PREFACE

This book was started many years ago, in the enthusiasm of expounding the theory of Lie groups for the first time to a class of advanced undergraduate and beginning graduate students. As it turned out, most of them did not have the equipment required for the customary path to Lie groups and representation theory, lacking things like differential geometry, functional analysis, or general topology. So I decided to look for an alternative route, starting straight from advanced calculus and linear algebra. Somewhat to my surprise, this lowly starting point turned out to be perfectly satisfactory, so much so that I have come to prefer it myself: it now seems natural and appropriate for an introduction. Over the years, when I taught Lie groups again, I might take shortcuts and add detours, but the basic course remained the same. The trail of lecture notes left on the way was gathered into a set of notes, which eventually became this book.

The route is through linear groups. By this I mean any group matrices, a notion which is both narrower and broader than is customary in Lie theory: it is narrower in that the groups are required to be linear and it is broader in that they are not required to be manifolds at the outset. (Nor indeed are manifolds themselves required at that point.) That it is possible to base a satisfactory theory on such simple foundations is not new: in the customary development it usually transpires at some point that presupposing a manifold structure is unnecessary in so far as every connected subgroup of a Lie group is of its own a manifold in a natural way. Yet, as far as I am aware, this circumstance has never been taken as basis for the theory. Apart from questions of taste and outlook, one explanation may be that the proof of the Lie correspondence between connected subgroups and Lie subalgebras customarily proceeds by Frobenius's theory of involutive differential systems, where the manifolds are appropriate and linearity is useless. Here an elementary argument, based essentially on the inverse function theorem, fills in what is needed. Be that as it may, a desire to shorten the list of prerequisites is not the only reason for the point of view taken: the restriction to linear groups seems desirable to me, even if the prerequisites are available; for it puts into focus from the beginning the essential aspects of the theory, free of technicalities. The help with matters of notation is also welcome, be it only writing matrix products instead of differentials of left and right translation. As further benefit, the presence of the matrix exponential allows a strikingly simple formulation of the basic fact of Lie theory: there is a natural one-to-one correspondence between connected linear groups and linear Lie algebras. Some care needs to be taken with the definition of the topology for these groups, but that is easily done, and quite appropriately in calculus style. Additional restrictions, such as closedness or even compactness, would not simplify the exposition, but would rather have the opposite effect.

Thus the prerequisites are modest. What is required is an understanding of linear algebra and multivariable calculus, such as might be gained from standard undergraduate courses, together with the rudiments of abstract group theory. This material can be found in most textbooks on hese subjects, for example in Hoffman-Kunze (1961), Marsden (1974), and Herstein (1964). The simplicity of the prerequisites is not an awkward constraint, but is entirely appropriate for the subject as presented here. Auxiliary material beyond these basics is developed as needed, less an interruption than an opportunity of acquiring new concepts in a meaningful setting, where they can be put to immediate use. In spite of its elementary nature, this book covers the essentials of the theory of Lie groups, so that it can serve not only as introduction to the subject for its own sake, but also as preparation for an advanced study of Lie groups and their representations.

Among the essentials of the theory of Lie groups I include the Cartan-Weyl theory of finite dimensional representations of semisimple (or reductive) groups. I present this theory "by example", discussing first U(n) and GL(n, $\mathbb{C}$), then the modifications needed for the other classical groups. I did not want to go into the general structure theory of semisimple Lie algebras, since there are already several expositions of the algebraic theory that could complement what is presented here. Nor did I want to get involved in the questions of functional analysis needed to prove the Peter-Weyl Theorem for arbitrary compact groups.

While the structure theory of semisimple groups has not been developed in general, the classical groups are discussed in considerable detail. Structural paraphernalia, such as roots, are introduced as auxiliary notions for the study of the classical groups. My hope is that this discussion will complement the abstract treatments found elsewhere.

Chapters I and II contain what is known as "Lie theory" in the context of linear groups. Together with parts of chapters III or IV, these chapters could serve for a one-semester course at an entirely elementary level. At a somewhat more advanced level, some of the material chapters I-IV could be omitted or left to the student and the emphasis placed on representation theory (chapter VI) instead. I have taught in this way one-semester courses based on chapter VI, covering only the bare essentials needed from previous chapters.

A course on Lie groups, as proposed here, could well be combined with a course on Lie algebras, based for example on Humphreys (1972), Jacobson (1962), or Serre (1966). The two courses could be given in sequence (in either order), or concurrently. There would be no duplication; the two courses would rather complement each other.

The numerous problems should not intimidate: they are generally not required within the text, but are intended to supplement the theory. They are not research projects, but exercises for the weekly problem set.

Here and there I have added some historical tidbits, not with scholarly intentions, but only because I found them remarkable or curious. The references, historical or otherwise, are kept to a minimum, certainly not intended as a guide to the literature.

Theorems, equations, figures, etc are numbered independently in each section. A parenthetical (QED) asks that a proof be supplied.