On Vervaat processes for sums and renewals in weakly dependent cases

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Abstract. We study the asymptotic behaviour of stochastic processes that are generated by sums of partial sums of (weakly dependent) random variables and their renewals, analogs of the so-called Bahadur-Kiefer process. We study their properly normalized integrals as Vervaat-type stochastic processes via strongly approximating them by a squared standard Wiener process.

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1 Introduction

We initiate our discussion by a brief review of the Vervaat (1972a,b) contributions to limit theorems for processes with positive drift and their inverses. Accordingly, along the lines of Vervaat (1972b), let Y be a non-negative stochastic process on $[0,\infty)$ such that almost all realizations $Y(\cdot,\omega):[0,\infty)\to[0,\infty)$ are non-decreasing unbounded functions and Y^{-1} be the generalized inverse of $Y(\cdot,\omega)$, i.e., $Y^{-1}(t,\omega):=\inf\{u:\ Y(u,\omega)>t\}$. Let $D_0[0,\infty)$ be the subset of non-decreasing, non-negative unbounded functions of $D[0,\infty)$, and let $C[0,\infty)$ be the subset of continuous functions of $D[0,\infty)$. For further use later on, we summarize Theorems 1 and 3 of Vervaat (1972b) for our convenience as follows (cf. also Theorems 3.2.3 and 3.2.4 of Vervaat (1972a)). Here and throughout $\stackrel{d}{\to}$ stands for convergence in distribution, while $\stackrel{\mathcal{D}}{\to}$ indicates weak convergence in an appropriate context. Also, $\stackrel{P}{\to}$ designates convergence in probability.

Theorem A *Let $Y_1,Y_2,\ldots$ be random elements in $D_0[0,\infty)$, $\widetilde{Y}$ a random element in $C[0,\infty)$ and $\zeta_1,\zeta_2,\ldots$ be positive random variables such that $\zeta_n\stackrel{P}{\to}0$ as $n\to\infty$. Then, as $n\to\infty$, the following two weak convergence statements are equivalent in $D[0,\infty)$ (endowed with the uniform topology on compact sets):*

$$\frac{Y_n-I}{\zeta_n}\stackrel{\mathcal{D}}{\to}\widetilde{Y}, \tag{1.1}$$

$$\frac{Y_n^{-1}-I}{\zeta_n}\stackrel{\mathcal{D}}{\to}-\widetilde{Y}, \tag{1.2}$$

where I denotes the identity map on $[0,\infty)$. Moreover, if any one of (1.1) and (1.2) holds, then

$$V(\cdot;Y_n)\stackrel{\mathcal{D}}{\to}\frac{1}{2}\widetilde{Y}^2, \tag{1.3}$$

in $C[0,\infty)$ (endowed with the uniform topology on compact sets), where

$$V(t;Y_n)=\frac{1}{\zeta_n^2}\int_0^t (Y_n(u)+Y_n^{-1}(u)-2u)\,du,\quad 0\le t<\infty. \tag{1.4}$$

In Theorem A it is assumed that $Y_1,Y_2,\ldots$ are random elements in $D_0[0,\infty)$. However, a convenient version of Theorem 2 of Vervaat (1972b) (cf. also Theorem 3.3.2 of Vervaat (1972a)) for $Y_1,Y_2,\ldots$ not necessarily in $D_0[0,\infty)$ reads as follows.

Theorem A* *Let $Y_1,Y_2,\ldots$ be random elements in $D[0,\infty)$, $\widetilde{Y}$ a random element in $C[0,\infty)$ and $\zeta_1,\zeta_2,\ldots$ be positive random variables as in* Theorem A. *If, as $n\to\infty$, we have*

$$\frac{Y_n-I}{\zeta_n}\stackrel{\mathcal{D}}{\to}\widetilde{Y}, \tag{1.5}$$

then

$$\frac{Y_n^{\uparrow} - I}{\zeta_n} \xrightarrow{\mathcal{D}} \widetilde{Y}, \tag{1.6}$$

where

$$Y_n^{\uparrow}(t) = \sup_{0 \le u \le t} Y_n(u), \quad t \ge 0. \tag{1.7}$$

Consequently, if we also assume that $Y_n^{\uparrow}$ are random elements in $D_0[0,\infty)$, then Theorem A *is applicable to $Y_n^{\uparrow}$, and hence we have also*

$$\frac{Y_n^{-1} - I}{\zeta_n} \xrightarrow{\mathcal{D}} -\widetilde{Y} \tag{1.8}$$

in $D[0,\infty)$, and

$$V(\cdot\,; Y_n^{\uparrow}) \xrightarrow{\mathcal{D}} \frac{1}{2}\widetilde{Y}^2 \tag{1.9}$$

as well, in $C[0,\infty)$.

In this paper we investigate these properties for partial sums of weakly dependent random variables and their inverses via strong invariance, including also the case of partial sums of i.i.d. random variables. In this context, we speak about weak dependence only if the partial sums in hand can be strongly approximated by a standard Wiener process (Brownian motion).

Let $X_1, X_2, \ldots$ be a sequence of random variables, not necessarily independent. Define the partial sum process by

$$S(t) := \sum_{i=1}^{[t]} X_i,$$

and the renewal counting process by

$$N(t) := \inf\{s : S(s) > t\}.$$

Define also

$$M(t) := \sup_{0 \le s \le t} S(s).$$

Our main result is as follows.

Theorem 1.1 *Let $\mu > 0$, and let $r(\cdot)$ be a positive non-decreasing function, regularly varying at infinity, for which $r(t) \ge t^{1/4}$ and*

$$r(t) = O((t \log\log t)^{1/2}), \qquad t \to \infty.$$

Assume that the following strong approximation holds:

$$\sup_{0\le t\le T} |S(t) - \mu t - W(t)| = o(r(T)) \quad a.s., \tag{1.10}$$

Then we have also

$$\sup_{0\le t\le T} |M(t) - \mu t - W(t)| = o(r(T)) \quad a.s., \tag{1.11}$$

and

$$\sup_{0\le t\le T} |\mu^{-1} t - N(t) - \mu^{-1} W(\mu^{-1} t)| = o(r(T)) \quad a.s., \tag{1.12}$$

where $\{W(t);\, t \ge 0\}$ *is a standard Wiener process on the line.*

If, moreover, $X_i \ge 0$, $i = 1, 2, \ldots,$ *and* $r(n) \ge n^{1/4}(\log\log n)^{3/4}$, *then* (1.10) *and* (1.12) *are equivalent and any of them implies*

$$\sup_{0\le t\le 1} \left|\mu^2 n^2 V_n(t) - \frac{1}{2} W^2(nt)\right| = O(r(n)\sqrt{n \log\log n}) \qquad a.s., \tag{1.13}$$

as $n \to \infty$.

As a consequence of (1.13), we conclude the weak convergence

$$2\mu^2 n V_n \xrightarrow{\mathcal{D}} W^2$$

in $C[0,1]$. Since W^2 is not differentiable, R_n, as the derivative of V_n, cannot converge weakly in $D[0,1]$ with any normalization. Further consequences are strong limit theorems, like the law of the iterated logarithm, Chung's law, etc.

$$\limsup_{n\to\infty} \frac{n\mu^2 \sup_{0\le t\le 1} |V_n(t)|}{\log\log n} = 1 \quad \text{a.s.}$$

$$\liminf_{n\to\infty} n\mu^2 \log\log n \sup_{0\le t\le 1} |V_n(t)| = \frac{\pi^2}{8} \quad \text{a.s.}$$

In Section 2 we present the proof of Theorem 1.1, while in Section 3 some typical examples will be given.

2 Proof of Theorem 1.1

First we show that (1.10) implies (1.11). Since $M(t) \geq S(t)$, obviously

$$\inf_{0\leq t\leq T}(M(t) - \mu t - W(t)) \geq -o(r(T)) \quad \text{a.s.,} \tag{2.1}$$

as $T \to \infty$. On the other hand, for $0 \leq s \leq t \leq T$, (1.10) implies

$$S(s) \leq \mu s + W(s) + o(r(T)) \leq \sup_{s\leq t}(\mu s + W(s)) + o(r(T)),$$

and hence

$$M(t) \leq \sup_{0\leq s\leq t}(\mu s + W(s)) + o(r(T)) = \mu t + W(t) + \sup_{0\leq s\leq t}(\mu(s-t) + W(s) - W(t)) + o(r(T)) \quad \text{a.s.}$$

On denoting $t - s = u$, $W(s) - W(t) + \mu(s-t) = W_1(u) - \mu u$, where $W_1(\cdot)$ is a standard Wiener process. Hence

$$\sup_{0\leq s\leq t}(\mu(s-t) + W(s) - W(t)) = \sup_{0\leq u\leq t}(W_1(u) - \mu u) \leq \sup_{0\leq u<\infty}(W_1(u) - \mu u),$$

that is an almost surely finite random variable, we have also

$$M(t) \leq \mu t + W(t) + o(r(T)) \quad \text{a.s.,}$$

i.e.,

$$\sup_{0\leq t\leq T}(M(t) - \mu t - W(t)) \leq o(r(T)) \quad \text{a.s.,} \tag{2.2}$$

as $T \to \infty$. Now (2.1) and (2.2) together imply (1.11).

The statement that (1.10) implies (1.12) follows from Theorem 2.1 of Horváth [15] (see also Theorem 1.3 in [8]).

In the rest of the proof we assume that $X_i \geq 0$, $i = 1, 2, \ldots$ Next we prove that in this case (1.12) implies (1.10). Let n be an integer and put $u = S(n)$, i.e., $N(u) = n$. Applying (1.12), we get

$$S(n) - \mu n = u - \mu N(u) = W\left(\frac{u}{\mu}\right) + o(r(u)) \quad \text{a.s.}$$

It follows that

$$\lim_{n\to\infty}\frac{S(n)}{n} = \lim_{u\to\infty}\frac{u}{N(u)} = \mu \quad \text{a.s.,}$$

and

$$\frac{u}{\mu} - n = \frac{u}{\mu} - N(u) = O((u \log\log u)^{1/2}) = O((n \log\log n)^{1/2}) \quad \text{a.s.},$$

hence

$$S(n) - \mu n = W(n + O((n \log\log n)^{1/2}) + o(r(n)) \quad \text{a.s.}$$

From large increments results for the Wiener process (cf. Csörgő and Révész [9]), we arrive at

$$W(n + O((n \log\log n)^{1/2})) = W(n) + O(n^{1/4}(\log\log n)^{3/4}), \quad \text{a.s.},$$

consequently,

$$S(n) - \mu n = W(n) + O(n^{1/4}(\log\log n)^{3/4}) + o(r(n)) \quad \text{a.s.}$$

Interpolating between n and $n+1$, we get also

$$S(t) - \mu t = W(t) + O(t^{1/4}(\log\log t)^{3/4}) + o(r(t)) \quad \text{a.s.},$$

implying (1.10).

Finally, we prove that (1.10) implies (1.13).

In [7] the following identity was shown:

$$V_n(t) = \frac{1}{2}\left(\frac{S(nt) - \mu nt}{\mu n}\right)^2 + A_n(t) - \frac{1}{2}M_n^2(t), \tag{2.3}$$

where

$$A_n(t) = -\bar{N}_n(t) \int_0^1 (\bar{S}_n(t + u\bar{N}_n(t)) - \bar{S}_n(t))\, du$$

and

$$M_n(t) = \frac{R_n(t)}{\mu\sqrt{n}},$$

$$\bar{S}_n(t) = \frac{S(nt) - \mu nt}{\mu n}, \qquad \bar{N}_n(t) = \frac{N(\mu nt) - nt}{n} \quad t \geq 0.$$

This, in fact, is an algebraic identity, and no properties of the random variables X_i were used in the proof. Hence we also have this identity in the present case.

Now we have to estimate $A_n(t)$ and $M_n(t)$. Obviously, it follows from (1.10) and (1.12) that, as $n \to \infty$,

$$M_n(t) = \frac{S(nt) + N(\mu nt) - 2\mu nt}{\mu\sqrt{n}} = o\left(\frac{r(n)}{n}\right) \quad \text{a.s.},$$

$$\bar{S}_n(t) = \frac{W(nt)}{\mu n} + o\left(\frac{r(n)}{n}\right), \quad \bar{N}_n(t) = O\left(\frac{\sqrt{\log\log n}}{\sqrt{n}}\right) \quad \text{a.s.,}$$

uniformly in $t \in [0,1]$. Hence

$$A_n(t) = -\bar{N}_n(t)\left(\int_0^1 \frac{W(nt + nu\bar{N}_n(t)) - W(nt)}{n}\,du + O\left(\frac{r(n)}{n}\right)\right).$$

Since $n\bar{N}_n(t) = O(\sqrt{n\log\log n})$, for the integrand we have, as before,

$$\frac{W(nt + nu\bar{N}_n(t)) - W(nt)}{n} = \frac{O(n^{1/4}(\log\log n)^{3/4})}{n} = O(n^{-3/4}(\log\log n)^{3/4}),$$

consequently,

$$\begin{aligned} A_n(t) &= O\left(\frac{\sqrt{\log\log n}}{\sqrt{n}}\right)\left(O\left(\frac{(\log\log n)^{3/4}}{n^{3/4}}\right) + O\left(\frac{r(n)}{n}\right)\right) \\ &= O\left(\frac{\log\log n)^{5/4}}{n^{5/4}} + \frac{r(n)(\log\log n)^{1/2}}{n^{3/2}}\right). \end{aligned}$$

Moreover,

$$\begin{aligned} (S(nt) - \mu nt)^2 = (W(nt) + 0(r(n)))^2 &\le W^2(nt) + |W(n)|o(r(n)) + o(r^2(n)) \\ &= W^2(nt) + o(r(n)\sqrt{n\log\log n}), \end{aligned}$$

from which (1.13) follows.

The proof of Theorem 1.1 is now complete. $\square$

3 Applications

In this Section we give some examples for the application of Theorem 1.1. There is a huge literature dealing with strong approximations of partial sums by a Wiener process both in the independent and dependent cases. For our present purpose, we restrict ourselves to the case when $\mu > 0$, or even when the summands themselves can only be non-negative, and the approximation is in terms of a standard Wiener process at time n, i.e., $W(n)$. In the case when the latter approximation takes place via $W(a_n)$, then we choose $a_n = n$, whenever possible. Note that if $a_n = cn$ with some constant $c > 0$, then, without loss of generality, it is possible to choose $c = 1$, by dividing X_i by $\sqrt{c}$.

Not aiming at completeness, we now give some typical examples. For further strong approximation results in the weakly dependent case, in addition to the works cited below, we refer to Eberlein [13], Philipp [20], Lin and Lu [18], Wu [31], and papers cited in these works.

3.1 i.i.d. sequences

Now assume that $\{X_i, i = 1, 2, \ldots\}$ is an i.i.d. sequence, $EX_i = \mu > 0$. The classical result of Strassen [25] gives $r(t) = \sqrt{t \log\log t}$ in case when $EX_i^2 < \infty$. On assuming that for a fixed $\alpha \geq 0$,

$$E(X_i^2(\log\log X_i^2)^\alpha) < \infty,$$

then it was shown by Jain et al. [16] that (1.10) holds with $r(t) = t^{1/2}(\log\log t)^{(1-\alpha)/2}$. We note that with $\alpha = 0$ this rate reduces to the just mentioned classical one in Strassen [25], and, for $\alpha = 2$, it yields that of Breiman [6]. In the case when $E|X_i|^\gamma < \infty$ with $2 < \gamma \leq 4$, Komlós et al. [17] combined with Major [19] yield $r(t) = t^{1/\gamma}$. In [7] the case $\gamma = 4$ is treated. In this case, for the Bahadur-Kiefer type process $R_n(\cdot)$, Deheuvels and Mason [11] proved

$$\lim_{n\to\infty} \frac{n^{1/2}\sup_{0\leq t\leq 1}|R_n(t)|}{(\log n)^{1/2}(\sup_{0\leq t\leq 1}|N(\mu nt) - nt|)^{1/2}} = \mu \quad \text{a.s.},$$

while Csörgő and Horváth [8] established

$$R_n(t) = n^{-1/4}\left(W(nt) - W\left(nt - \frac{W(nt)}{\mu}\right)\right) + o(n^{-1/4}) \quad \text{a.s.}$$

and for fixed $t > 0$

$$n^{1/4}R_n(t) \xrightarrow{d} t^{1/4}\mu^{-1/2}N_1\sqrt{|N_2|},$$

where N_1 and N_2 are independent standard normal random variables.

For further results for Bahadur-Kiefer type process when the 4-th moment does not exist, we refer to Deheuvels and Steinebach [12].

3.2 Martingales

Let $\{X_i - \mu, i = 1, 2, \ldots\}$, $\mu > 0$, be a stationary ergodic sequence of martingale differences, i.e., $E(X_i - \mu \mid X_1, \ldots, X_{i-1}) = 0$ almost surely, for each $i \geq 2$. Assume that $EX_i^2 = 1$ and put

$$V_n = \sum_{i=1}^n E((X_i - \mu)^2 \mid X_1, \ldots, X_{i-1}).$$

On assuming that, as $n \to \infty$,

$$|V_n - n| = o(n(\log\log n)^{-\alpha}),$$

almost surely for some $\alpha \geq 0$, and

$$E(X_i^2(\log X_i^2)(\log\log X_i^2)^\alpha) < \infty,$$

then Theorem 4.3 of Jain et al. [16] yields (1.10) with $r(t) = t^{1/2}(\log\log t)^{(1-\alpha)/2}$. The latter result amounts to spelling out a specific rate in the stationary case for Strassen's classical approximation theorem in [26] for sums of martingale differences.

For further results for martingales along similar lines, we refer to Philipp and Stout [22].

3.3 Mixing sequences

There are several types and results for mixing sequences. Define the following mixing coefficients:

- (1) ϕ-mixing:

$$\phi(n) := \sup \frac{|P(AB) - P(A)P(B)}{P(A)|} \to 0, \quad n \to \infty,$$

  where sup is taken for $A \in \mathcal{F}_1^k$, $B \in \mathcal{F}_{k+n}^\infty$, $k \geq 1$, $P(A) > 0$, where $\mathcal{F}_j^\ell$ is the sigma algebra generated by $(X_j, \ldots, X_\ell)$.

- (2) α-mixing:

$$\alpha(n) := \sup |P(AB) - P(A)P(B)| \to 0, \quad n \to \infty,$$

  where sup is taken as in (1).

- (3) ρ-mixing

$$\rho(n) := \sup |\rho(X,Y)| \to 0, \quad n \to \infty,$$

  where $\rho(X,Y)$ is the correlation coefficient of X and Y and sup is taken for X and Y, measurable with respect to $\mathcal{F}_1^k$ and $\mathcal{F}_{k+n}^\infty$, resp., $k \geq 1$.

For the stationary ϕ-mixing case, a result of Philipp and Stout [21] implies that under the conditions

$$\sum_n \phi^{1/2}(n) < \infty, \quad \lim_{n\to\infty} \frac{Var(S(n))}{n} = 1,$$

$E|X_k|^{2+\delta} < \infty$, $E(X_k) = \mu > 0$, we have (1.10) with $r(t) = t^{1/2-\lambda}$, $\lambda < \delta/(24+12\delta)$.

Again, for the stationary ϕ-mixing case, it follows from Berkes and Philipp [3] and Dabrowski [10] that under the conditions

$$\phi(n) = O((\log n)^{-(1+\varepsilon)(1+2/\delta)}), \quad \lim_{n\to\infty} \frac{Var(S(n))}{n} = 1,$$

$E|X_k|^{2+\delta} < \infty$, $E(X_k) = \mu > 0$, we have (1.10) with $r(t) = t^{1/2}(\log t)^{-1/4}$.

Further results in the ϕ-mixing case can be found in Shao and Lu [24]. In particular, if $E(S^2(n)) = n$, $\sup_k E|X_k|^4 < \infty$, $\phi(n) = O(1/n)$, then (1.10) is true with $r(t) = t^{1/4}(\log t)^{9/4+\delta}$, $\delta > 0$.

In the stationary α-mixing case a result of Bradley [5] implies that under

$$\sup_n \frac{E|S(n)|^{2+\delta}}{(VarS(n))^{(2+\delta)/2}} < \infty, \quad \alpha(n) = O((\log n)^{-\lambda}),$$

$\delta > 0$, $\lambda > 1 + 3/\delta$, $EX_k = \mu > 0$ we have (1.10) with $r(t) = t^{1/2}(\log\log t)^{-1/2}$.

A further result of Shao and Lu [24] for the α-mixing case implies that under the conditions

$$\sup_n E|X_k|^\beta < \infty, \quad \sum_n (\alpha(n))^{1/2+\delta-1/\beta} < \infty,$$

for some $0 < \delta \le 2$, $\beta > 2 + \delta$, (1.10) holds with $r(t) = t^{1/(2+\delta)}(\log t)^{1+(1+\lambda)/(2+\delta)}$.

For the stationary ρ-mixing case the results of Shao [23] imply that under the conditions $\rho(n) = O((\log n)^{-\gamma})$, $\gamma > 1$, $E(S^2(n)) = n$, we have (1.10) with $r(t) = (t \log\log t)^{1/2}$ if $EX_k^2 < \infty$ and $r(t) = t^{1/2}(\log n)^{-\delta}$, $0 < \delta < \gamma/2 - 1/4$, if $E|X_k|^{2+\delta} < \infty$.

3.4 Associated sequences

A sequence $X_1, X_2, \ldots$ of random variables is called associated if

$$Cov(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \ge 0, \quad n \ge 1$$

for coordinatewise non-decreasing functions f, g. It follows from the result of Yu [29] that under the assumptions

$$\sup_{n\ge1} E|X_n|^{2+\gamma} < \infty, \quad \inf_{n\ge1,k\ge0} \frac{E(S_{n+k} - S_k)^2}{n} > 0,$$

$$u(n) = \sup_{k\ge1} \sum_{|j-k|\ge n} Cov(X_j, X_k) = O(e^{-\lambda n}), \quad EX_1^2 + 2\sum_{i=2}^{\infty} E(X_1 X_i) < \infty,$$

$E(X_k) = \mu > 0$, $ES^2(n) = n$, (1.10) holds with $r(t) = t^{1/2-\delta}$.

Wang [30] treated the quasi-associated case when

$$Cov(f(X_1, \ldots, X_i), g(X_{i+1}, \ldots, X_n)) \ge 0,$$

for $1 \le i \le n-1$, and coordinatewise non-decreasing functions f, g. It follows that under the above conditions, with the modified assumption $u(n) = O(n^{-\gamma})$, (1.10) holds with $r(t) = t(\log t)^{-d}$ and any $d > 0$.

3.5 Lacunary series

For lacunary Walsh series it follows from the result proved by Berkes [1] that, if $\{w_n(x),\ n = 1, 2, \ldots\}$ is the series of Walsh functions, then (1.10) is true for $X_k = w_{n_k}(\omega) + \mu$ with $n_{k+1}/n_k \geq 1 + ck^{-\alpha}$, $0 \leq \alpha < 1/2$, $r(t) = t^{1/2-\delta}$, some $\delta > 0$, where ω is a uniform (0,1) random variable.

For lacunary trigonometric series, a consequence of a result of Philipp and Stout [21] reads as follows. (1.10) is true for $X_k = \sqrt{2}\cos(2\pi n_k \omega) + \mu$, where $\mu > 0$, ω is a uniform (0,1) random variable, and $n_{k+1}/n_k > q > 1$, with $r(t) = t^{5/12+\delta}$, any $\delta > 0$. A related result of Berkes [2] for lacunary trigonometric series implies that (1.10) holds for the above X_k under the gap condition $n_{k+1}/n_k \geq 1 + k^{-\alpha}$, $0 \leq \alpha < 1/2$ with $r(t) = t^{1/2-\delta}$, some $\delta > 0$.

For results along these lines for general lacunary series, we refer to Berkes and Philipp [4].

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