Modeling avian influenza using Filippov systems to determine culling of infected birds and quarantine

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\textbf{A B S T R A C T}

The growing number of reported avian influenza cases has prompted awareness of the effectiveness of pharmaceutical or/and non-pharmaceutical interventions that aim to suppress the transmission rate. We propose two Filippov models with threshold policy: the avian-only model with culling of infected birds and the SIIR (Susceptible–Infected–Infected–Recovered) model with quarantine. The dynamical systems of these two models are governed by nonlinear ordinary differential equations with discontinuous right-hand sides. The solutions of these two models will converge to either one of the two endemic equilibria or the sliding equilibrium on the discontinuous surface. We prove that the avian-only model achieves global stability. Moreover, by choosing an appropriate quarantine threshold level $I_c$ in the SIIR model, this model converges to an equilibrium in the region below $I_c$ or a sliding equilibrium, suggesting the outbreak can be controlled. Therefore a well-defined threshold policy is important for us to combat the influenza outbreak efficiently.

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1. Introduction

Recently, a new bird flu H7N9 has been reported as a threat to the public health across China. As an early stage of precaution, the China Health and Family Planning Commission has alerted the WHO (World Health Organization) about this infection [1,2]. Further, epidemiological investigations have been carried out to identify the root of the infection so that the disease can be controlled in the most effective and efficient way [2]. The public are also advised to take care of their personal hygiene, avoiding any contact with the sick or bird carcasses, reducing contact with wild birds and limiting unnecessary visits to poultry farms [3,2]. Humans can be infected by avian influenza through direct contact with dead or infected poultry and wild birds. People who have been infected by avian influenza may initially develop several symptoms such as fever, sore throat, muscle aches, cough, having breathing difficulties and conjunctivitis [4–6].

The spread of the new highly pathogenic avian influenza A viruses has not only triggered a major loss of life but has also cost a significant amount of money. Governments worldwide have spent billions of dollars to treat the infected patients and invest in prevention to control the disease [7]. Thus it is crucial to identify any possible effective control measures that...
can eradicate the disease or at least to bring down the impact of the outbreak to a minimum level. That is, minimizing the number of infected is always a priority.

A significant number of mathematical modeling studies have been initiated to evaluate the effectiveness and the role of control measures in combating avian influenza [8–13]. Ferguson et al. [14] examined the effectiveness of targeted prophylaxis antiviral drug and social distancing measures in fighting an emerging influenza outbreak in Southeast Asia. Nuño et al. [15] assessed the basic public-health control strategies (such as using protective tools like gloves and masks, isolation in hospital wards and quarantine of suspected patients) in order to minimize the infection rate in hospitals and communities. The use of antiviral drugs and vaccination in combating a potential flu pandemic had also been discussed. Gulbudak and Martcheva [16] incorporated various approaches to culling of domestic birds: mass, modified and selective culling approaches. They concluded that, besides culling of domestic birds, timely employment of temporary control methods such as separation of poultry from wild birds, increasing biosecurity and prohibiting poultry movement and hatching eggs will either reduce the number of infected domestic birds or eradicate the disease in poultry.

Further, Agusto [17] applied optimal control theory to a system of ordinary differential equations to describe the transmission of two-strain avian influenza. Isolation of individuals with avian and mutant strains is represented by a pair of control variables. Moreover, cost-effectiveness of all possible combinations of the control measures is calculated. The results show that the combination strategy of isolating individuals with both avian and mutant strains is the most cost-effective and provides more benefits towards disease eradication compared to only using one control strategy. Chong et al. [18] suggested that a combination of pharmaceutical (vaccination) and non-pharmaceutical (personal protection and isolation) interventions can combat avian influenza more effectively.

Several conventional control methods such as pharmaceutical or non-pharmaceutical interventions may be employed if the number of infected individuals exceeds a certain tolerant threshold, say $I_c$, in order to control or suppress the transmission rate of an emerging infectious disease. Thus, whenever the number of infected is below the threshold level $I_c$, the infection is considered tolerable. However, once the number of infected reaches $I_c$, we assume that an outbreak might occur. Henceforth, we call this type of disease management strategy a threshold policy [19–21].

Xiao et al. [22] extended the classical SIR model to a Filippov SIR model incorporating behavioral change of general individuals and implementation of necessary control measures by public authorities. They showed that the model solutions will either converge to one of the two endemic equilibria or the sliding equilibrium on the discontinuous surface. In order to preclude the outbreak or to stabilize the infection at a desired level, Xiao et al. suggested that choosing a proper combination of threshold level and control intensities is crucial.

Tang et al. [19] designed a piecewise HIV virus dynamic model with CD4$^+$ T cell counts to evaluate the strategies of structured treatment interruptions (STIs) of antiretroviral therapies. The dynamic models for drug-on and drug-off states with a single threshold and two thresholds (i.e., threshold window) are studied. Both models for STIs with single threshold and threshold window show that the CD4$^+$ T cell counts are preserved above a safe level. However, numerical results show that, by picking different lower and upper tolerant thresholds, it will either converge to a stable level or fluctuate. To conclude, an appropriate tolerant threshold of CD4$^+$ T cell counts and an individualized STI strategy based on the initial value of CD4$^+$ T cell counts for each individual patient are essential to compute the duration of drug on/off states for a patient.

In addition, Zhao et al. [23] proposed two Filippov plant disease models with cultural control strategy; a plant-disease model with replanting and roguing, and a Lotka–Volterra Filippov plant disease model with proportional planting rate. For the former model, a roguing rate that is proportional to the number of infected plants is considered. The global dynamic behavior of these models is discussed. Further, the global stability of five types of equilibria is thoroughly investigated.

An HPAI (highly pathogenic avian influenza) outbreak brings losses to the poultry business especially in commercialized poultry-processing industries. Besides the great loss in these business ventures, a significant number of birds will be destroyed [24,25]. The H5N1 outbreak in Hong Kong during 1997 caused an estimated loss of $13 million and the culling of 1.4 millions birds. In the 2001 H5N1 outbreak in Hong Kong, 1.2 million birds were killed, resulting in a total loss of $3.8 million. The H7N7 outbreak in 2003 in several European countries caused a loss of $314 million and 30 million birds [26,27].

HPAI viruses (H5 and H7 subtypes) usually cause infection among common bird species, such as chickens, ducks, pigeons, quails, turkeys and others. HPAI viruses can result in a very high mortality rate (90%–100%). Avian influenza viruses can be found mostly in the feces, saliva and nasal secretions of birds. Due to limited space of birds in the farm, avian influenza viruses can be spread easily among poultry flocks through aerosol or fecal-oral route [8,26,28]. Poultry, mainly chicken meat and eggs, are a valuable source of protein for many people, especially for lower-income groups, since chicken meat is the cheapest of all farm animals [29]. Hence, it is important for us to study avian influenza infections.

Here we would like to propose two mathematical models with piecewise control strategy that relate to threshold policy: an avian-only model with culling of infected birds as a control strategy in Section 2 and an SIIR model with quarantine as the control measure in Section 3. The dynamical systems of these two models are governed by nonlinear ordinary differential equations with discontinuous right-hand sides. The local asymptotic stability of disease-free and endemic equilibria in the regions below and above the threshold level are analyzed in each model. Further, the existence of a sliding mode, its dynamics and the global stability of the equilibria (if it exists) will also be investigated in each model. Finally, we will discuss the implications of our results in Section 4.
2. The avian-only model with culling of infected domestic birds

In this section, we consider an avian-only model incorporating culling of infected birds as a control strategy. Here we only consider domestic birds for the avian population. In order to manage the disease, the number of infected birds is used as an index of reference in applying the control strategy. The disease is considered to be manageable and the implementation of control methods is not required if the number of infected birds is below the tolerant threshold \( I_T \). However, the action of culling the infected birds has to be employed immediately when the number of the infected birds exceeds the threshold level \( I_T \). This action is essential to control the outbreak before the situation becomes more severe.

The avian-only model is driven by two compartments: susceptible domestic birds \((S_d)\) and infected domestic birds \((I_d)\). The total population of domestic birds, \(N_d(t)\), is the sum of \(S_d(t)\) and \(I_d(t)\) at time \(t\). Here, we represent the bird inflow, natural death and disease death by the parameters \(A_d\), \(\mu_d\) and \(d_d\), respectively. The differential equations for this model are formulated as follows:

\[
\begin{align*}
S_d'(t) &= A_d - \beta_d S_d I_d - \mu_d S_d \\
I_d'(t) &= \beta_d S_d I_d - (\mu_d + d_d) I_d - u_d c I_d
\end{align*}
\]

with

\[
u_d = \begin{cases} 
0 & \text{for } I_d < I_T \Leftrightarrow \sigma_d(I_d) = I_d - I_T < 0 \\
1 & \text{for } I_d > I_T \Leftrightarrow \sigma_d(I_d) = I_d - I_T > 0,
\end{cases}
\]

(2.1)

where \( I_T > 0 \) is the tolerance threshold, \( \beta_d \) is the rate at which domestic birds contract avian influenza and \( c \) is the culling rate of infected domestic birds.

Moreover, we divide \((S_d, I_d) \in \mathbb{R}^2_+\) into three regions as follows:

\[
G_{1d} := \{(S_d, I_d) \in \mathbb{R}^2_+; I_d < I_T\}
\]

\[
G_{2d} := \{(S_d, I_d) \in \mathbb{R}^2_+; I_d > I_T\}
\]

\[
M_d := \{(S_d, I_d) \in \mathbb{R}^2_+; I_d = I_T\}.
\]

We define the normal vector perpendicular to \(M_d\) as \(n_d = (0, 1)^T\) and the right-hand sides of (2.1) in region \(G_{1d}\) are denoted by \(f_{1d}\) for \(i = 1, 2\), where

\[
f_{1d} = f_{1d}(S_d, I_d) = \begin{pmatrix}
A_d - S_d(\beta_d + \mu_d) \\
I_d[\beta_d S_d - (\mu_d + d_d)]
\end{pmatrix}
\]

\[
f_{2d} = f_{2d}(S_d, I_d) = \begin{pmatrix}
A_d - S_d(\beta_d + \mu_d) \\
I_d[\beta_d S_d - (\mu_d + d_d + c)]
\end{pmatrix}
\]

(2.2)

Lemma 2.1. The set \(D_d = \{(S_d, I_d) \in \mathbb{R}^2_+; S_d + I_d \leq \frac{A_d}{\mu_d}\}\) is a positively invariant and attracting region for model (2.1) with any given initial conditions in \(\mathbb{R}^2_+\).

Proof. By adding both \(S_d'(t)\) and \(I_d'(t)\) of model (2.1), we get

\[
N_d' = A_d - \mu_d S_d - (\mu_d + d_d) I_d - u_d c I_d \leq A_d - \mu_d N_d.
\]

(2.3)

Solving (2.3) by using an integrating factor, we obtain

\[
\int_0^t \frac{d}{d\xi} \left(N_d e^{\mu_d \xi}\right) d\xi = \int_0^t A_d e^{\mu_d \xi} d\xi
\]

\[
N_d(t) e^{\mu_d t} = N_d(0) + \frac{A_d}{\mu_d} \left(e^{\mu_d t} - 1\right)
\]

\[
N_d(t) \leq \frac{A_d}{\mu_d} \text{ if } N_d(0) = S_d(0) + I_d(0) \leq \frac{A_d}{\mu_d}.
\]

Thus we obtain \(N_d(t) \leq \frac{A_d}{\mu_d}\) if \(N_d(0) \leq \frac{A_d}{\mu_d}\). Hence the region \(D_d\) is positively invariant.

Next, to show that \(D_d\) is an attracting region for model (2.1), let \(N_d(t) > \frac{A_d}{\mu_d}\) and \(\frac{A_d}{\mu_d} = \psi_d \Rightarrow A_d = \mu_d \psi_d\). From (2.3), we have

\[
N_d' \leq A_d - \mu_d N_d = \mu_d(\psi_d - N_d) < 0.
\]

We infer that the total population of domestic birds (i.e., \(N_d = S_d + I_d\)) of (2.1) is bounded by \(\frac{A_d}{\mu_d}\). Moreover, every solution of model (2.1) with initial conditions in \(D_d\) will remain in \(D_d\) for \(t > 0\). It is noteworthy to mention that every solution with initial conditions in \(\mathbb{R}^2_+ \setminus D_d\) will approach \(D_d\) as \(t \to \infty\). Hence the \(\omega\)-limit sets of (2.1) are contained in \(D_d\). \(\blacksquare\)
Since $D_1$ is a positively invariant and attracting region for model (2.1), the solution of model (2.1) exists in $D_1 \forall t > 0$ and this model is mathematically and epidemiologically well-posed in $D_1$ [30]. So it is sufficient to consider the dynamics of this model in $D_1$.

2.1. Analysis in region $G_{1d}$

In this section, we begin with the calculation of the basic reproduction number and then analyze the stability of the equilibria in region $G_{1d}$. The dynamics in region $G_{1d}$ can be described by the following nonlinear ordinary differential equations:

\[
\begin{pmatrix}
S'_d(t) \\
I'_d(t)
\end{pmatrix} = \begin{pmatrix}
\Lambda_d - \beta_d S_d I_d - \mu_d S_d \\
\beta_d S_d I_d - (\mu_d + d_d) I_d
\end{pmatrix} \equiv f_{1d}.
\]  

(2.4)

There are two equilibria involved in (2.4), the DFE (disease-free equilibrium), $E_{10d} = (S_d, I_d) = \left( \frac{\Lambda_d}{\mu_d}, 0 \right)$ and a unique positive EE (endemic equilibrium), $E_{11d} = \left( \frac{\mu_d + d_d}{\beta_d}, \frac{\Lambda_d \beta_d - \mu_d (\mu_d + d_d)}{\beta_d (\mu_d + d_d)} \right)$. The basic reproduction number (see [31,32] for further details) for model (2.4), $R_{1d}$, is given as follows:

\[
R_{1d} = \frac{\Lambda_d \beta_d}{\mu_d (\mu_d + d_d)}.
\]

In addition, we would like to show that the DFE and EE of model (2.4) achieve local asymptotic stability in the following theorems, and the Jacobian matrix for this model is

\[
J_{1d}(S_d, I_d) = \begin{pmatrix}
-\beta_d I_d - \mu_d & -\beta_d S_d \\
\beta_d S_d & \beta_d S_d - (\mu_d + d_d)
\end{pmatrix}.
\]

**Theorem 2.2.** The DFE, $E_{10d}$, of model (2.4) is locally asymptotically stable if $R_{1d} < 1$.

**Proof.** By solving the characteristic equation $|J_{1d}(E_{10d}) - \lambda I| = 0$, we obtain

\[
(-\mu_d - \lambda) \begin{pmatrix}
\Lambda_d \beta_d \\
-\mu_d + \Lambda_d \beta_d
\end{pmatrix} = 0 \implies \lambda = -\mu_d < 0
\]

if $R_{1d} < 1$. We conclude that, at the DFE, all eigenvalues of (2.4) are negative if $R_{1d} < 1$. Hence $E_{10d}$ is locally asymptotically stable if $R_{1d} < 1$. ■

**Theorem 2.3.** The EE, $E_{11d}$, of model (2.4) is locally asymptotically stable if $R_{1d} > 1$.

**Proof.** The eigenvalues of $J_{1d}(E_{11d})$ are

\[
\lambda = 1 + \frac{1}{2} \left( -\frac{\Lambda_d \beta_d}{\mu_d + d_d} \pm \sqrt{\Delta_{1d}} \right) \text{ where } \Delta_{1d} = \left( \frac{\Lambda_d \beta_d}{\mu_d + d_d} \right)^2 - 4 \left[ \Lambda_d \beta_d - \mu_d (\mu_d + d_d) \right].
\]

If $R_{1d} > 1$, we obtain $\Lambda_d \beta_d - \mu_d (\mu_d + d_d) > 0$. Thus all $\lambda$ are complex eigenvalues with negative real parts if $\Delta_{1d} < 0$ since all associated parameters are positive. Otherwise, if $\Delta_{1d} > 0$, then $\lambda_{1d} < \left( \frac{\Lambda_d \beta_d}{\mu_d + d_d} \right)^2$, so all $\lambda$ are negative real numbers.

It follows that $E_{11d}$ is either a stable spiral or stable node. Hence $E_{11d}$ achieves local asymptotic stability whenever $R_{1d} > 1$. ■

2.2. Analysis in region $G_{2d}$

A similar analysis as shown in Section 2.1 will be carried out in this section. The following equations describe the dynamics in region $G_{2d}$.

\[
\begin{pmatrix}
S'_d(t) \\
I'_d(t)
\end{pmatrix} = \begin{pmatrix}
\Lambda_d - \beta_d S_d I_d - \mu_d S_d \\
\beta_d S_d I_d - (\mu_d + d_d + c) I_d
\end{pmatrix} \equiv f_{2d}.
\]  

(2.5)

In $G_{2d}$, we found two equilibria: the DFE, $E_{20d} = (S_d, I_d) = \left( \frac{\Lambda_d}{\mu_d}, 0 \right)$, and a unique positive EE, $E_{21d} = \left( \frac{\mu_d + d_d + c}{\beta_d}, \frac{\Lambda_d \beta_d - \mu_d (\mu_d + d_d + c)}{\beta_d (\mu_d + d_d + c)} \right)$. Moreover, the basic reproduction number (refer to [31,32] for further details) for model (2.5), $R_{2d}$, is thus

\[
R_{2d} = \frac{\Lambda_d \beta_d}{\mu_d (\mu_d + d_d + c)}.
\]
Further, the local asymptotic stability of the DFE and EE of model (2.5) are shown in the following theorems.

**Theorem 2.4.** The DFE $E_{20d}$ of model (2.5) is locally asymptotically stable if $R_{2d} < 1$.

We use a similar method as in the proof of Theorem 2.2 to demonstrate that all eigenvalues of (2.5) at $E_{20d}$ are negative or have negative real parts whenever $R_{2d} < 1$. Therefore, we claim that $E_{20d}$ is locally asymptotically stable if $R_{2d} < 1$.

**Theorem 2.5.** The EE $E_{21d}$ of model (2.5) is locally asymptotically stable if $R_{2d} > 1$.

The same method as Theorem 2.3 can be used to prove Theorem 2.5, so we omit the proof here.

2.3. Existence of a sliding mode and its dynamics

**Definition 2.1** ([23]). If $\langle n_d, f_{1d} \rangle > 0$ and $\langle n_d, f_{2d} \rangle < 0$ on $\Omega_d \subset M_d$, then $\Omega_d$ is the sliding region.

Types of regions on discontinuity surfaces are given in Appendix A.

The existence of a sliding mode is assured if $\langle n_d, f_{1d} \rangle > 0$ and $\langle n_d, f_{2d} \rangle < 0$. In this case, we have

$$\langle n_d, f_{1d} \rangle > 0 \iff \frac{\mu_d + d_d}{\beta_d} \quad \text{and} \quad \langle n_d, f_{2d} \rangle < 0 \iff \frac{\mu_d + d_d + c}{\beta_d}.$$  

Note that we have $h_{1d} < h_{2d}$ whenever $c > 0$. So the sliding domain $\Omega_d \subset M_d$ is defined as follows:

$$\Omega_d = \left\{ (S_d, I_d) \in M_d : \frac{\mu_d + d_d}{\beta_d} < S_d < \frac{\mu_d + d_d + c}{\beta_d} \right\} = \{(S_d, I_d) \in M_d ; h_{1d} < S_d < h_{2d} \}.$$  

Next, we find the sliding mode equations using Filippov convex method [33,34], which is demonstrated as follows:

$$f_d = \alpha f_{1d} + (1 - \alpha)f_{2d} \quad \text{where} \quad f_d = \left( \frac{S_d'(t)}{I_d'(t)} \right) \quad \text{and} \quad \alpha = \frac{\langle n_d, f_{2d} \rangle}{\langle n_d, f_{2d} - f_{1d} \rangle}.$$  

Since the sliding mode only exists on $\Omega_d \in M_d$ and there is no change of $I_d$ with respect to time $t$, we can rewrite (2.6) on $\Omega_d$ in following manner.

$$S_d'(t) = \Lambda_d - \beta_d S_d I_d - \mu_d S_d.$$  

The sliding equilibrium, $E_d = \left( \frac{\Lambda_d}{\beta_d I_d + \mu_d}, I_d \right)$, is a unique pseudoequilibrium (refer to Appendix B for further discussion of types of equilibrium points for aFilippov system) if

$$\frac{\mu_d + d_d}{\beta_d} < \frac{\Lambda_d}{\beta_d I_d + \mu_d} < \frac{\mu_d + d_d + c}{\beta_d}.$$  

By manipulating (2.8), we infer that $E_d$ lies on $\Omega_d$ if

$$h_{3d} \equiv \frac{\Lambda_d \beta_d - \mu_d (\mu_d + d_d + c)}{\beta_d (\mu_d + d_d + c)} < I_d < \frac{\Lambda_d \beta_d - \mu_d (\mu_d + d_d)}{\beta_d (\mu_d + d_d)} \equiv h_{4d}.$$  

In conclusion, $E_d$ is locally asymptotically stable on $\Omega_d$ since $\frac{\partial}{\partial S_d} (\Lambda_d - \beta_d S_d I_d - \mu_d S_d) = -\beta_d I_d - \mu_d < 0$ where $\mu_d, \beta_d, I_d > 0$; i.e., the eigenvalue of (2.7) is negative.

2.4. Global stability of the endemic equilibria

We divide $(S_d, I_d) \in \mathbb{R}^2_+$ into three regions, $G_{1d}, M_d$ and $G_{2d}$. For each region, there exists equilibrium points, $E_d, E_{11d}$ and $E_{21d}$, which are located in regions $M_d, G_{1d}$ and $G_{2d}$, respectively. In this section, we represent $E_d, E_{11d}$, $E_{21d}$ and the initial point in Figs. 2–6 by symbols $\circ, \bullet, \times$ and $\square$, respectively. Next, the stability of equilibria $E_d, E_{11d}$ and $E_{21d}$ is discussed in the following subsections and some numerical simulations have been shown to depict the stability of the equilibrium point. All parameters are given in Table 1, unless otherwise stated.

2.4.1. Case 1: $E_{11d}$ and $E_{21d}$ are virtual equilibria if $h_{3d} < I_d < h_{4d}$

Let us denote the virtual equilibria $E_{11d}$ and $E_{21d}$ as $E_{11d}^v$ and $E_{21d}^v$. These two equilibria are located in regions $G_{2d}$ and $G_{1d}$, respectively. In this case, we claim that $E_d \in \Omega_d \subset M_d$ is globally asymptotically stable if $h_{3d} < I_d < h_{4d}$ in the following theorem. So if a limit cycle does not exist in model (2.1), then our claim is valid.
Theorem 2.6. \( E_d \in \Omega_d \subset M_d \) is globally asymptotically stable if \( h_{1d} < l_T < h_{4d} \).

Proof. Let \( g_1 = \Lambda_d - \beta_d S_d d - 2 \mu_d S_d, g_2 = \beta_d S_d (\Lambda_d - (\mu_d + d_a) I_d), g_3 = \beta_d S_d (\Lambda_d - (\mu_d + d_a + u_d) c) I_d \) and \( g_4 = \beta_d S_d I_d - (\Lambda_d + d_a + c) I_d \). Consider a Dulac function, \( B(S_d, I_d) = \frac{1}{S_d} \) for regions \( I_d < l_T \) and \( I_d > l_T \) where \( l_T > 0 \) and \((S_d, I_d) \) \( \in \mathbb{R}_+^2 \).

For regions \( I_d < l_T \) and \( I_d > l_T \), we obtain

\[
\frac{\partial (Bg_1)}{\partial S_d} + \frac{\partial (Bg_3)}{\partial I_d} = \frac{\partial}{\partial S_d}\left( \frac{\Lambda_d}{S_d} - \beta_d - \frac{\mu_d}{I_d} \right) + \frac{\partial}{\partial I_d}\left( \beta_d - \frac{\mu_d + d_a + u_d c}{I_d} \right)
\]

\[
= - \frac{\Lambda_d}{S_d^2 I_d}
\]

\[
< 0 \quad \forall (S_d, I_d) \in \mathbb{R}_+^2 \setminus M_d.
\]

We refer to [22], which has demonstrated that Dulac’s theorem (see Theorem C.3 in Appendix C for more details) can be used to prove the non-existence of a limit cycle for a discontinuous dynamical system. In this case, the dynamical system (2.1) with (2.2) is discontinuous at the line \( l_T = l_1 \) and (2.9) is satisfied for \( l_T \neq l_1 \). In order to show the non-existence of limit cycle \( \Gamma \) that surrounds the sliding equilibrium \( E_d \), we have to show that \( \int_G dS = 0 \) for \( i = 1, 2, \) by Green’s Theorem. We would like to show this by contradiction. Assume that there exists a limit cycle \( \Gamma \) that passes through the discontinuous manifold \( M_d \) containing \( E_d \) and the sliding domain \( \Omega_d \) in its interior. Suppose this limit cycle \( \Gamma \) has period \( T \) and direction as shown in Fig. 1. Let us denote the intersection points of \( \Gamma \) and \( M_d \) (i.e., the line \( l_d = l_T \)) as \( P_t \) and \( Q_t \), the intersection points of \( \Gamma \) and the line \( l_d = l_T - \delta \) as \( P_t = P + a_1(\delta) \) and \( Q_t = Q - a_2(\delta) \), and the intersection points of \( \Gamma \) and the line \( l_d = l_T + \delta \) as \( P_2 = P + b_1(\delta) \) and \( Q_2 = Q - b_2(\delta) \) where \( \delta > 0 \) is sufficiently small. Moreover, we assume that \( a_1(\delta) \), \( a_2(\delta) \), \( b_1(\delta) \) and \( b_2(\delta) \) are continuous with respect to \( \delta \) and \( \lim_{\delta \to 0} a_1(\delta) = \lim_{\delta \to 0} a_2(\delta) = \lim_{\delta \to 0} b_1(\delta) = 0 \) are satisfied. The region \( G_{id} \) is bounded by \( \Gamma_1 \) and segment \( P_t Q_t \), whereas the region \( G_{2d} \) is bounded by \( \Gamma_2 \) and segment \( P_2 Q_2 \). Furthermore, the nonlinear ordinary differential equation in region \( G_{id} \) are denoted by \( g_1 \) and \( g_2 \). Let \( \partial g_{id} \) denote the boundary of \( G_{id} \). By Green’s Theorem, we obtain the following:

\[
\int_{G_{id}} \left[ \frac{\partial (Bg_1)}{\partial S_d} + \frac{\partial (Bg_2)}{\partial I_d} \right] dS_d dI_d = \int_{G_{id}} \frac{\partial (Bg_1)}{\partial S_d} dS_d dI_d + \int_{G_{id}} \frac{\partial (Bg_2)}{\partial I_d} dS_d dI_d
\]

\[
= \int_{\Gamma_1} (Bg_1) dt - \int_{\Gamma_1} (Bg_2) dS_d
\]

\[
= \int_{\Gamma_1} Bg_1 dt + \int_{Q_1 P_1} Bg_1 dI_d - \left( \int_{\Gamma_1} Bg_2 dS_d + \int_{Q_1 P_1} Bg_2 dS_d \right)
\]

\[
= \int_{\Gamma_1} (Bg_1 - Bg_2 - g_1) dt - \int_{Q_1 P_1} Bg_2 dS_d
\]

\[
= - \int_{Q_1 P_1} Bg_2 dS_d
\]

where \( \frac{dS_d}{dt} = g_1 \implies dS_d = g_1 dt, \frac{dl_d}{dt} = g_2 \implies dl_d = g_2 dt \) and there are no changes of \( l_d \) in the segment \( P_1 Q_1 \implies \int_{Q_1 P_1} Bg_1 dI_d = \int_{l_T - \delta}^{l_T + \delta} Bg_1 dI_d = 0 \).

Similarly, in \( G_{2d} \), the dynamical system is represented by \( g_1 \) and \( g_4 \). By Green’s Theorem, we have

\[
\int_{G_{2d}} \left[ \frac{\partial (Bg_1)}{\partial S_d} + \frac{\partial (Bg_4)}{\partial I_d} \right] dS_d dI_d = - \int_{P_2 Q_2} Bg_4 dS_d.
\]

Suppose \( G_{20} \subset G_{2d} \). Let \( \xi = \int_{G_{20}} \left[ \frac{\partial (Bg_1)}{\partial S_d} + \frac{\partial (Bg_4)}{\partial I_d} \right] dS_d dI_d = \int_{G_{20}} (Bg_1 dI_d - Bg_4 dS_d) < 0 \) from (2.9). Thus we have

\[
0 > \xi > -\left( \int_{Q_1 P_1} Bg_2 dS_d + \int_{P_2 Q_2} Bg_4 dS_d \right).
\]
Fig. 1. Limit cycle $\Gamma$.

Fig. 2. $E_d \in \Omega_d \subset M_d$ is globally asymptotically stable if $h_{3d} < I_T < h_{4d}$.

Moreover, by taking the limit $\delta \to 0$ of the addition of (2.10) and (2.11), we obtain

$$\lim_{\delta \to 0} \left( -\int_{Q_1P_1}^{Q_2P_2} Bg_d dS_d - \int_{P_2Q_2} Bg_d dS_d \right)$$

$$= \lim_{\delta \to 0} \left[ \int_{P + a_1(\delta)}^{Q - a_1(\delta)} \left( \beta_d - \frac{\mu_d + d_d}{S_d} \right) dS_d - \int_{P + h_1(\delta)}^{Q - h_1(\delta)} \left( \beta_d - \frac{\mu_d + d_d + c}{S_d} \right) dS_d \right]$$

$$= \left[ \beta_d S_d - (\mu_d + d_d) \ln S_d \right]_P^Q - \left[ \beta_d S_d - (\mu_d + d_d + c) \ln S_d \right]_P^Q$$

$$= c(\ln Q - \ln P) > 0$$

since $Q > P$, which contradicts (2.12). Thus there are no limit cycles surrounding the sliding domain $\Omega_d$ and the sliding equilibrium $E_d$. Hence $E_d \in \Omega_d \subset M_d$ is globally asymptotically stable if $h_{3d} < I_T < h_{4d}$. ■

Fig. 2 shows that all the trajectories with arbitrary initial conditions in $\mathbb{R}^2_+$ will converge to $E_d \in \Omega_d \subset M_d$ if $h_{3d} < I_T < h_{4d}$, as per Theorem 2.6. We pick $I_T = 20$ in this figure. Trajectories denoted by (a) will hit and slide to the right of $\Omega_d$ before converging to $E_d$. Meanwhile, trajectories (b) will hit and slide to the left of $\Omega_d$ and then move towards $E_d$.

Since $\frac{\Delta d}{\mu_d}$ (from Table 1) is large, it is unlikely we can show clearly that Case 1 will remain in the positively invariant and attracting region, $D_d = \left\{ (S_d, I_d) \in \mathbb{R}^2_+ : I_d + S_d \leq \frac{\Delta d}{\mu_d} \right\}$, as $t \to \infty$. For this reason, we increase $\mu_d$ to 0.3 in Fig. 3 to depict the convergence of solutions of Case 1 in region $D_d$ and define $I_T = 8$. From Fig. 3, we found that the possible trajectories for this case are

(a) a trajectory that hits $\Omega_d$ from the region $G_{1d}$ slides to the left of $\Omega_d$ and moves towards $E_d$.

(b) a trajectory with initial point located either inside or outside the attraction region $D_d$ will cross $M_d$ from $G_{1d}$ to $G_{2d}$. Then the trajectory hits and slides to the left of $\Omega_d$ before converging to $E_d$.

(c) a trajectory with initial point located in $G_{2d}$ either inside or outside of the attraction region $D_d$ will hit and slide to the right of $\Omega_d$ before moving towards $E_d$. 
Fig. 3. All trajectories move towards $E_d \in \Omega_d \subset M_d$ in the positively invariant and attracting region $D_d = \{(S_d, I_d) \in \mathbb{R}_+^2 : I_d + S_d \leq \frac{\Lambda_d}{\mu_d}\}$ if $h_{1d} < I_T < h_{4d}$ is fulfilled.

Fig. 4. $E_{11d}^R \in G_{1d}$ is globally asymptotically stable if $I_T > h_{4d}$.

Fig. 5. All solutions of Case 2, where $I_T > h_{4d}$, will approach $E_{11d}^R \in G_{1d}$ in region $D_d = \{(S_d, I_d) \in \mathbb{R}_+^2 : I_d + S_d \leq \frac{\Lambda_d}{\mu_d}\}$ as $t \to \infty$.

Fig. 6. $E_{21d}^R \in G_{2d}$ is globally asymptotically stable if $I_T < h_{3d}$.
2.4.2. Case 2: $E_{11d}$ is a real equilibrium, whereas $E_{21d}$ is a virtual equilibrium if $I_T > h_{ad}$

Let us denote $E_{11d}^R$ as a real equilibrium and $E_{21d}^V$ as a virtual equilibrium. Both of these equilibria are located in region $G_{1d}$ and there is no equilibrium lying in region $G_{2d}$. Further, we claim that $E_{11d}^R$ achieves global asymptotic stability if $I_T > h_{ad}$. In order to show the global behavior of $E_{11d}^R$ in this case, we would like to consider the following Lyapunov functions for model (2.1), which have given rise to Theorem 2.7:

$$V_1 = V_1(S_d, I_d) = S_d - \frac{\mu_d + d_d}{\beta_d} - \frac{\mu_d + d_d}{\beta_d} \ln \left( \frac{\beta_d S_d}{\mu_d + d_d} \right) + I_d - \frac{\Lambda_d \beta_d - \mu_d (\mu_d + d_d)}{\beta_d (\mu_d + d_d)}$$

and

$$V_2 = V_2(S_d, I_d) = S_d - \frac{\mu_d + d_d + c}{\beta_d} - \frac{\mu_d + d_d + c}{\beta_d} \ln \left( \frac{\beta_d S_d}{\mu_d + d_d + c} \right) + I_d - \frac{\Lambda_d \beta_d - \mu_d (\mu_d + d_d + c)}{\beta_d (\mu_d + d_d + c)}$$

(2.13)

**Theorem 2.7.** The function

$$V(S_d, I_d) = \begin{cases} 
V_1(S_d, I_d); & I_d < I_T \\
V_1(S_d, I_T) + V_2(S_d, I_d) - V_2(S_d, I_T); & I_d = I_T \text{ and } S_d \leq \frac{\mu_d + d_d}{\beta_d} \\
V_1(S_d, I_T); & I_d = I_T \text{ and } S_d > \frac{\mu_d + d_d}{\beta_d} \\
V_1(S_d, I_d); & I_d > I_T \end{cases}$$

is a Lyapunov function on $\mathbb{R}^2_+$ for (2.1) and $E_{11d}^R$ is globally asymptotically stable if $I_T > h_{ad}$.

**Proof.** If $I_T > h_{ad}$, it follows that $\Lambda_d \beta_d < \beta_d (\mu_d + d_d) I_T + (\mu_d + d_d) \Leftrightarrow \Lambda_d \beta_d < (\mu_d + d_d) (\beta_d I_T + \mu_d)$.

(a) We want to show that if $(S_d, I_d) \in G_{1d} := \{(S_d, I_d) \in \mathbb{R}^2_+; I_d < I_T\}$, then $\langle \nabla V, f_{1d} \rangle \leq 0$.

In this particular case, we have the fact that $V_1(S_d, I_d) > 0 \forall (S_d, I_d) \in G_{1d}$ and $V_1(E_{11d}^R) = 0$. Then

$$\langle \nabla V, f_{1d} \rangle = \langle \nabla V_1, f_{1d} \rangle = \frac{[\beta_d S_d - (\mu_d + d_d)][\Lambda_d - S_d(\beta_d I_T + \mu_d)]}{\beta_d S_d} + \frac{[(\mu_d + d_d)(\beta_d I_T + \mu_d) - \Lambda_d \beta_d][\beta_d S_d - (\mu_d + d_d)]}{\beta_d (\mu_d + d_d)}$$

$$= -\frac{\Lambda_d [\beta_d S_d - (\mu_d + d_d)]^2}{\beta_d S_d (\mu_d + d_d)} \leq 0 \quad \forall (S_d, I_d) \in G_{1d}$$

where $\langle \nabla V, f_{1d} \rangle = 0$ when $S_d = \frac{\mu_d + d_d}{\beta_d}$. Otherwise, $\langle \nabla V, f_{1d} \rangle < 0$.

(b) We claim that if $(S_d, I_d) \in \left\{(S_d, I_d) \in M_d; S_d \leq \frac{\mu_d + d_d}{\beta_d} \right\}$ is satisfied, then we obtain $\sup_{\theta \leq a \leq 1} \langle \nabla V, \alpha f_{1d} + (1 - \alpha) f_{2d} \rangle = 0$.

For $I_d = I_T$ and $S_d \leq \frac{\mu_d + d_d}{\beta_d}$, we have $V_1(S_d, I_T) + V_2(S_d, I_d) - V_2(S_d, I_T) > 0$. We find that, when $I_d = I_T$,

$$\langle \nabla V, f_{1d} \rangle = \frac{\Lambda_d [\beta_d S_d - (\mu_d + d_d)][(\mu_d + d_d + c) - \beta_d S_d]}{\beta_d S_d (\mu_d + d_d + c)} \leq 0$$

where, for all $S_d \leq \frac{\mu_d + d_d}{\beta_d}$, we have $\beta_d S_d - (\mu_d + d_d) \leq 0$ and $(\mu_d + d_d + c) - \beta_d S_d > 0$. It follows that $\langle \nabla V, f_{1d} \rangle = 0$ when $S_d = \frac{\mu_d + d_d}{\beta_d}$. Otherwise, $\langle \nabla V, f_{1d} \rangle < 0$. 


Thus, we have
\[
\langle \nabla_V, f_{2d} \rangle = \frac{[\beta_d S_d - (\mu_d + d_d)][\Lambda_d - S_d(\beta_d l_d + \mu_d)]}{\beta_d S_d} + \frac{[(\mu_d + d_d + c)(\beta_d l_d + \mu_d) - \Lambda_d \beta_d][\beta_d S_d - (\mu_d + d_d + c)]}{\beta_d (\mu_d + d_d + c)}
\]
\[
< \frac{[(\mu_d + d_d + c)[\beta_d S_d - (\mu_d + d_d)][\Lambda_d - S_d(\beta_d l_d + \mu_d)]}{\beta_d S_d(\mu_d + d_d + c)} + \frac{S_d[(\mu_d + d_d + c)(\beta_d l_d + \mu_d) - \Lambda_d \beta_d][\beta_d S_d - (\mu_d + d_d)]}{\beta_d S_d(\mu_d + d_d + c)}
\]
where \( l_d = l_T, (\mu_d + d_d + c)(\beta_d l_d + \mu_d) - \Lambda_d \beta_d > 0 \) and
\[
< \frac{S_d[(\mu_d + d_d + c)(\beta_d l_d + \mu_d) - \Lambda_d \beta_d][\beta_d S_d - (\mu_d + d_d)]}{\beta_d S_d(\mu_d + d_d + c)}
\]
where \( \beta_d S_d - (\mu_d + d_d) \leq 0, (\mu_d + d_d + c) - \beta_d S_d > 0 \forall S_d \leq \frac{\mu_d + d_d}{\beta_d} \) and \( \langle \nabla_V, f_{2d} \rangle = 0 \) when \( S_d = \frac{\mu_d + d_d}{\beta_d} \).

(c) We claim that, under the condition that
\[
(S_d, l_d) \in M_d; S_d > \frac{\mu_d + d_d}{\beta_d}
\]
where \( S_d \) is the solution to the equation $\sum_{i=1}^{2} f_{id}(S_d, l_d) = 0$, we have \( \sup_{0 \leq \alpha \leq 1} \langle \nabla_V, \alpha f_{1d} + (1 - \alpha) f_{2d} \rangle \leq 0 \).

(d) We want to show that, whenever the condition $(S_d, l_d) \in G_{2d} := \{(S_d, l_d) \in \mathbb{R}_+^2; l_d > l_T\}$ is satisfied, we obtain \( \langle \nabla_V, f_{2d} \rangle < 0 \).

For \( l_d > l_T \), it follows that \( V_1(S_d, l_d) > 0 \). Next,
\[
\langle \nabla_V, f_{2d} \rangle = \langle \nabla_V, f_{2d} \rangle _{= \langle \nabla_V, f_{2d} \rangle} 
\]
\[
\langle \nabla_V, f_{2d} \rangle = \frac{[\beta_d S_d - (\mu_d + d_d)][\Lambda_d - S_d(\beta_d l_d + \mu_d)]}{\beta_d S_d} \text{ where } l_d = l_T
\]
\[
< \frac{S_d[(\mu_d + d_d + c)(\beta_d l_d + \mu_d) - \Lambda_d \beta_d][\beta_d S_d - (\mu_d + d_d)]}{\beta_d S_d(\mu_d + d_d + c)} \text{ where } (\beta_d l_d + \mu_d) < \frac{\Lambda_d \beta_d}{\mu_d + d_d}
\]
\[
< \frac{S_d[(\mu_d + d_d + c)(\beta_d l_d + \mu_d) - \Lambda_d \beta_d][\beta_d S_d - (\mu_d + d_d)]}{\beta_d S_d(\mu_d + d_d + c)} \leq 0 \forall S_d > \frac{\mu_d + d_d}{\beta_d}
\]
Hence, \( \sup_{0 \leq \alpha \leq 1} \langle \nabla_V, \alpha f_{1d} + (1 - \alpha) f_{2d} \rangle < 0 \).

Thus, $V(S_d, l_d)$ is a Lyapunov function on $D_d$ and, by Lemma 2.1, $D_d$ is compact. Let $\Sigma_{1d} := \{(S_d, l_d) \in \mathbb{R}_+^2; \hat{V}^* = 0 \} = G_{1d} \cup \left( \frac{\mu_d + d_d}{\beta_d}, l_T \right)$. So the largest positively invariant subset of $\Sigma_{1d}$ is $[\mathbb{R}_+^2]$. Hence, by LaSalle’s Invariance Principle and
The function $V$ is a Lyapunov function for (2.1) and $\{E^R_{11d}\}$ is globally asymptotically stable if $I_T < h_{sd}$.

**Theorem 2.8.** The function $V_2(S_d, I_d)$ (2.13) is a Lyapunov function on $R^2_+$ for (2.1) and $\{E^R_{21d}\}$ is globally asymptotically stable if $I_T < h_{sd}$.

The proof of Theorem 2.8 is similar to that of Theorem 2.7.

We depict Theorem 2.8 numerically in Fig. 6. It is clearly shown that every solution of Case 3 will approach $E^R_{21d}$ as $t \to \infty$ with arbitrary initial conditions in $R^2_+$. Trajectories, which are depicted in Fig. 6, are

(a) a trajectory that starts in region $G_{1d}$ or $G_{2d}$ will hit and slide to the right of $\Omega_d$ before moving towards $E^R_{21d}$.

(b) a trajectory with initial condition in $G_{2d}$ will approach $E^R_{21d}$ as $t \to \infty$.

(c) a trajectory with initial point in $G_{1d}$ may pass through $M_d$ and then proceed towards $E^R_{21d}$ in region $G_{2d}$.

We increase the parameter $\mu_d$ to 0.3 in Fig. 7 to show that the numerical solutions of Case 3 remain in region $D_d$ and converge to $E^R_{21d}$ as $t \to \infty$. In this simulation, we select $I_T = 1.2$. From Fig. 7, we can see that

(a) a trajectory with initial point located in $G_{1d}$ within $D_d$ will hit and slide to the right of $\Omega_d \subset M_d$ before moving towards $E^R_{21d}$ in region $G_{2d}$.

(b) a trajectory with initial condition located in $G_{2d}$ and either within or outside of the attraction region $D_d$ will approach $E^R_{21d}$ directly.

(c) a trajectory that begins from $G_{2d}$ might hit $\Omega_d \subset M_d$ and slide to the right before moving towards $E^R_{21d}$.

For Fig. 8, we set $A_d = 100, \mu_d = 0.3, \beta_d = 0.01, d_d = 0.05, c = 0.5$ and $I_T = 50$. We observe that all trajectories with arbitrary initial conditions converge to $E^R_{21d}$, which agrees with the theoretical result shown in Theorem 2.8.

In conclusion, the solutions of model (2.1) will converge to either one of the two endemic equilibria (i.e., either $E^R_{11d}$ in $G_{1d}$ or $E^R_{21d}$ in $G_{2d}$) or the sliding equilibrium $E_d$ on sliding domain $\Omega_d \subset M_d$ if the requirement of (2.8) is met. We do not have to apply any control methods whenever $h_{sd} < I_T < h_{sd}$ (Case 1) or $I_T > h_{sd}$ (Case 2) is satisfied. This is due to the number of infected birds, which always remain below the given threshold level $I_T$ since we have proclaimed that the infection is tolerable. Therefore, in this particular case, the trajectory of model (2.1) either converges to $E^R_{11d}$ in $G_{1d}$ or stabilizes at $E_d$. 
threshold level. Control methods will be triggered as the number of infected birds reaches the critical level (i.e., greater than the tolerance level) beyond which we proclaim that an outbreak will occur. In order to inhibit the occurrence of an outbreak or stabilize the infection at a satisfactory level, by virtue of Theorem 2.6, we need a proper combination of control intensity and tolerance level. Hence, in order to combat an outbreak effectively, we require a well-defined threshold policy.

3. The SIIR model with quarantine as a control measure

When six people were reported dead and 18 people infected by H5N1 in Hong Kong in 1997, it changed the general belief that avian influenza viruses were believed to be non-infectious to humans. Most avian influenza viruses do not spread to humans; however, H5N1, H7N2, H7N3, H7N7 and H7N9 are known to cause severe infections in humans [26,38,39]. Avian influenza viruses transmit easily to humans through direct contact with dead or infected birds. However, there are some reported cases that humans might be infected by the lethal virus indirectly via contaminated water, food that has been stained by the virus or other objects contaminated with infected birds’ feces [26,40].

There are many types of control methods that have been employed to reduce the infection rate of avian influenza, such as practicing personal protection, isolation, prescription of antiviral drugs and vaccination [18,14,15]. So in this section, we would like to consider a Filippov SIIR avian influenza model incorporating quarantine as a control measure. This model consists of susceptibles (S), humans infected with avian strain (I_a), humans infected with mutant strain (I_m) and humans who have recovered from avian and mutant strains (R). Here, we assume that when the total number of infected humans, I_a + I_m, is greater than some threshold level I_c, infected humans with either avian or mutant strain will be isolated from susceptibles. In other words, quarantine will be implemented in order to control the spread of the disease and the quarantined individuals will not return to the susceptible population; that is, the immunity was permanent. However, if the total number of infected humans is below the tolerance threshold I_c, then quarantine is not required. The SIIR model equations can be expressed as:

\[
\begin{align*}
S'(t) &= \Lambda - \beta_a (1 - qu)SI_a - \beta_m (1 - qu)SI_m - \mu S \\
I_a'(t) &= \beta_a (1 - qu)SI_a - (\mu + d + \gamma + \epsilon)I_a \\
I_m'(t) &= \beta_m (1 - qu)SI_m + \epsilon I_a - (\mu + d + \gamma)I_m \\
R'(t) &= \sigma (I_a + I_m) - \mu R
\end{align*}
\]

(3.1)

with

\[
\begin{align*}
u &= \begin{cases} 
0 & \text{for } I_a + I_m < I_c \Rightarrow \sigma (I_a, I_m) = I_a + I_m - I_c < 0 \\
1 & \text{for } I_a + I_m > I_c \Rightarrow \sigma (I_a, I_m) = I_a + I_m - I_c > 0.
\end{cases}
\end{align*}
\]

(3.2)

where q is the quarantine rate and I_c > 0 is the critical threshold of the total number of infected humans. Table 2 shows the descriptions of the associated parameters in model (3.1) and sample values.

Since R decouples from the remaining equations in model (3.1), we consider only the first three equations of model (3.1) with (3.2). It should be noted that R always preserves local stability; i.e., the associated eigenvalue is \( \lambda = -\mu < 0 \) where \( \mu > 0 \). We further assume that \( \beta_a > \beta_m [37] \). Furthermore, we define

\[
\begin{align*}
G_1 &= \{(S, I_a, I_m) \in \mathbb{R}_+^3; I_a + I_m < I_c\} \\
G_2 &= \{(S, I_a, I_m) \in \mathbb{R}_+^3; I_a + I_m > I_c\} \\
M &= \{(S, I_a, I_m) \in \mathbb{R}_+^3; I_a + I_m = I_c\}.
\end{align*}
\]
3.1. Analysis in region $G$

Model (3.1) is mathematically and epidemiologically well-posed in $D$. We can use a similar method as shown in Lemma 2.1 to prove Lemma 3.1; hence we omit the proof of this lemma.

Description of the associated parameters in SIIR model (3.1) and sample values.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Sample value</th>
<th>Units</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>Human recruitment rate</td>
<td>1000000</td>
<td>Individuals per day</td>
<td>[36]</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Natural mortality rate of humans</td>
<td>0.0075</td>
<td>per day</td>
<td>[36]</td>
</tr>
<tr>
<td>$\beta_a$</td>
<td>Transmission rate of human-to-human with avian strain</td>
<td>0.4</td>
<td>per individual per day</td>
<td>[37]</td>
</tr>
<tr>
<td>$\beta_m$</td>
<td>Transmission rate of human-to-human with mutant strain</td>
<td>0.3 $\times$ $\beta_a$</td>
<td>per individual per day</td>
<td>[37]</td>
</tr>
<tr>
<td>$d$</td>
<td>Additional disease death rate of humans due to avian influenza</td>
<td>0.15</td>
<td>per day</td>
<td>[36]</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Recovery rate of humans with avian influenza</td>
<td>0.2669</td>
<td>per day</td>
<td>[41]</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>Mutation rate</td>
<td>0.01</td>
<td>per day</td>
<td>[37]</td>
</tr>
<tr>
<td>$q$</td>
<td>Quarantine rate</td>
<td>0.6</td>
<td>Assumed</td>
<td></td>
</tr>
</tbody>
</table>

Table 2

The manifold $M$ is a discontinuous surface and it divides $\mathbb{R}^4_+$ into two regions, $G_1$ and $G_2$. We denote the normal vector that is perpendicular to $M$ as $n = (0, 1, 1, 1)^T$ and all the right-hand sides of (3.1) in region $G_i$ by $f_i$ for $i = 1, 2$. The dynamical systems in regions $G_1$ and $G_2$ are thus represented by

$$
\begin{align*}
  f_1 &= f_1(S, I_a, I_m) = \begin{pmatrix}
    A - \beta_a S I_a - \beta_m S I_m - \mu S \\
    \beta_a S I_a - (\mu + d + \gamma + \epsilon) I_a \\
    \beta_m S I_m + \epsilon I_a - (\mu + d + \gamma) I_m
  \end{pmatrix}, \\
  f_2 &= f_2(S, I_a, I_m) = \begin{pmatrix}
    A - (1 - q) \beta_a S I_a - (1 - q) \beta_m S I_m - \mu S \\
    (1 - q) \beta_a S I_a - (\mu + d + \gamma + \epsilon) I_a \\
    (1 - q) \beta_m S I_m + \epsilon I_a - (\mu + d + \gamma) I_m
  \end{pmatrix}.
\end{align*}
$$

Lemma 3.1. The set $D = \{(S, I_a, I_m, R) \in \mathbb{R}^4_+; N = S + I_a + I_m + R \leq \frac{A}{\mu}\}$ is a positively invariant and attracting region for (3.1) with any initial conditions in $\mathbb{R}^4_+$.

We can use a similar method as shown in Lemma 2.1 to prove Lemma 3.1; hence we omit the proof of this lemma.

Since $D$ is a positively invariant and attracting region for model (3.1), the solution of (3.1) exists in $D \forall t > 0$ and model (3.1) is mathematically and epidemiologically well-posed in $D$ [30]. Thus it is sufficient to consider the dynamics of this model in $D$.

3.1. Analysis in region $G_1$

The dynamical systems in region $G_1$ can be described by the following nonlinear ordinary differential equations.

$$
\begin{align*}
  \begin{pmatrix}
    S'(t) \\
    I_a'(t) \\
    I_m'(t)
  \end{pmatrix} &= \begin{pmatrix}
    A - \beta_a S I_a - \beta_m S I_m - \mu S \\
    \beta_a S I_a - (\mu + d + \gamma + \epsilon) I_a \\
    \beta_m S I_m + \epsilon I_a - (\mu + d + \gamma) I_m
  \end{pmatrix} := f_1.
\end{align*}
$$

There are two equilibria in $G_1$, the DFE $E_{10} = (S, I_a, I_m) = \left(\frac{A}{\mu}, 0, 0\right)$ and a unique positive EE $E_{11} = (E_{11} S, E_{11} I_a, E_{11} I_m)$ where

$$
\begin{align*}
  E_{11} S &= \frac{\mu + d + \gamma + \epsilon}{\beta_a}, \\
  E_{11} I_a &= \frac{\epsilon [A \beta_a - \mu (\mu + d + \gamma + \epsilon)]}{(\beta_a - \beta_m)(\mu + d + \gamma)(\mu + d + \gamma + \epsilon)}, \\
  E_{11} I_m &= \frac{\beta_a (\beta_a - \beta_m)(\mu + d + \gamma + \epsilon) - \epsilon \beta_m}{\beta_a (\beta_a - \beta_m)(\mu + d + \gamma + \epsilon) - \epsilon \beta_m} E_{11} I_m.
\end{align*}
$$

In $G_1 := \{(S, I_a, I_m) \in \mathbb{R}^3_+; I_m < -I_a + I_c\}$, we have $E_{11} \in \mathbb{R}^3_+$, and this implies that

$$
\begin{align*}
  E_{11} S &= \frac{\mu + d + \gamma + \epsilon}{\beta_a} > 0, \\
  E_{11} I_a &= \frac{\epsilon [A \beta_a - \mu (\mu + d + \gamma + \epsilon)]}{(\beta_a - \beta_m)(\mu + d + \gamma)(\mu + d + \gamma + \epsilon)} > 0.
\end{align*}
$$
which implies $\Lambda \beta_a - \mu (\mu + d + \gamma + \epsilon) > 0$ since $\beta_a > \beta_m$ and

$$E_{11}l_a = \frac{[\mu + d + \gamma)(\beta_a - \beta_m) - \epsilon \beta_m]}{\epsilon \beta_a} E_{11}l_m > 0,$$  
(3.6)

which implies $(\mu + d + \gamma)\beta_a - (\mu + d + \gamma + \epsilon)\beta_m > 0$ since $\beta_a > \beta_m$ and $E_{11}l_m > 0$.

The transmission matrix $F_1$ and transition matrix $V_1$ of model (3.4) are defined as

$$F_1 = \begin{pmatrix} \beta_S & S \beta_m & 0 \\ 0 & \beta_m S \\ 0 & \beta_m S \end{pmatrix}$$  
and

$$V_1 = \begin{pmatrix} \mu + d + \gamma + \epsilon & 0 \\ 0 & \mu + d + \gamma \\ 0 & \mu + d + \gamma \end{pmatrix},$$  
respectively.

At the DFE, we have

$$F_1 V_1^{-1} = \begin{pmatrix} \frac{\Lambda \beta_a}{\mu (\mu + d + \gamma + \epsilon)} & 0 & 0 \\ \frac{\Lambda \beta_m}{\mu (\mu + d + \gamma + \epsilon)} & 0 & 0 \\ \frac{\Lambda \beta_m}{\mu (\mu + d + \gamma + \epsilon)} & \frac{\Lambda \beta_m}{\mu (\mu + d + \gamma + \epsilon)} & \frac{\Lambda \beta_m}{\mu (\mu + d + \gamma + \epsilon)} \end{pmatrix}$$

and the basic reproduction number (see [31,32] for more details) of $G_1$ is given as follows:

$$R_1 := \max \left\{ \frac{\Lambda \beta_a}{\mu (\mu + d + \gamma + \epsilon)} : \frac{\Lambda \beta_m}{\mu (\mu + d + \gamma + \epsilon)} \right\} = \max \{R_{1a}, R_{1m}\}$$

where $R_{1a} = \frac{\Lambda \beta_a}{\mu (\mu + d + \gamma + \epsilon)}$ and $R_{1m} = \frac{\Lambda \beta_m}{\mu (\mu + d + \gamma + \epsilon)}$.

The Jacobian matrix of model (3.4) is

$$J_1(l_a, l_m) = \begin{pmatrix} -\beta_a l_a - \beta_m l_m - \mu & -\beta_a l_a & -\beta_a l_a - \beta_m l_m - \mu \\ \beta_a l_a & -\beta_a l_a & -\beta_a l_a - \beta_m l_m - \mu \\ \beta_m l_m & \beta_m l_m & -\beta_m l_m - \beta_m l_m - \mu \end{pmatrix}$$

Further, the local asymptotic stability of $E_{10}$ and $E_{11}$ is shown in the following theorems.

**Theorem 3.2.** For model (3.4), the DFE $E_{10}$ is locally asymptotically stable if $R_1 < 1$.

As in the proof of Theorem 2.2, we can show that all eigenvalues of (3.4) at $E_{10}$ are negative if $R_1 < 1$. Hence $E_{10}$ achieves local asymptotic stability whenever $R_1 < 1$.

**Theorem 3.3.** For model (3.4), the endemic equilibrium $E_{11}$ is locally asymptotically stable if $R_1 > 1$, $a_1, a_2, a_3 > 0$ and $a_1 a_2 > a_3$, where

$$a_1 = \frac{\Lambda \beta_a}{\mu (\mu + d + \gamma + \epsilon)}, a_2 = \frac{\beta_a}{\mu (\mu + d + \gamma + \epsilon)} - \beta_m (\mu + d + \gamma + \epsilon), a_3 = \frac{\beta_m (\mu + d + \gamma + \epsilon) - \beta_m (\mu + d + \gamma + \epsilon)}{\beta_a}.$$

**Proof.** At $E_{11}$, the Jacobian matrix is

$$J_1(E_{11}) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

where

$$A_{11} = \frac{\beta_a}{\epsilon \beta_a}, A_{12} = -\beta_m (\mu + d + \gamma + \epsilon), A_{13} = -\beta_m (\mu + d + \gamma + \epsilon), A_{21} = \frac{\beta_a - \beta_m (\mu + d + \gamma + \epsilon)}{\beta_a}, A_{22} = \frac{\beta_a - \beta_m (\mu + d + \gamma + \epsilon)}{\beta_a} - \beta_m (\mu + d + \gamma + \epsilon), A_{23} = \frac{\beta_a - \beta_m (\mu + d + \gamma + \epsilon)}{\beta_a} - \beta_m (\mu + d + \gamma + \epsilon), A_{31} = \frac{\beta_a - \beta_m (\mu + d + \gamma + \epsilon)}{\beta_a} - \beta_m (\mu + d + \gamma + \epsilon),$$

By solving the characteristic equation $|J_1(E_{11}) - \lambda I| = 0$, we obtain

$$\lambda^3 + \left[ \frac{\beta_a}{\mu + d + \gamma + \epsilon} + \frac{(\mu + d + \gamma)\beta_a - (\mu + d + \gamma + \epsilon)\beta_m}{\beta_a} \right] \lambda^2$$

$$+ \left[ \frac{\beta_a - \beta_m (\mu + d + \gamma + \epsilon)}{\beta_a} (\mu + d + \gamma) - \beta_m (\mu + d + \gamma + \epsilon) \right] \lambda$$

$$+ \frac{(\mu + d + \gamma)\beta_a - (\mu + d + \gamma + \epsilon)\beta_m}{\beta_a} = 0.$$  
(3.7)

If $R_1 > 1 \implies R_{1a} > 1 \implies \beta_a - \mu (\mu + d + \gamma + \epsilon) > 0$ and $\beta_a (\mu + d + \gamma) - \beta_m (\mu + d + \gamma + \epsilon) > 0$ from (3.6) $\implies \beta_a (\mu + d + \gamma) - \beta_m (\mu + d + \gamma + \epsilon) > 0$, then we obtain $a_1, a_2, a_3 > 0$. Moreover, if we also have $a_1 a_2 > a_3$, then, by the Routh–Hurwitz Criterion [42], all roots of (3.7) are negative or have negative real parts. Hence $E_{11}$ is locally asymptotically stable if $R_1 > 1$ and $a_1 a_2 > a_3$. ■
3.2. Analysis in region $G_2$

The dynamics in region $G_2$ can be represented by nonlinear ordinary differential equations as follows:

$$
\begin{pmatrix}
S'(t) \\
I_g'(t) \\
I_m'(t)
\end{pmatrix} = \begin{pmatrix}
(\Lambda - \beta_0 (1-q) S_l - \beta_m (1-q) S_l - \mu S) \\
(\beta_0 (1-q) S_l - (\mu + d + \gamma + \epsilon) I_a) \\
(\beta_m (1-q) S_l + \epsilon a - (\mu + d + \gamma) I_m)
\end{pmatrix} =: f_2.
\tag{3.8}
$$

In $G_2$, we have two equilibria: the DFE, $E_{20} = (S, I_a, I_m) = \left( \frac{\Lambda}{\mu}, 0, 0 \right)$, and a unique positive EE, $E_{21} = (E_{21} S, E_{21} I_a, E_{21} I_m)$.

where

$$
E_{21} S = \frac{\mu + d + \gamma + \epsilon}{\beta_a (1-q)}
$$

$$
E_{21} I_m = \frac{\epsilon [A \beta_a (1-q) - \mu (\mu + d + \gamma + \epsilon)]}{(1-q)(\beta_a - \beta_m)(\mu + d + \gamma + \epsilon)}
$$

$$
E_{21} I_a = \frac{[\beta_a (1-q) - \mu (\mu + d + \gamma + \epsilon)][(\mu + d + \gamma) \beta_a - (\mu + d + \gamma + \epsilon) \beta_m]}{(\mu + d + \gamma)(\beta_a - \beta_m)(\mu + d + \gamma + \epsilon)} E_{21} I_m
$$

Furthermore, we have $E_{21} \in \mathbb{R}^3_+$, and this implies that

$$
E_{21} S = \frac{\mu + d + \gamma + \epsilon}{\beta_a (1-q)} > 0
$$

$$
E_{21} I_m = \frac{\epsilon [A \beta_a (1-q) - \mu (\mu + d + \gamma + \epsilon)]}{(1-q)(\beta_a - \beta_m)(\mu + d + \gamma + \epsilon)} > 0,
$$

which implies $A \beta_a (1-q) - \mu (\mu + d + \gamma + \epsilon) > 0$, where $0 < 1 - q < 1$ and $\beta_a > \beta_m \implies \beta_a - \beta_m > 0$ and

$$
E_{21} I_a = \frac{[\mu + d + \gamma] \beta_a - (\mu + d + \gamma + \epsilon) \beta_m}{\epsilon \beta_a} E_{21} I_m
$$

which implies $(\mu + d + \gamma) \beta_a - (\mu + d + \gamma + \epsilon) \beta_m > 0$ with $E_{21} I_m > 0$.

The transmission matrix, $F_2$, and transition matrix, $V_2$, of model (3.8) are

$$
F_2 = \begin{pmatrix}
\beta_a (1-q) S_l & 0 \\
0 & \beta_m (1-q) S_l
\end{pmatrix}
$$

and

$$
V_2 = \begin{pmatrix}
\mu + d + \gamma + \epsilon & 0 \\
-\epsilon & \mu + d + \gamma
\end{pmatrix},
$$

respectively.

At the DFE $E_{20}$, we have

$$
F_2 V_2^{-1} = \begin{pmatrix}
\frac{A \beta_a (1-q)}{\mu (\mu + d + \gamma + \epsilon)} & 0 \\
\frac{\epsilon A \beta_m (1-q)}{\mu (\mu + d + \gamma + \epsilon)} & \frac{A \beta_m (1-q)}{\mu (\mu + d + \gamma + \epsilon)}
\end{pmatrix}
$$

and the basic reproduction number (see [31,32] for further details) of $G_2$ is

$$
R_2 := \max \left\{ \frac{A \beta_a (1-q)}{\mu (\mu + d + \gamma + \epsilon)}, \frac{A \beta_m (1-q)}{\mu (\mu + d + \gamma + \epsilon)} \right\} = \max \{R_{2a}, R_{2m}\}
$$

where $R_{2a} = \frac{A \beta_a (1-q)}{\mu (\mu + d + \gamma + \epsilon)}$ and $R_{2m} = \frac{A \beta_m (1-q)}{\mu (\mu + d + \gamma + \epsilon)}$.

In addition, the Jacobian matrix of model (3.8) is

$$
J_2 (S, I_a, I_m) = \begin{pmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23} \\
B_{31} & B_{32} & B_{33}
\end{pmatrix}
$$

where $B_{11} = -\beta_a (1-q) I_a - \beta_m (1-q) I_m - \mu, B_{12} = -\beta_a (1-q) S, B_{13} = -\beta_m (1-q) S, B_{21} = \beta_a (1-q) I_a, B_{22} = \beta_a (1-q) S - (\mu + d + \gamma + \epsilon), B_{23} = 0, B_{31} = \beta_m (1-q) I_m, B_{32} = \epsilon$ and $B_{33} = \beta_m (1-q) S - (\mu + d + \gamma)$.

Furthermore, the local asymptotic stability of $E_{21}$ and $E_{21}$ is shown in the following theorems.

**Theorem 3.4.** For model (3.8), the DFE $E_{20}$ is locally asymptotically stable if $R_2 < 1$. 
We use a similar method as shown in Theorem 3.2 to prove Theorem 3.4; i.e., to show that all eigenvalues of model (3.8) at \( E_0 \) are negative if \( R_2 < 1 \).

**Theorem 3.5.** For model (3.8), the endemic equilibrium \( E_2 \) is locally asymptotically stable if \( R_2 > 1 \), \( b_1, b_2, b_3 > 0 \) and \( b_1 b_2 > b_3 \), where

\[
\begin{align*}
    b_1 &= \frac{\Delta \beta_a (1 - q)}{\mu + d + \gamma + \epsilon} + \frac{(\mu + d + \gamma) \beta_a - (\mu + d + \gamma + \epsilon) \beta_m}{\beta_a} \\
    b_2 &= A \beta_a (1 - q) - \mu (\mu + d + \gamma + \epsilon) \beta_a (\mu + d + \gamma) - \epsilon \beta_m \beta_a (\mu + d + \gamma) + A \beta_a (1 - q) [\beta_a (\mu + d + \gamma) - \beta_m] (\mu + d + \gamma + \epsilon) \\
    b_3 &= \frac{\Delta \beta_a (1 - q) - \mu (\mu + d + \gamma + \epsilon) [\mu + d + \gamma] \beta_a - (\mu + d + \gamma + \epsilon) \beta_m}{\beta_a}
\end{align*}
\]

Similar methods as Theorem 3.3 can be used to demonstrate the proof of Theorem 3.5; thus we omit the proof of this theorem.

### 3.3. Existence of sliding mode and its dynamical systems

We need to compute

\[
(n, f_1) = \left( \begin{array}{c}
0 \\
1
\end{array} \right), \quad \left( \begin{array}{c}
A - \beta_a S \lambda_a - \beta_m \lambda_m - \mu S \\
\beta_a \lambda_a - (\mu + d + \gamma + \epsilon) I_a \\
\beta_m \lambda_m + \epsilon I_a - (\mu + d + \gamma) I_m
\end{array} \right)
\]

\[
= \beta_a \lambda_a + \beta_m \lambda_m - (\mu + d + \gamma) (I_a + I_m)
\]

\[
= \beta_a \lambda_a + \beta_m \lambda_m - (\mu + d + \gamma) I_c
\]

where, on \( M \), we have \( I_m = -I_a + I_c \) and

\[
(n, f_2) = \left( \begin{array}{c}
0 \\
1
\end{array} \right), \quad \left( \begin{array}{c}
A - \beta_a (1 - q) S \lambda_a - \beta_m (1 - q) S \lambda_m - \mu S \\
\beta_a (1 - q) S \lambda_a - (\mu + d + \gamma + \epsilon) I_a \\
\beta_m (1 - q) S \lambda_m + \epsilon I_a - (\mu + d + \gamma) I_m
\end{array} \right)
\]

\[
= \beta_a (1 - q) S \lambda_a + \beta_m (1 - q) S \lambda_m - (\mu + d + \gamma) (I_a + I_m)
\]

\[
= \beta_a (1 - q) S \lambda_a + \beta_m (1 - q) S (I_c - I_a) - (\mu + d + \gamma) I_c
\]

\[
= (\beta_a - \beta_m) (1 - q) S \lambda_a + \beta_m (1 - q) S (I_c - I_a) - (\mu + d + \gamma) I_c.
\]

A sliding mode exists if \( (n, f_1) > 0 \) and \( (n, f_2) < 0 \). Thus

\[
(n, f_1) > 0 \quad \text{if} \quad S > h_1 (I_a) := \frac{\mu + d + \gamma) I_c}{(\beta_a - \beta_m) I_a + \beta_m I_c}
\]

\[
(n, f_2) < 0 \quad \text{if} \quad S < h_2 (I_a) := \frac{(\mu + d + \gamma) I_c}{(1 - q) [(\beta_a - \beta_m) I_a + \beta_m I_c]}
\]

where \( 0 < 1 - q < 1 \). Since \( \beta_a > \beta_m, 0 < 1 - q < 1 \) and \( I_a, I_c > 0 \), then we obtain \( h_2 (I_a) = \frac{h_1 (I_a)}{1 - q} \) and \( h_1 (I_a) < h_2 (I_a). \) So the sliding domain \( \Omega \subset M \) is defined as

\[
\Omega := \{ (S, I_a, I_m) \in M; \ h_1 (I_a) < S < h_2 (I_a), I_a + I_m = I_c \}.
\]

Further, we can find sliding mode equations by using the Utkin equivalent control method [43]. From (3.2), we have

\[
\sigma (I_a, I_m) = I_a + I_m - I_c.
\]

Then,

\[
\frac{d \sigma}{d t} = \frac{\partial \sigma}{\partial I_a} \cdot \frac{d I_a}{d t} + \frac{\partial \sigma}{\partial I_m} \cdot \frac{d I_m}{d t}
\]

\[
= (1 - q) S (\beta_a I_a + \beta_m I_m) - (\mu + d + \gamma) (I_a + I_m)
\]

from (3.1).

By setting \( \frac{d \sigma}{d t} = 0 \) and solving for \( u \), we obtain

\[
u = \frac{S (\beta_a - \beta_m) I_a + \beta_m I_c - (\mu + d + \gamma) I_c}{q S (\beta_a - \beta_m) I_a + \beta_m I_c}
\]

where, on \( M \), we have \( I_m = -I_a + I_c \).
From $\frac{d\alpha}{dt} = 0$, we also have $l'_e(t) + l''_e(t) = 0$. By substituting (3.11) into (3.1), we have

$$ S'(t) = \Lambda - (\mu + d + \gamma)k_e - \mu S; $$

$$ l'_e(t) = \beta_0 l_a \left[ \frac{(\mu + d + \gamma)k_e}{(\beta - \beta_m)l_a + \beta_m k_e} \right] - (\mu + d + \gamma + \epsilon)l_e. $$

(3.12)

So the sliding mode equations on $\Omega \subset M$ are

$$ S'(t) = \Lambda - (\mu + d + \gamma)k_e - \mu S; $$

$$ l'_e(t) = \beta_0 l_a \left[ \frac{(\mu + d + \gamma)k_e}{(\beta - \beta_m)l_a + \beta_m k_e} \right] - (\mu + d + \gamma + \epsilon)l_e; $$

(3.13)

$$ l''_e(t) = -l'_e(t). $$

For model (3.13), there exists a unique positive pseudoequilibrium point, $E_s = (E_s, E_{l_a}, E_{l_m})$, where $E_s = \frac{\Lambda - (\mu + d + \gamma)k_e}{\mu}$, $E_{l_a} = \frac{k(\beta_0(\mu + d + \gamma) - \beta_m(\mu + d + \gamma + \epsilon))}{(\beta - \beta_m)l_a}$, and $E_{l_m} = \frac{\epsilon \beta_0 l_a}{(\beta - \beta_m)(\mu + d + \gamma + \epsilon)}$. $E_s$ is in $\Omega \subset M$ if the following constraint is satisfied.

$$ h_1(l_e) < E_s < h_2(l_e) \iff h_1(l_e) < \frac{\Lambda - (\mu + d + \gamma)l_e}{\mu} < h_2(l_e). $$

A reduced dynamical system of (3.13) is defined as in (3.12), and the local asymptotic stability of $E_s$ is shown in the following theorem.

**Theorem 3.6.** $E_s \in \Omega$ is locally asymptotically stable if $\beta_0(\mu + d + \gamma) - \beta_m(\mu + d + \gamma + \epsilon) > 0$.

A similar approach as in Theorem 2.2 can be employed to demonstrate that all eigenvalues of (3.12) at $E_s$ are negative if $\beta_0(\mu + d + \gamma) - \beta_m(\mu + d + \gamma + \epsilon) > 0$, so we omit the proof of this theorem.

3.4. Local stability of the endemic equilibria

$(S, l_a, l_m) \in \mathbb{R}^3_+$ is divided into three regions, $G_1$, $M$, and $G_2$. There exists an equilibrium point in each region, $E_{11}$, $E_1$, and $E_{21}$ in regions $G_1$, $\Omega \subset M$ and $G_2$, respectively. In this section, let us denote the real and virtual equilibria with superscripts $R$ and $V$, respectively. We will discuss the stability of $E_1, E_{11}$ and $E_{21}$ in the following subsections. Note that, in order to illustrate the theoretical results, some numerical simulations are carried out in this section. All parameters shown in Table 2 are used in the numerical simulations, unless otherwise stated.

3.4.1. Case 1: $E_{11}$ and $E_{21}$ are virtual equilibria

If (3.14) is satisfied, then both $E_{11}$ and $E_{21}$ are virtual equilibria.

$$ E_{11}l_a + E_{11}l_m > l_e \quad \text{and} \quad E_{21}l_a + E_{21}l_m < l_e. $$

(3.14)

Here $E_{11}^R$ and $E_{21}^R$ are located in regions $G_2$ and $G_1$, respectively. In this case, we have $E_s \in \Omega \subset M$, which is locally asymptotically stable. All trajectories will converge to $E_s$ if (3.14) is satisfied.

**Theorem 3.7.** The pseudoequilibrium $E_s$ cannot coexist with $E_{11}^R$ and $E_{21}^R$. In addition, $E_s \in \Omega \subset M$ is locally asymptotically stable if it exists.

**Proof.** Note that $E_s - h_1(E_s) > (\epsilon)0 \implies \Lambda \beta_0 - \mu(\mu + d + \gamma + \epsilon) > (\epsilon)\beta_0(\mu + d + \gamma)l_e, E_s - h_2(E_s) > (\epsilon)0 \implies \Lambda \beta_0(1 - q) - \mu(\mu + d + \gamma + \epsilon) > (\epsilon)\beta_0(1 - q)(\mu + d + \gamma)l_e$ where $0 < 1 - q < 1$ and all associated parameters are positive, $E_{11}l_a + E_{11}l_m = \frac{\Lambda \beta_0 - \mu(\mu + d + \gamma + \epsilon)}{\beta_0(\mu + d + \gamma)}$ and $E_{21}l_a + E_{21}l_m = \frac{\Lambda \beta_0(1 - q) - \mu(\mu + d + \gamma + \epsilon)}{\beta_0(1 - q)(\mu + d + \gamma)}$. We refer to [44] to prove that the pseudoequilibrium $E_s$ cannot coexist with $E_{11}^R$ and $E_{21}^R$. So we have to show that (a) if $E_s \in \Omega \subset M$ is a pseudoequilibrium, then $E_{11}$ and $E_{21}$ are virtual equilibria, and (b) if $E_s$ is not a pseudoequilibrium, then $E_{11}$ and $E_{21}$ are real equilibria.

(a) If $E_s \in \Omega \subset M$ is a pseudoequilibrium (i.e., $h_1(E_s) < E_s < h_2(E_s) \implies E_s - h_1(E_s) > 0$ and $E_s - h_2(E_s) < 0$), then $E_{11}l_a + E_{11}l_m > l_e$ and $E_{21}l_a + E_{21}l_m < l_e$ indicate that $E_{11}$ and $E_{21}$ are virtual equilibria.

$$ E_{11}l_a + E_{11}l_m = \frac{\Lambda \beta_0 - \mu(\mu + d + \gamma + \epsilon)}{\beta_0(\mu + d + \gamma)} > \frac{\beta_0(\mu + d + \gamma)l_e}{\beta_0(\mu + d + \gamma)} = l_e $$

where $E_s - h_1(E_s) > 0 \implies \Lambda \beta_0 - \mu(\mu + d + \gamma + \epsilon) > \beta_0(\mu + d + \gamma)l_e$.

$$ E_{21}l_a + E_{21}l_m = \frac{\Lambda \beta_0(1 - q) - \mu(\mu + d + \gamma + \epsilon)}{\beta_0(1 - q)(\mu + d + \gamma)} < \frac{\beta_0(1 - q)(\mu + d + \gamma)l_e}{\beta_0(1 - q)(\mu + d + \gamma)} = l_e $$

where $E_s - h_2(E_s) < 0 \implies \Lambda \beta_0(1 - q) - \mu(\mu + d + \gamma + \epsilon) < \beta_0(1 - q)(\mu + d + \gamma)l_e$. Thus the existence of pseudoequilibrium $E_s$ implies the non-existence of real equilibria $E_{11}$ and $E_{21}$. 
If $E_3$ achieves local asymptotic stability. Both stability in Theorem 3.6. For any choice of threshold level $I$, equilibria $E_1$ and $E_2$ exist in region $G$. So $E_1$ and $E_2$ are real equilibria whenever $E_3 \notin G \subset M$. Therefore, the pseudoequilibrium $E_3$ cannot coexist with the real equilibria $E_1$ and $E_2$.

Next, we would like to discuss the stability of $E_3 \in G \subset M$. We have shown that $E_3 \notin G \subset M$ achieves local asymptotic stability in Theorem 3.6. For any choice of threshold level $I$, in between $E_1 I_a + E_2 I_m$ and $E_1 I_a + E_1 I_m$, the local asymptotic stability of $E_3$ in the sliding domain always holds. Hence, $E_3$ is locally asymptotically stable in the sliding domain $\Omega \subset M$ if it exists.

Since the difference between $E_1 I_a = E_1 I_m$ and $E_2 I_a + E_2 I_m$ (i.e., $\frac{q \mu(\mu+d+\gamma+\epsilon)}{\beta_1(1-q)(\mu+d+\gamma)}$) with $\mu = \frac{1}{65 \times 365}$ is considerably small (0.0001619), then we select $\mu = 0.3$ and $I_a = 2.5$ while other parameters are defined in Table 2 in order to depict Case 1 clearly; i.e., $E_3 \in G \subset M$ achieves local asymptotic stability if (3.14) is fulfilled. Fig. 9 shows that any trajectory that begins either in region $G_1$ or $G_2$ will converge to $E_3 \in G \subset M$ if (3.14) is satisfied.

3.4.2. Case 2: $E_1$ is a real equilibrium, whereas $E_2$ is a virtual equilibrium

If the following constraint is satisfied, then $E_1$ is a real equilibrium and $E_2$ is a virtual equilibrium.

$$E_1 I_a + E_1 I_m < I_c \quad \text{and} \quad E_2 I_a + E_2 I_m < I_c.$$

(3.15)

Both $E_{1R}$ and $E_{2V}$ are located in region $G_1$. In this case, we have an equilibrium point located in $G_1$ (i.e., $E_1$) and there is no equilibrium point located in region $G_2$. If (3.15) is satisfied, then all trajectories in this case will converge to $E_{1R}$. Hence, $E_{1R}$ achieves local asymptotic stability.

**Theorem 3.8.** $E_{1R}$ is locally asymptotically stable if (3.15) is satisfied.
(a) A trajectory with initial point in $G_2$ will hit and slide to the right on $\Omega \subset M$ before moving towards $E_{11}^R$.  
(b) A trajectory will cross the region $S < h_1$ on $M$ from the direction of $G_2$ before moving towards $E_{11}$.  
(c) A trajectory that begins in region $G_1$ will converge to $E_{11}^R$ without hitting or passing through manifold $M$.  
(d) A trajectory will pass through $M$ moving towards $G_2$ from $G_1$ and hit manifold $M$ again from the direction of $G_2$. Then it will slide down on $\Omega \subset M$ before converging to $E_{11}^R$ in $G_1$.

Fig. 10. $E_{11}^R \in G_1$ is locally asymptotically stable if (3.15) is fulfilled.

We discover that $E_{11}^R$ is located in region $G_1$ if (3.15) is fulfilled. Since we have proved that the equilibrium point $E_{11} \in G_1$ achieves local asymptotic stability in Theorem 3.3, we omit the proof of Theorem 3.8.

Case 2 is depicted in Fig. 10 with $I_c = 8$. Any trajectory with initial point in region $G_1$ or $G_2$ will converge directly to $E_{11}^R$ either without hitting the manifold $M$ or it will hit the manifold $M$, slide and then move towards the equilibrium $E_{11}^R$.

3.4.3. Case 3: $E_{21}$ is a real equilibrium, whereas $E_{11}$ is a virtual equilibrium

$E_{21}$ is a real equilibrium and $E_{11}$ is a virtual equilibrium if (3.16) is satisfied.

$$E_{11}l_a + E_{11}l_m > I_c \quad \text{and} \quad E_{21}l_a + E_{21}l_m > I_c. \quad (3.16)$$

In this case, both $E_{11}^R$ and $E_{21}^R$ are located in region $G_2$. There is no equilibrium point that can be found in region $G_1$, but there is one equilibrium point (i.e., $E_{21}$) that lies in region $G_2$. All trajectories will converge to $E_{21}^R$ if (3.16) is fulfilled. So $E_{21}^R$ achieves local asymptotic stability in this case.

**Theorem 3.9.** $E_{21}^R$ achieves local asymptotic stability if the requirement of (3.16) is met.

Note that $E_{21}^R$ is located in region $G_2$ if (3.16) is satisfied. In Theorem 3.5, we have proved that the equilibrium point $E_{21} \in G_2$ is locally asymptotically stable. So we omit the proof of Theorem 3.9.

The result of Theorem 3.9 is illustrated in Fig. 11. All trajectories in this case with $I_c = 6$ will either hit or do not hit the manifold $M$ before converging to $E_{21}^R$.

4. Conclusion and discussion

Two Filippov models that are governed by nonlinear ordinary differential equations with discontinuous right-hand sides have been proposed; notably the avian-only model with culling of infected birds and the SIIR model with quarantine as control measure. At the initial stage of an outbreak, many people are not aware of the existence of the disease. This usually leads to rapid disease outbreak since no disease preventions have been practiced by the public. When the emerging infectious disease has reached a critical stage, known as the “threshold level”, people may start to take necessary precautions to prevent themselves from being infected [22]. Sliding mode control is one of the desirable methods to depict this type of disease-management phenomenon [21].

An HPAI outbreak in avian population can create havoc in the poultry industry; a large number of birds will have to be killed since culling birds is one of the primary strategy to eradicate an avian flu outbreak, especially among the infected avian population. Studies on culling have been carried out to identify the most effective approach to eradicating the disease.
A trajectory will hit $M$ from the direction of $G_2$. Then it will slide to right on $\Omega \subset M$ before moving towards $E_{21}^d$ in $G_2$.

A trajectory with initial point in region $G_2$ converges to $E_{21}^d$ directly.

A trajectory hits $\Omega \subset M$ from $G_1$ and then moves up on $\Omega$ before converging to $E_{21}^1$ in $G_2$.

A trajectory crosses manifold $M$ from $G_1$ to $G_2$ and then moves towards $E_{21}^d$.

Fig. 11. $E_{21}^d \in G_2$ is locally asymptotically stable if (3.16) is satisfied.

and reducing the socio-economic impact [16,45]. Hence it is essential for us to look closely at which culling threshold level should be chosen in order to eliminate the disease or at least to stabilize the infection. For instance, in the avian-only Filippov model (2.1), whenever the trajectory is found to be converging to $E_{11d}$ in $G_{1d}$ or $E_d \in \Omega_d \subset M_d$, we proclaim that the infection of avian influenza in the avian population is still bearable. However, if the solution of model (2.1) converges to $E_{21d}$ in $G_{2d}$, we assume that an outbreak is emerging. As a response to the outbreak, control methods have to be implemented in order to suppress the transmission and contain the disease. In addition, the theoretical results and numerical simulations in Section 2 show that model (2.1) achieves global asymptotic stability.

Due to the influenza pandemic history, HPAI outbreaks, mainly H5N1, have caused severe infections in humans and resulted in many human deaths [46]. Many types of interventions have been applied to minimize the impact of avian influenza. Quarantine is one of the conventional control methods that has been widely used, especially in the absence of medicines and vaccines, during the onset of the outbreak to reduce the transmission rate of the disease. However, quarantine policy (e.g., location of quarantine, timeframe, who can set up quarantine, the use of legal orders and who has the authority to issue the orders and so on), limitations of resources (e.g., food, clean drinking water and medical equipments) and the lack of health-care workers are some of the most critical issues for public-health authorities [47,48]. Hence, an SIIR model with quarantine as a control measure is designed to assess an appropriate quarantine threshold level that will lead to disease elimination. In Section 3, it is shown that the solutions of model (3.1) will converge to either one of the two endemic equilibria or the sliding equilibrium. In order to inhibit an outbreak or to stabilize the infection, we have to choose a suitable tolerance threshold $I_c$ such that the trajectory of model (3.1) is approaching $E_{11}$ in $G_1$ or a sliding equilibrium $E_s$ on $\Omega \subset M$.

There are several limitations of these two models that should be mentioned here. Throughout the model simulations, fixed constants of bird inflow and human recruitment have been applied in avian-only and SIIR models. We have made assumptions that the immunity of humans was permanent (i.e., recovered humans will not move to susceptible class) and the human-to-human transmission rate with avian strain is greater than the human-to-human transmission rate with mutant strain. For the avian population, infected birds are presumed to stay infected; i.e., infected birds will not move to other classes such as susceptible and recovered compartments. It is also noteworthy that we assumed humans with avian and mutant strains have the same values of recovery and additional disease death rate.

Our findings show that we can either preclude the influenza outbreak or stabilize the infection at a desired level by choosing an appropriate threshold level. A well-defined threshold policy is essential to us in order to combat an outbreak effectively and efficiently.

Acknowledgments

NSC acknowledges support from the Ministry of Higher Education, Malaysia, and School of Informatics and Applied Mathematics, Universiti Malaysia Terengganu. RJS is supported by an NSERC Discovery Grant. For citation purposes, please note that the question mark in "Smith?" is part of his name.
Appendix A. Types of regions on a discontinuity surface $M$

Suppose an ordinary differential equation
\[ \dot{x} = f(x, t) \] 
with threshold policy is discontinuous on a surface $M$ that is defined by equation
\[ \sigma(x) = 0 \]
where $x \in \mathbb{R}^n$. The surface $M$ separates the $x$ space into domains $G^-$ and $G^+$. Let us denote the differential equations that represent the dynamics in the regions $G^-$ and $G^+$ as $f^-(x, t)$ and $f^+(x, t)$, respectively.

There are three types of regions on $M$: sliding, sewing and escaping regions [23], which are defined as follows.

**Definition A.1** ([23]).
(a) If $|n, f^-| > 0$ and $|n, f^+| < 0$ on $\Omega \subset M$, then $\Omega$ is known as a sliding region.
(b) If $|n, f^-| \cdot |n, f^+| > 0$, i.e., $|n, f^-|$ and $|n, f^+|$ have the same signs on $\Omega_2 \subset M$, then $\Omega_2$ is called as a sewing region.
(c) If $|n, f^-| < 0$ and $|n, f^+| > 0$ on $\Omega_3 \subset M$, then $\Omega_3$ is known as an escaping region.

Note that a stable virtual equilibrium will not be achieved as the dynamics will change once the trajectory hits the discontinuous manifold [23].

Appendix B. Types of equilibrium points for a Filippov system

In this appendix, we will use similar notations as in Appendix A. Let us denote the sliding mode equation that describes the motion in the sliding region $\Omega \subset M$ by $f^S(x, t)$. Suppose there exists an equilibrium point in each region $G^-$, $G^+$ and $\Omega$, denoted by $E_1$, $E_2$ and $E_3$, respectively. There are four types of equilibria that might exist in a model of ordinary differential equations with threshold policy: real, virtual, pseudo-equilibrium and boundary equilibria [23].

**Definition B.1** ([23]).
(a) $E^R$ is a real equilibrium if $f^-(E^R) = 0$ and $\sigma(E^R) < 0$ or $f^+(E^R) = 0$ and $\sigma(E^R) > 0$.
(b) $E^V$ is a virtual equilibrium if $f^-(E^V) = 0$ and $\sigma(E^V) > 0$ or $f^+(E^V) = 0$ and $\sigma(E^V) < 0$.
(c) $E^P$ is a pseudo-equilibrium if $f^-(E^P) = 0$ and $\sigma(E^P) = 0$ or $f^+(E^P) = 0$ and $\sigma(E^P) = 0$.
(d) $E^B$ is a boundary equilibrium if $E^B$ is an equilibrium point on the sliding mode; i.e., $f^S(E^B) = 0$ and $\sigma(E^B) = 0$.

Note that a stable virtual equilibrium will not be achieved as the dynamics will change once the trajectory hits the discontinuous manifold [23].

Appendix C. Lyapunov function and theories on global stability of the Filippov system

Consider a differential equation (A.1) with $f \in C^1(G)$ where $G$ is an open subset of $\mathbb{R}^n$. The solution $\phi(t, x_0)$ of the initial-value problem (A.1) with $x_0 \in G$ will be a dynamical system on $G$ if and only if $\forall x_0 \in G, \phi(t, x_0)$ is defined $\forall t \in \mathbb{R}$. The function $\phi(\cdot, x) : \mathbb{R} \to G$ for $x \in G$ defines a solution curve, trajectory or orbit of (A.1) with initial point $x_0 \in G$. A trajectory with $x_0 \in G$ can be described as a motion along the curve $\Gamma = \{x \in G; x = \phi(t, x_0), t \in \mathbb{R}\}$, which is defined by (A.1) (refer to [49] for further details).

**Definition C.1** ([49]). A point $E \in G$ is an $\omega$-limit point of the trajectory $\phi(\cdot, x)$ of (A.1) if there is a sequence $t_n \to \infty$ such that $\lim_{n \to \infty} \phi(t_n, x) = E$. The set of all $\omega$-limit points of a trajectory $\Gamma$ is called the $\omega$-limit set of $\Gamma$ and it is denoted by $\omega(\Gamma)$.

**Definition C.2** ([49]). Let $G$ be an open subset of $\mathbb{R}^n$, $f \in C^1(G)$ and $\phi_t : G \to G$ be the flow of the differential equation (A.1) defined $\forall t \in \mathbb{R}$. Then a set $S \subset G$ is called invariant with respect to the flow $\phi_t$ if $\phi_t(S) \subset S$ $\forall t \in \mathbb{R}$ and $S$ is called positively invariant with respect to the flow $\phi_t$ if $\phi_t(S) \subset S$ $\forall t \geq 0$.

Let $\Gamma_1(t) := \{x \in \mathbb{R}^n_+; x = \phi(t, x_0) \text{ for some } x_0 \in G\}$ and $\zeta(G) := \bigcup_{t \geq 0} \Gamma_1(t)$.

**Definition C.3** ([50,23]). A function $V \in C^1(\mathbb{R}^n)$ is called a Lyapunov function of (A.1) on $G \subset \mathbb{R}^n$ if it is non-negative on $G$ and, $\forall x \in G$,
\[
\dot{V}(x) := \max_{\eta \in G(x)} \langle \nabla V(x), \eta \rangle \leq 0 \quad \text{where}
\]
\[
g(x) := \begin{cases} 
    f^-(x); & x \in G^- \\
    \alpha f^+(x) + (1 - \alpha) f^-(x); & x \in M \text{ where } \alpha \in [0, 1] \\
    f^+(x); & x \in G^+. 
\end{cases}
\]
Proposition C.1 ([50, 23, LaSalle’s Invariance Principle]). Suppose $G \subset \mathbb{R}^n$ is an open set that satisfies $\omega(G) := \bigcup_{x \in G} \omega(x) \subset \zeta(G)$. Let every Filippov solution $\phi(t, x_0)$ of (A.1) be unique and defined $\forall t \geq 0$ and $x_0 \in G$. Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function of (A.1) on $G$. Then $\omega(G)$ is a subset of the largest positively invariant subset of $\mathcal{S}$ where $\Sigma := \{x \in \bar{G}; \dot{V}(x) = 0\}$.

Corollary C.2 ([50, 23]). Assume that $G$ and $V : \mathbb{R}^n \to \mathbb{R}$ satisfy Proposition C.1 and $\mathbb{R}^n \setminus G$ is repelling in the sense that all solutions stay in $\mathbb{R}^n \setminus G$ for only a finite time. Let $\omega(\mathbb{R}^n) = \omega(G)$ be bounded. Then $\omega(\mathbb{R}^n)$ is globally asymptotically stable.

Theorem C.3 ([49, Dulac’s Theorem]). Suppose
\begin{equation}
\frac{dx}{dt} = f(x, y) \quad \text{and} \quad \frac{dy}{dt} = g(x, y),
\end{equation}
where $f(x, y)$ and $g(x, y)$ are assumed to be $C^1$ functions. If there exists a $C^1$ function $B(x, y)$ (where $B(x, y)$ is also known as a Dulac function) in a simply connected region $R$ such that $\frac{\partial B}{\partial x} + \frac{\partial B}{\partial y}$ has constant sign and is not identically zero in any subregion, then system (C.1) does not have a periodic orbit lying entirely in $R$.

References