I_T^*$, then there is no equilibrium on $\Sigma_8 \subset M_1$, $E_1 \notin G_1$, $E_2 \in G_2$;*
- *if $I_T^* < I_T < I_2^*$, then $E_{s2}^- \notin \Sigma_8 \subset M_1$, $E_{s2}^+ \notin \Sigma_8 \subset M_1$, $E_1 \notin G_1$, $E_2 \in G_2$;*
- *if $I_2^* < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_8 \subset M_1$, $E_{s2}^+ \notin \Sigma_8 \subset M_1$, $E_1 \notin G_1$, $E_2 \notin G_2$;*
- *if $I_T > I_1^*$, then $E_{s2}^- \notin \Sigma_8 \subset M_1$, $E_{s2}^+ \notin \Sigma_8 \subset M_1$, $E_1 \in G_1$, $E_2 \notin G_2$.*

Proposition 6.4. *Suppose $S_T^* < S_2^*$.*

(i) *Suppose that $I_2^* > I_1^*$. Then*

- *if $I_T < I_T^*$, then there is no equilibrium on $\Sigma_8 \subset M_1$, $E_1 \notin G_1$, $E_2 \in G_2$;*
- *if $I_T^* < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_8 \subset M_1$, $E_{s2}^+ \in \Sigma_8 \subset M_1$, $E_1 \notin G_1$, $E_2 \in G_2$;*
- *if $I_1^* < I_T < I_2^*$, then $E_{s2}^- \notin \Sigma_8 \subset M_1$, $E_{s2}^+ \in \Sigma_8 \subset M_1$, $E_1 \in G_1$, $E_2 \in G_2$;*
- *if $I_T > I_2^*$, then $E_{s2}^- \notin \Sigma_8 \subset M_1$, $E_{s2}^+ \notin \Sigma_8 \subset M_1$, $E_1 \in G_1$, $E_2 \notin G_2$.*

(ii) *Suppose that $I_2^* < I_1^*$. Then*

- *if $I_T < I_T^*$, then there is no equilibrium on $\Sigma_8 \subset M_1$, $E_1 \notin G_1$, $E_2 \in G_2$;*
- *if $I_T^* < I_T < I_2^*$, then $E_{s2}^- \in \Sigma_8 \subset M_1$, $E_{s2}^+ \in \Sigma_8 \subset M_1$, $E_1 \notin G_1$, $E_2 \in G_2$;*
- *if $I_2^* < I_T < I_1^*$, then $E_{s2}^- \in \Sigma_8 \subset M_1$, $E_{s2}^+ \notin \Sigma_8 \subset M_1$, $E_1 \notin G_1$, $E_2 \notin G_2$;*
- *if $I_T > I_1^*$, then $E_{s2}^- \notin \Sigma_8 \subset M_1$, $E_{s2}^+ \notin \Sigma_8 \subset M_1$, $E_1 \in G_1$, $E_2 \notin G_2$.*

All above discussions can be summarized into four situations.

I. $E_{s2}^- \notin \Sigma_8 \subset M_1$, $E_{s2}^+ \notin \Sigma_8 \subset M_1$, $E_1 \notin G_1$, $E_2 \in G_2$, when (S_T, I_T) belongs to $\Psi_1 \cup \Psi_2$:

$$\Psi_1 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_T > S_2^*,\ I_T < I_2^*,\ \text{if } S_T^* > S_2^*\},$$
$$\Psi_2 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_T > S_2^*,\ I_T < I_T^*,\ \text{if } S_T^* < S_2^*\}.$$

E_2 is globally asymptotically stable, denoted by E_2^R. All solutions with any initial conditions in $\mathbb{R}_+^2$ will approach E_2^R as t increases. The phase portrait is shown in Fig. 11(A).

II. $E_{s2}^- \in \Sigma_8 \subset M_1$, $E_{s2}^+ \notin \Sigma_8 \subset M_1$, $E_1 \notin G_1$, $E_2 \notin G_2$, when (S_T, I_T) belongs to Ψ_3:

$$\Psi_3 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_T > S_2^*,\ I_2^* < I_T < I_1^*,\ \text{if } I_2^* < I_1^*\}.$$

E_{s2}^- is a globally asymptotically pseudoequilibrium on $\Sigma_8 \subset M_1$. All trajectories with any initial values in $\mathbb{R}_+^2$ will converge to E_{s2}^- , as displayed in Fig. 11(B).

III. $E_{s2}^- \in \Sigma_8 \subset M_1$, $E_{s2}^+ \in \Sigma_8 \subset M_1$, $E_1 \notin G_1$, $E_2 \in G_2$, when (S_T, I_T) belongs to Ψ_4:

$$\Psi_4 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_T > S_2^*,\ I_T^* < I_T < \min\{I_1^*, I_2^*\},\ \text{if } S_T^* < S_2^*\}.$$

$E_{s2}^- \in \Sigma_8 \subset M_1$ is a stable pseudoequilibrium, whereas $E_{s2}^+ \in \Sigma_8 \subset M_1$ is unstable. All solutions will eventually approach E_{s2}^- or E_2^R. The phase portrait is given in Fig. 11(C).

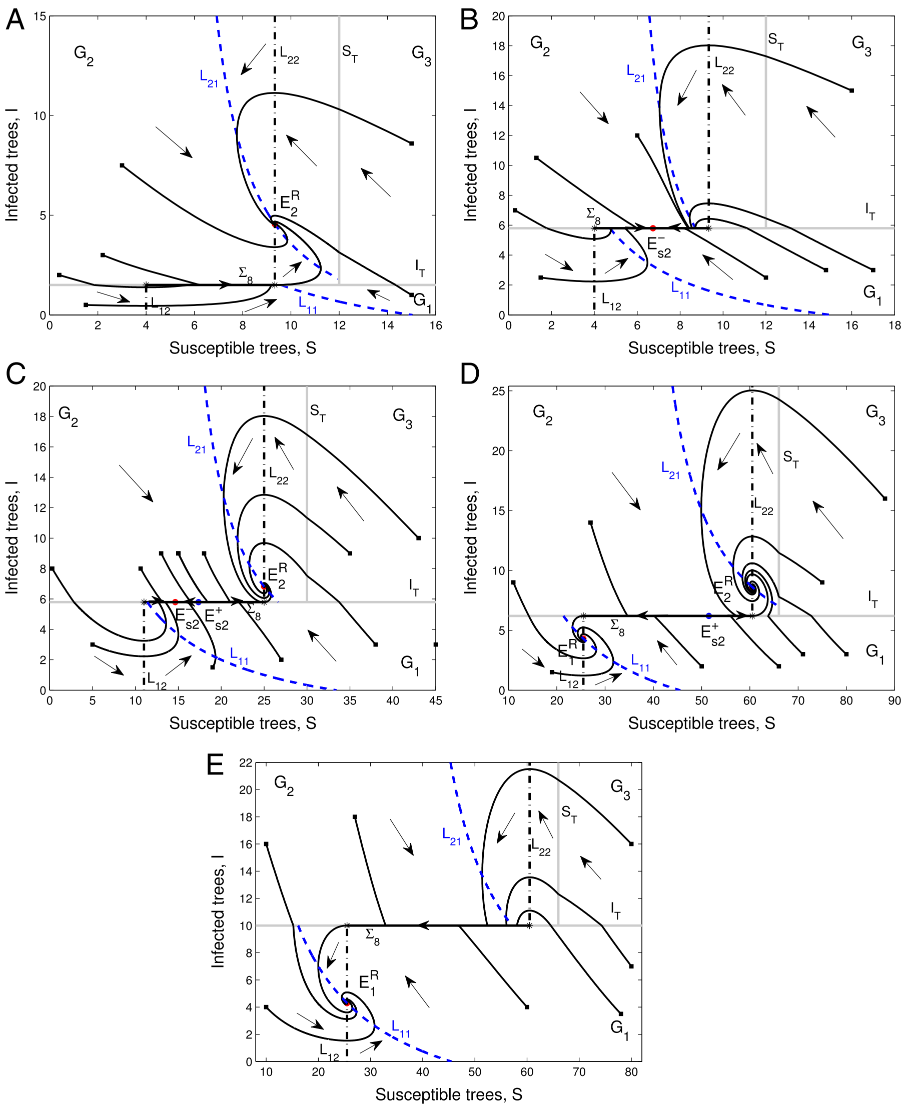

Fig. 11. Basic behaviour of solutions of system (2.1) with (2.2) in Case 4. Parameters are fixed as $\Lambda = 6$, $\beta = 0.15$, $\mu = 0.4$, $\alpha = 0.2$, $r_1 = 0.05$, $c_1 = 0.8$, $c_2 = 0.7$. $S_T = 12$. For (A), we choose $I_T = 1.5$ such that situation I is satisfied. For (B), we choose $I_T = 5.8$ such that situation II is satisfied. For (C), we choose $\Lambda = 5$, $\beta = 0.05$, $\mu = 0.15$, $\alpha = 0.4$, $r_1 = 0.1$, $c_1 = 0.7$, $c_2 = 0.6$. $S_T = 30$ and $I_T = 5.79$ such that situation III is satisfied. For (D), we choose $\Lambda = 5$, $\beta = 0.02$, $\mu = 0.11$, $\alpha = 0.4$, $r_1 = 0.1$, $c_1 = 0.7$, $c_2 = 0.6$. $S_T = 66$ and $I_T = 6.2$ such that situation IV is satisfied. For (E), we choose the same parameter values as (D) except $I_T = 10$ such that situation V is satisfied.

IV. $E_{s2}^- \notin \Sigma_8 \subset M_1$, $E_{s2}^+ \in \Sigma_8 \subset M_1$, $E_1 \in G_1$, $E_2 \in G_2$, when (S_T, I_T) belongs to Ψ_5:

$$\Psi_5 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_T > S_2^*,\ I_1^* < I_T < I_2^*,\ \text{if } S_T^* < S_2^*, I_2^* > I_1^*\}.$$

E_{s2}^+ is an unstable pseudoequilibrium on $\Sigma_8 \subset M_1$. E_1 and E_2 are real equilibria, denoted by E_1^R and E_2^R. All trajectories will finally approach E_1^R or E_2^R, as illustrated in Fig. 11(D).

V. $E_{s2}^- \notin \Sigma_8 \subset M_1$, $E_{s2}^+ \notin \Sigma_8 \subset M_1$, $E_1 \in G_1$, $E_2 \notin G_2$, when (S_T, I_T) belongs to Ψ_6:

$$\Psi_6 = \{(S_T, I_T) \in \mathbb{R}_+^2 : S_T > S_2^*,\ I_T > \max\{I_1^*, I_2^*\}\}.$$

E_1 is globally asymptotically stable, denoted by E_1^R. All solutions with any initial conditions in $\mathbb{R}_+^2$ will approach E_1^R as t increases. The phase portrait is shown in Fig. 11(E).

Table 1
Main results of system (2.1) with (2.2).

$S_T < S_1^*$	$S_1^* < S_T < S_3^*$	$S_3^* < S_T < S_2^*$	$S_T > S_2^*$
$I_T < I_3^*$: (I)	$I_T < I_3^*$: (I)	$(S_T, I_T) \in \Phi_1 \cup \Phi_2$: (I)	$(S_T, I_T) \in \Psi_1 \cup \Psi_2$: (I)
$I_1^* < I_T < I_3^*$: (II)	$(S_T, I_T) \in \Omega_1 \cup \Omega_2$: (II)	$(S_T, I_T) \in \Phi_3 \cup \Phi_4$: (II)	$(S_T, I_T) \in \Psi_3$: (II)
	$(S_T, I_T) \in \Omega_3 \cup \Omega_4 \cup \Omega_5$: (II)	$(S_T, I_T) \in \Phi_5$: (III)	$(S_T, I_T) \in \Psi_4$: (IV)
	$(S_T, I_T) \in \Omega_6$: (II)	$(S_T, I_T) \in \Phi_6$: (II)	$(S_T, I_T) \in \Psi_5$: (IV)
	$(S_T, I_T) \in \Omega_7$: (III)	$(S_T, I_T) \in \Phi_7$: (IV)	$(S_T, I_T) \in \Psi_6$: (II)
$I_T > I_1^*$: (II)	$(S_T, I_T) \in \tilde{\Omega}_1$: (II)	$(S_T, I_T) \in \tilde{\Phi}_1$: (II)	
	$(S_T, I_T) \in \tilde{\Omega}_2$: (III)	$(S_T, I_T) \in \tilde{\Phi}_2$: (III)	
		$(S_T, I_T) \in \tilde{\Phi}_3$: (IV)	

7. Discussion

We proposed a mathematical model of fire blight with discontinuous right-hand sides, resulting in a Filippov system, using a threshold policy consisting of cutting off infected branches and replanting susceptible trees. Between these two control measures, cutting off infected branches plays a leading role in reducing fire-blight infection, while the strategy of replanting susceptible trees contributes to minimizing economic losses and maximizing fruit production. Hence our main purpose is to use the Filippov system to model the threshold policy and establish conditions that not only lead the number of infected trees to a tolerable threshold level but also minimize economic losses. To achieve this aim, our formulation employed the number of infected and susceptible trees to be the threshold levels to determine whether or not we need to implement control strategies. No control strategy is necessary when the number of infected trees is less than the infected threshold value I_T; when above I_T, the infected branches are removed at a rate of c_1 and susceptible trees are replanted at a rate of r_1 if the number of susceptible trees is below the susceptible threshold level S_T; we only remove infected branches at a rate of c_2 if $S > S_T$. Therefore the Filippov fire-blight model (2.1) with (2.2) constructed here can be used to describe the spread of fire blight in an orchard associated with such a threshold policy.

Making use of the analysis of the dynamics of system (2.1) with (2.2), we summarize the main results in Table 1 associated with following outcomes.

(I) The infected threshold value I_T is sufficiently small, so it is impossible to avoid an outbreak. The number of infected trees will increase above I_T to reach the level of a globally asymptotically stable equilibrium, as shown in Figs. 3, 6, 9(A) and 11(A).

(II) System (2.1) with (2.2) has a unique globally asymptotically stable equilibrium E_1^R, a unique globally asymptotically stable pseudoequilibrium (E_{s2}^- or E_{s1}) on the manifold M_1 or no equilibrium. For these choices of the threshold values S_T and I_T, the number of infected trees will converge to the globally asymptotically stable equilibrium that lies below I_T or on $I = I_T$, or converge to $E_T = (S_T, I_T)$, as represented in Figs. 4, 5, 7(A), 7(B), 7(C), 8(A), 9(B), 9(D), 10(A), 11(B) and 11(E).

(III) There is a locally asymptotically stable pseudoequilibrium E_{s2}^- on the manifold M_1 or a locally asymptotically stable E_1^R and an unstable pseudoequilibrium E_{s2}^+ on the manifold M_1. So the number of infected trees can eventually approach the locally asymptotically stable equilibrium E_{s2}^- or E_1^R, or converge to E_T, as shown in Figs. 7(D), 8(B), 9(C) and 10(B).

(IV) All solutions of system (2.1) with (2.2) will converge to a locally asymptotically stable equilibrium (E_{s3} or E_2^R) that lies above I_T or approach a locally asymptotically stable equilibrium (E_1^R or E_{s2}^-) that lies below I_T or on $I = I_T$, depending on initial conditions, as illustrated in Figs. 9(E), 10(C), 11(C) and 11(D).

The global dynamics of the Filippov fire-blight system (2.1) with (2.2) have been investigated and summarized. Note that our control objective is to reduce the number of infected plants below or equal to the infected threshold value I_T. These results show that the choice of the susceptible and infected threshold values is quite important and guides decisions as to whether to undertake control strategies. The infected threshold value I_T is chosen sufficiently small in Case (I), which indicates that the number of infected trees will eventually be larger than I_T. However, the eventual number of infected trees may be less than the original infected number, in which case undertaking control strategies may waste resources. In reality, the value of I_T should be chosen appropriately. For Cases (II) and (III), our control objective can be achieved eventually and there is no need to modify the threshold policy. However, in Case (IV), the threshold policy may need to be modified, depending on the initial numbers of susceptible and infected trees. All results obtained here could be beneficial in choosing the threshold values and designing a corresponding threshold policy.

This work is a first approach to investigate the spread of fire blight when the threshold policy is applied. There are several limitations of the Filippov model that should be acknowledged. We took fire-blight infection through the environment into account and did not pay attention to the transmission through pollinating insects that act as a vector, which is also a very effective way for bacteria to spread. These need careful consideration, and we leave these for further investigation. Note that, even though cutting off infected branches is a good method of reducing fire-blight infection, it also leads to high costs with crop loss and labour. Another preventative strategy is spraying, which is applied by farmers to target the bacteria to reduce the spread of infection and chance of an outbreak, but it is costly and time-consuming to maintain. The most effective way to control fire blight is likely through a combination of cutting off infected branches and spraying, both of which should be done in moderation. We used the number of susceptible trees as a proxy for the available funds for the replanting rate, but other possibilities could be modelled, such as a constant replanting rate. We leave these explorations for future work.

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