Dependence in Lag for Markov chains on partially ordered state spaces with applications to degradable networks

Rafał Kulik*    Cornelia Wichelhaus†

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Abstract

We study the property of dependence in lag for Markov chains on countable partially ordered state spaces and give conditions which ensure that a process is monotone in lag. In case of linearly ordered state spaces proofs are based on the Lorentz inequality. However, we show that on partially ordered spaces Lorentz inequality is only true under additional assumptions. By using supermodular-type stochastic orders we derive comparison inequalities which compare the internal dependence structure of processes with that of their speeding-down versions.

Applications of the results are presented for degradable exponential networks in which the nodes are subject to random breakdowns and repairs. We obtain comparison results for the breakdown processes as well as for the queue length processes which are not even Markovian on their own.

Key words: dependence ordering, supermodular functions, Lorentz inequality, dependence in lag, Jackson networks, networks with breakdowns and repairs

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*rkulik@mail.usyd.edu.au, School of Mathematics & Statistics F07, University of Sydney NSW 2006, Australia and Math. Institute, Wrocław University, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
†wichelhaus@statlab.uni-heidelberg.de, Institute of Applied Mathematics, University of Heidelberg, Im Neuenheimer Feld 294, 69120 Heidelberg, Germany
1 Introduction

Let $Y = (Y_t : t \geq 0)$ be a stationary homogeneous Markov chain in continuous time with a partially ordered countable state space $E$. We assume that $E$ is a lattice ordered Polish space equipped with a partial ordering $\prec$. Especially, we consider $E = \mathbb{N}^J$ with coordinate wise ordering $\leq'$, and $E = \mathcal{P}(J)$, the power set of $J = \{1, \ldots, J\}$, with inclusion ordering $\subseteq$, where $\subseteq$ means "subset or equal".

In this paper we show that under stochastic monotonicity, reversibility and up-and-down assumptions dependence in our Markov processes decreases in time in the sense of supermodular-type ordering. More precisely, for all $s < t$ and all functions $f$ from a particular class we have

$$
\mathbb{E}[f(Y_0, Y_t)] \leq \mathbb{E}[f(Y_0, Y_s)].
$$

We call $Y$ then monotone in lag and shall refer to such inequalities as to covariance inequalities. Particularly, (1) implies for all increasing, real-valued functions $g, h$ on $E$ for $s < t$: $\text{Cov}[g(Y_0), h(Y_t)] \leq \text{Cov}[g(Y_0), h(Y_s)]$, provided that the covariances exist.

Furthermore, we are able to compare two Markov chains with the same stationary distribution but such that the second process is obtained by “speeding down” the underlying environment. This is a counterpart to the results for totally ordered spaces, in particular in the context of single queueing systems, see Hu and Joe (1995), Bäuerle and Rolski (1998), Hu and Pan (2000), Miyoshi and Rolski (2004) and Rolski (2005). The first result concerning covariance inequalities for stochastically monotone Markov chains and, in particular, one server queueing systems, has been obtained by Daley. It is Daley’s longstanding conjecture that similar results should be obtained for $k$-server queueing systems. We refer to Daley (1968a), (1968b) for the details. McNickle (1991) studies correlation in lag for two-node Jackson networks via the randomization method.

Applications to network processes we have in mind require methods and theory for non linearly ordered spaces. Up to now in the literature only processes with totally ordered state spaces have been investigated with respect to internal dependence. Techniques to compare internal dependence in Markov processes on partially ordered, not
necessary countable spaces, were derived in Daduna and Szekli (2006) and developed further in Daduna et al. (2006). These techniques involve comparison of integrals with respect to supermodular-type functions. In the latter paper, the isotone differences ordering was introduced and comparison results for Markov processes and their so called $\varepsilon$-transformations were obtained. Applications are in the framework of standard and degradable queueing systems. This paper is a continuation of the study and explores the dependence in lag property for Markov chains on partially ordered spaces.

We give a review of dependence orderings based on supermodular-type functions on partially ordered spaces in Section 2.1. One of the important tools to study internal dependence properties like monotonicity in lag in case of totally ordered state spaces is the so called Lorentz inequality. However, as we show in Section 2.2, this inequality is in general not valid in the partially ordered case. Thus, we have to explore which classes of monotone Markov chains fulfill this inequality. It turns out that sufficient conditions are, among others, reversibility and up-and-down property. For Markov chains $Y$ we study their speeding-down versions $\hat{Y}^c$ and compare their internal dependence structures.

A prominent example of a Markov chain on a partially ordered state space which is not totally ordered is the joint queue length process of a standard exponential network. Clearly, this process is not up-and-down. Therefore we are not able to establish covariance inequalities for it. However, in Section 3 we show how to compare two degradable exponential networks in which the nodes may break down. We consider networks which only differ in the speed of the chains which control the breakdowns and repairs of the nodes. We obtain covariance inequalities for these breakdown and repair processes which are Markovian on their own and show that they are monotone in lag. Furthermore, we derive covariance inequalities for the marginal queue length processes in the degradable networks which enables us to compare the internal dependence structure of the networks’ performance processes. It is to emphasize that these processes are not Markovian themselves and do not offer a direct access.
2 Covariance inequalities for Markov processes on lattices

2.1 Dependence orderings

Stochastic orders are powerful tools for the comparison of stochastic processes and models which often lead to good approximations and bounds. Many stochastic orders for random variables \( Y \) and \( Y' \) on the same space \( E \) are characterized by integral inequalities with respect to a class \( \mathcal{F}(E) \) of real-valued measurable functions on \( E \), i.e. we say \( Y \prec_{\mathcal{F}(E)} Y' \) if \( \mathbb{E}[f(Y)] \leq \mathbb{E}[f(Y')] \) for all \( f \in \mathcal{F}(E) \). These orders \( \prec_{\mathcal{F}(E)} \) are called integral stochastic orders and the set \( \mathcal{F}(E) \) is the generator of the ordering. We refer to the book of Müller and Stoyan (2002) for a comprehensive study of notions, properties and applications of stochastic orders.

We consider in this paper random vectors \( Y = (Y_1, \ldots, Y_n) \) and \( Y' = (Y'_1, \ldots, Y'_n) \) on lattice ordered product spaces \( E^{(n)} = \times_{i=1}^{n} E_i \) with partially ordered Polish spaces \( E_i \), with a closed partial ordering \( \prec_i \) each, \( i = 1, \ldots, n \), and endowed with the Borel-\( \sigma \)-algebra \( \mathcal{E}^{(n)} \) on \( E^{(n)} \). A partial ordering \( \prec^{(n)} \) on \( E^{(n)} \) is defined coordinate wise, i.e., for \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \) \( \in E^{(n)} \) we have \( x \prec^{(n)} y \) if and only if \( x_i \prec_i y_i \) for all \( i = 1, \ldots, n \). If \( (E_i, \prec_i) = (E, \prec) \) for \( i = 1, \ldots, n \), we write \( E^n \) instead of \( E^{(n)} \) and \( \prec^n \) instead of \( \prec^{(n)} \). A function \( f : E \to \mathbb{R} \) is \( \prec \)-increasing if for all \( x, y \in E \), \( x \prec y \) implies \( f(x) \leq f(y) \).

If \( Y \) and \( Y' \) have values in \( E^{(n)} = \mathbb{R}^n \) with the same marginal distributions and the generator \( \mathcal{F}(E^{(n)}) \) of the order \( \prec_{\mathcal{F}(E^{(n)})} \) is such that it contains the functions \( f(x_1, \ldots, x_n) = x_i x_j \) for all \( 1 \leq i < j \leq n \), then \( Y \prec_{\mathcal{F}(E^{(n)})} Y' \) implies \( \text{Cov}(Y_i, Y_j) \leq \text{Cov}(Y'_i, Y'_j) \) for all \( i < j \). The order \( \prec_{\mathcal{F}(E^{(n)})} \) then is called naturally a dependence order. An important dependence order is defined by the class of supermodular functions. We generalize this concept to the case of partially ordered spaces \( E^{(n)} : \)

**Definition 2.1 (supermodular order)** Let \( Y = (Y_1, \ldots, Y_n) \) and \( Y' = (Y'_1, \ldots, Y'_n) \) be random vectors with values in a lattice ordered space \( (E^{(n)}, \mathcal{E}^{(n)}, \prec^{(n)}) \). We call a function \( f : E^{(n)} \to \mathbb{R} \) supermodular if \( f(x \wedge y) + f(x \vee y) \geq f(x) + f(y) \) for all \( x, y \in E^{(n)} \). We denote by \( \mathcal{L}_{\text{sm}}(E^{(n)}, \prec^{(n)}) \) the class of all real-valued measurable supermodular...
functions on $E^{(n)}$. We say that $Y$ and $Y'$ are ordered in the supermodular order, written $Y <_{\text{sm}(\prec^{(n)})} Y'$, if $E[f(Y)] \leq E[f(Y')]$ for all functions $f \in \mathcal{L}_{\text{sm}}(E^{(n)}, \prec^{(n)})$ such that both expectations exist.

If the spaces $E_i$ are linearly ordered the class of supermodular functions $\mathcal{L}_{\text{sm}}(E^{(n)}, \prec^{(n)})$ on $(E^{(n)}, \mathcal{E}^n, \prec^{(n)})$ agrees with the class $\mathcal{L}_{\text{idif}}(E^{(n)}, \prec^{(n)})$ of functions with isotone differences and the corresponding orders are equivalent:

**Definition 2.2 (isotone differences order)** Let $Y = (Y_1, \ldots, Y_n)$ and $Y' = (Y'_1, \ldots, Y'_n)$ be random vectors with values in a partially ordered space $(E^{(n)}, \mathcal{E}^n, \prec^{(n)})$. A function $f : E^{(2)} \to \mathbb{R}$ has isotone differences if for all $x_1 <_1 x'_1$, $x_2 <_2 x'_2$ we have

$$f(x'_1, x'_2) - f(x_1, x'_2) \geq f(x'_1, x_2) - f(x_1, x_2).$$

(2)

A function $f : E^{(n)} \to \mathbb{R}$ has isotone differences if (2) is satisfied for any pair $(i, j)$ of coordinates, whereas the remaining variables are fixed. We denote the class of all measurable functions with isotone differences on $(E^{(n)}, \mathcal{E}^{(n)})$ by $\mathcal{L}_{\text{idif}}(E^{(n)}, \prec^{(n)})$. We say that $Y$ and $Y'$ are ordered in the isotone differences order, written $Y <_{\text{idif}(\prec^{(n)})} Y'$, if $E[f(Y)] \leq E[f(Y')]$ for all $f \in \mathcal{L}_{\text{idif}}(E^{(n)}, \prec^{(n)})$, such that both expectations exist.

In case of partially ordered state spaces we have to distinguish between supermodular functions and functions with isotone differences. It was demonstrated in Daduna et al. (2006) that in case of partially ordered state spaces the class of supermodular functions is not appropriate to study dependence orderings because of the lack of many closure properties. However, as was shown there the class of functions with isotone differences possesses all important closure properties and hence is appropriate to study dependence orderings. The notion of dependence ordering for random processes is defined as follows:

**Definition 2.3** Let $Y = (Y_t : t \in T)$, $Y' = (Y'_t : t \in T)$, $T \subseteq \mathbb{R}$, be stochastic processes with partially ordered state space $(E, \mathcal{E}, \prec)$. Then we write $Y <_{\text{idif}(\prec^{(\infty)})} Y'$ if for all $n \geq 2$ and all $t_1 < \cdots < t_n \in T$ we have $(Y_{t_1}, \ldots, Y_{t_n}) <_{\text{idif}(\prec^{(n)})} (Y'_{t_1}, \ldots, Y'_{t_n})$.

For simplicity, we will omit $(\prec^{(\infty)})$ and $(\prec^{(n)})$ in the above notations if it is clear which orderings $\prec_i$, $i = 1, \ldots, n$, we mean.
Consider now stationary homogeneous continuous-time Markov chains $Y = (Y(t) : \ t \geq 0)$ and $Y' = (Y'(t) : \ t \geq 0)$ with a common countable state space $(E, \mathcal{E}, \prec)$ and families of transition kernels $P^Y := (P_t^Y : E \times \mathcal{E} \to [0, 1] : \ t \geq 0)$ and $P^{Y'} := (P_t^{Y'} : E \times \mathcal{E} \to [0, 1] : \ t \geq 0)$ respectively. By $P^Y$ and $P^{Y'}$ we denote the families of transition kernels for the time-reversed processes. $Q^Y = (q^Y(x,y) : \ x, y \in E)$, $Q^{Y'} = (q^{Y'}(x,y) : \ x, y \in E)$, $Q^Y = (q^Y_-(x,y) : \ x, y \in E)$, $Q^{Y'} = (q^{Y'}_-(x,y) : \ x, y \in E)$ are the corresponding infinitesimal generators. We always assume that all processes are uniform (i.e. their generators are bounded and conservative). Assume further that $\pi = (\pi(x) : \ x \in E)$ is the stationary distribution common for both $Y$ and $Y'$.

A Markov process $Y$ is called (stochastically) $\prec$-monotone if all kernels $P^Y_t$ of its transition kernel family $P^Y := (P^Y_t : E \times \mathcal{E} \to [0, 1] : \ t \geq 0)$ are $\prec$-monotone. Recall that a transition kernel $P^Y_t$ is $\prec$-monotone if the map $x \mapsto \int_E P^Y_t(x,dy)f(y)$ is $\prec$-increasing provided that $f$ is $\prec$-increasing. From now on we assume that either $P^Y$ and $\widehat{P}^Y_-$ or $P^Y_-$ and $\widehat{P}^Y$ are families of $\prec$-monotone kernels. We are not aware of any results concerning supermodular-type ordering of Markov processes without this restriction, except of the recent papers by Kulik (2004) and Daduna and Szekli (2006).

2.2 Lorentz inequality on partially ordered state spaces

Consider first a stationary monotone discrete-time Markov process $Y = (Y_n : \ n \in \mathbb{N})$ with state space $E$. In case of $E = \mathbb{R}$ it is shown in Hu and Pan (2000) that the process $Y$ is monotone in lag, which in the framework of totally ordered state spaces means that for all supermodular functions $f \in \mathcal{L}_{sm}(\mathbb{R}^2, \leq_2)$ we have $E(f(Y_0, Y_m)) \leq E(f(Y_0, Y_n))$ for all $n \geq m$. Thus, internal dependencies decrease in time in the sense of supermodular ordering.

The proof of monotonicity of $\varphi(n) := \mathbb{E}(f(Y_0, Y_n))$, $n \geq 0$, for each supermodular function $f$, is based on the Lorentz inequality, see Müller and Stoyan (2002):

*If $Y_0$ and $Y_1$ are random variables on $(\mathbb{R}, \leq)$ with the same distribution then for all supermodular functions $f \in \mathcal{L}_{sm}(\mathbb{R}^2, \leq_2)$: \[ \mathbb{E}(f(Y_0, Y_1)) \leq \mathbb{E}(f(Y_0, Y_0)). \]*

In fact, this inequality here is necessary for the monotonicity of $\varphi(n)$. 
In this section we investigate the tempting question whether it is possible to prove the following analogue in case of partially ordered spaces \((E, \prec)\):

If \(Y_0\) and \(Y_1\) are random variables on \((E, \prec)\) with the same distribution then for all functions \(f \in \mathcal{L}_{\text{idif}}(E^2, \prec^2)\):

\[
\mathbb{E}(f(Y_0, Y_1)) \leq \mathbb{E}(f(Y_0, Y_0)).
\]

It turns out that the situation is completely different if \(E\) is not totally ordered and that this inequality does not hold in general on only partially ordered spaces \(E\) as the following counter-example shows.

**Example 2.4** Let the space \(E = \{(0, 0), (0, 1), (1, 0), (1, 1)\}\) be endowed with a partial ordering \(\prec\) which is given by \((0, 0) \prec (0, 1) \prec (1, 1)\) and \((0, 0) \prec (1, 0) \prec (1, 1)\).

Consider the two-dimensional space \(E^2\) with the ordering \(\prec^2\) on \(E^2\) defined coordinate wise. Define functions \(f_i : E \to \mathbb{R}, i = 1, 2\), as \(f_1(x) := 1_{\{x \in \{(0,1),(1,1)\}\}}, f_2(x) := 1_{\{x \in \{(1,0),(1,1)\}\}}\). Clearly, \(f_i \in \mathcal{I}^*(E, \prec)\) for \(i = 1, 2\), and, \(f = f_1f_2 \in \mathcal{L}_{\text{idif}}(E^2, \prec^2)\).

Let now \((Y_0, Y_1)\) be a random vector with state space \((E^2, \prec^2)\) given by the distribution: \(\mathbb{P}\left((Y_0, Y_1) = ((0, 1), (1, 0))\right) = \mathbb{P}\left((Y_0, Y_1) = ((1, 0), (0, 1))\right) = \frac{1}{2}\). By computing the marginal distributions we see that the random variables \(Y_0\) and \(Y_1\) have the same distribution on \(E\): \(\mathbb{P}(Y_i = (0,1)) = \mathbb{P}(Y_i = (1,0)) = \frac{1}{2}, \quad i = 0, 1\).

It follows:

\[
\mathbb{P}\left((Y_0, Y_0) = ((0, 1), (0, 1))\right) = \mathbb{P}\left((Y_0, Y_0) = ((1, 0), (1, 0))\right) = \frac{1}{2}.
\]

Thus,

\[
\mathbb{E}(f(Y_0, Y_1)) = \mathbb{E}\left(f_1(Y_0)f_2(Y_1)\right) = \mathbb{P}\left((Y_0, Y_1) = ((0, 1), (1, 0))\right) = \frac{1}{2}
\]

and

\[
\mathbb{E}(f(Y_0, Y_0)) = \mathbb{E}\left(f_1(Y_0)f_2(Y_0)\right) = \mathbb{P}\left((Y_0, Y_0) = ((0,1), (1,0))\right) = 0.
\]

Hence, Lorentz inequality does not hold.

Thus, the question to be answered here is for which class of processes \(Y\) the property of monotonicity in lag can be proved. We present the answer in the framework of continuous-time Markov chains \(Y = \{Y_t : t \geq 0\}\) with generator \(Q^Y\) on a countable partially ordered state space \((E, \mathcal{E}, \prec)\). Recall that the process \(Y\) is always assumed to be uniform, \(\prec\)-monotone and stationary with steady state distribution \(\pi = (\pi(x): x \in E)\).

For the process \(Y\) we define for each \(c \in (0,1)\) a process \(\tilde{Y}^c\) via the intensity matrix \(Q^{\tilde{Y}^c} = (q^{\tilde{Y}^c}(x,y) : x, y \in E)\) determined by \(q^{\tilde{Y}^c}(x,y) := cq^Y(x,y)\) for all \(x, y \in E\). Thus,
\( \tilde{Y}^c \) develops more slowly than \( Y \). The properties of the process \( Y \) like being uniform, \( \prec \)-monotone and stationary with steady state distribution \( \pi = (\pi(x) : x \in E) \) are transferred to the process \( \tilde{Y}^c \). For our purposes it is necessary to assume that the time-reversed process of \( \tilde{Y}^c \) is also \( \prec \)-monotone.

A Markov chain \( Y \) is called up-and-down if \( q^Y(x,y) = 0 \) if \( x \) and \( y \) are not comparable and the chain \( Y \) is reversible if \( \pi(x) q^Y(x,y) = \pi(y) q^Y(y,x) \) for all \( x, y \in E \).

**Theorem 2.5** Let \( Y = (Y_t : t \geq 0) \) be a stationary Markov chain with generator \( Q^Y = (q^Y(x,y) : x, y \in E) \) on the partially ordered state space \((E, \mathcal{E}, \prec)\). If \( Y \) is reversible and up-and-down then the processes \( Y \) and \( \tilde{Y}^c \) are ordered in the isotone differences ordering, i.e., for all \( f \in \mathcal{L}_{\text{idif}}(E^2, \prec^2) \) we have:

\[
\mathbb{E}(f(Y_0,Y_t)) \leq \mathbb{E}(f(\tilde{Y}^c_0,\tilde{Y}^c_t)) \text{ for all } t \geq 0.
\]

**Proof:** Let \( f \in \mathcal{L}_{\text{idif}}(E^2, \prec^2) \). Then we have:

\[
\sum_{x \in E} \pi(x) \sum_{y \in E} (1 - c) q^Y(x,y) f(x,y) = \sum_{x \in E} \pi(x) \sum_{y \in E : y \neq x} (1 - c) q^Y(x,y) (f(x,y) - f(x,x))
\]

\[
\overset{(1)}{=} \sum_{x \in E} \pi(x) \sum_{y \in E : x \prec y} (1 - c) q^Y(x,y) (f(x,y) - f(x,x)) + \sum_{x \in E} \sum_{y \in E : x \prec y} (1 - c) \pi(x) q^Y(x,y) (f(x,y) - f(x,x))
\]

\[
\overset{(2)}{=} \sum_{x \in E} \pi(x) \sum_{y \in E : x \prec y} (1 - c) q^Y(x,y) (f(x,y) - f(x,x)) + \sum_{y \in E} \sum_{x \in E : x \prec y} (1 - c) q^Y(y,x) (f(y,x) - f(y,y))
\]

\[
\overset{(3)}{=} \sum_{x \in E} \pi(x) \sum_{y \in E : x \prec y} (1 - c) q^Y(x,y) (f(x,y) - f(x,x)) + \sum_{x \in E} \sum_{y \in E : x \prec y} (1 - c) q^Y(x,y) (f(y,x) - f(y,y))
\]

\[
= \sum_{x \in E} \pi(x) \sum_{y \in E : x \prec y} (1 - c) q^Y(x,y) (f(x,y) + f(y,x) - f(x,x) - f(y,y)) \overset{(4)}{\leq} 0.
\]
Corollary 2.6 Let $Y = (Y_t : t \geq 0)$ be a stationary Markov chain. If $Y$ is reversible and up-and-down then we have for all $0 < s < t : (Y_0, Y_t) \preceq_{\text{idif}} (Y_0, Y_s)$.

Proof: We have $s = ct$ for some $c \in (0, 1)$. The result follows from Theorem 2.5 since the transition kernels of the chains $Y$ and $\hat{Y}^c$ are related by $P_t^{\hat{Y}^c} = P_t^Y$ for all $t \geq 0$.

An analogue of Lorentz inequality on partially ordered spaces directly follows from Corollary 2.6. However, we have to restrict ourselves here to bounded $f \in L_{\text{idif}}(E^2, \preceq^2)$:

Corollary 2.7 If the Markov chain $Y$ is reversible and up-and-down then for all bounded $f \in L_{\text{idif}}(E^2, \preceq^2)$ we have $\mathbb{E}[f(Y_0, Y_1)] \leq \mathbb{E}[f(Y_0, Y_0)]$ for all $t \geq 0$.

Proof: Due to Corollary 2.6 it is enough to show for all bounded $f \in L_{\text{idif}}(E^2, \preceq^2)$:

$$\lim_{k \to \infty} \mathbb{E}[f(Y_0, Y_{k+1})] = \mathbb{E}[f(Y_0, Y_0)].$$

Denote by $p_k^Y(x; z)$ the $\frac{1}{k}$-step transition probability from $x$ to $z$ of $Y$ and by $\delta_{\{x\}}(\cdot)$ the dirac measure in $\{x\}$. Then we have due to the bounded convergence theorem:

$$\lim_{k \to \infty} \mathbb{E}[f(Y_0, Y_{k+1})] = \lim_{k \to \infty} \left( \sum_{x \in E} \pi(x) \sum_{z \in E} p_k^Y(x; z) f(x, z) \right)$$

$$= \sum_{x \in E} \pi(x) \lim_{k \to \infty} \left( \sum_{z \in E} p_k^Y(x; z) f(x, z) \right) = \sum_{x \in E} \pi(x) \sum_{z \in E} \left( \lim_{k \to \infty} p_k^Y(x; z) \right) f(x, z)$$

$$= \sum_{x \in E} \pi(x) \sum_{z \in E} \delta_{\{x\}}(z) f(x, z) = \mathbb{E}[f(Y_0, Y_0)].$$

Standard technical arguments as in the totally ordered case yield:

Corollary 2.8 If $Y$ is reversible and up-and-down then for all $t_i \geq 0$ and all bounded $f \in L_{\text{idif}}(E^{k+1}, \preceq^{k+1})$, the map $(t_1, \ldots, t_k) \to \mathbb{E}[f(Y_0, Y_{t_1}, Y_{t_1+t_2}, \ldots, Y_{t_1+\ldots+t_k})]$ is decreasing, i.e. for all $(t_1, \ldots, t_k) \leq^k (t_1', \ldots, t_k')$ we have

$$\mathbb{E}[f(Y_0, Y_{t_1}, Y_{t_1+t_2}, \ldots, Y_{t_1+\ldots+t_k})] \geq \mathbb{E}[f(Y_0, Y_{t_1'}, Y_{t_1'+t_2'}, \ldots, Y_{t_1'+\ldots+t_k'})].$$
Following the lines of Def. 3 in Miyoshi and Rolski (2004) we call $Y$ then $\mathcal{L}_{idif}$-regular.

3 Covariance inequalities for unreliable networks

We apply the results of the preceding sections to open exponential networks of unreliable nodes which we sketch here. For a detailed description and for the discussion of more general classes of degradable networks of product form we refer to Sauer (2006).

The model is as follows: Consider a Jackson network of $J$ numbered nodes, denoted by $J = \{1, \ldots, J\}$. Station $j \in J$, is a single server queue with infinite waiting room under FCFS regime. Indistinguishable customers arrive in a Poisson stream with intensity $\lambda > 0$ and are sent to node $j$ with probability $r_{0j} \sum_{j=1}^{J} r_{0j} = r \leq 1$. Customers arriving at node $j$ request a service which is exponentially distributed with mean $\mu_j^{-1}$. All service times and the arrival process are assumed to be independent.

A customer departing from node $j$ immediately proceeds to node $i$ with probability $r_{ji}$ or departs from the network with probability $r_{j0}$. The routing is independent of the past of the system given the momentary node where the customer is. Let $J_0 := J \cup \{0\}$. We assume that the stochastic matrix $R_0 := (r_{ij} : i, j \in J_0)$ is irreducible. Denote by $\eta = (\eta_1, \ldots, \eta_J)$ the unique solution of the traffic equation $\eta_j = \lambda r_{0j} + \sum_{i=1}^{J} \eta_i r_{ij}$, $j \in J$.

The servers at the nodes are unreliable, i.e., the nodes may break down.

Control of breakdowns and repairs is as follows: Let $I \subset J$ be the set of nodes in down status and $H \subset J \setminus I$ with $H \neq \emptyset$, be some subset of nodes in up status. Then the nodes of $H$ break down concurrently with intensity $\alpha(I, I \cup H)$. Nodes in down status neither accept new customers nor continue serving the old customers who will wait for the server’s return. Therefore, the routing has to be changed so that customers attending to join a node in down status are rerouted to nodes in up status or to the outside. We will give three rerouting schemes below. These originate from mechanisms applied e.g. in production theory or in the theory of information blocking. Assume the nodes in $I$ are under repair, $I \neq \emptyset$. Then the nodes of $H \subset I, H \neq \emptyset$, return from repair as a batch with intensity $\beta(I, I \setminus H)$ and immediately resume services. We adapt
routing then anew. The intensities for occurrence of breakdowns and repairs have to be set under constraints. A versatile class of intensities is defined as follows.

**Definition 3.1** Let $I$ be the set of nodes in down status. The intensities for breakdowns, resp. repairs for $H \neq \emptyset$ are defined by

$$\alpha(I, I \cup H) := \frac{a(I \cup H)}{a(I)}, \text{ resp. } \beta(I, I \setminus H) := \frac{b(I)}{b(I \setminus H)},$$

where $a$ and $b$ are any functions, $a, b : P(J) \to \mathbb{R}_+ = [0, \infty)$ with $a(\emptyset) = b(\emptyset) = 1$. We set $0^0 := 0$. The above intensities are assumed to be finite.

**Definition 3.2** Rerouting matrices of interest are as follows.

1. **Blocking.** Assume that the routing matrix $R = (r_{ij} : i, j \in J)$ is reversible. Assume the nodes in $I$ are presently under repair. Then the routing probabilities are redefined on $J_0 \setminus I$ according to

$$\tilde{r}_{ij}^1 = r_{ij} \text{ for } i, j \in J_0 \setminus I \text{ with } i \neq j, \text{ and, } \tilde{r}_{ij}^1 = r_{ii} + \sum_{k \in I} r_{ik} \text{ for } i \in J_0 \setminus I \text{ with } i = j.$$

2. **Stalling.** If there is any breakdown of some node, then the arrival stream to the network and all service processes are completely interrupted and resumed only when all nodes are repaired again.

3. **Skipping.** Assume that the nodes in $I$ are the nodes presently under repair. Then the routing matrix is redefined on $J_0 \setminus I$ according to:

$$\tilde{r}_{jk}^1 = r_{jk} + \sum_{i \in I} r_{ji} r_{ik}^1, \text{ for } k \in J_0 \setminus I, j \in J.$$

**State space and Markov process** describing the time evolution of the degradable exponential network are constructed as follows:

Let $X_j(t)$ be the number of customers present at node $j$ at time $t \geq 0$. Then $X := (X(t), t \geq 0)$ with $X(t) = (X_1(t), \ldots, X_J(t))$ is the joint queue length process on the state space $(E, \prec) := (\mathbb{N}^J, \leq^J)$. Note that $X$ is not Markovian. The availability status of the network at time $t \geq 0$ is indicated by $Y(t) = I$, where $I \subseteq J$ is the set
of nodes that are broken down (under repair). $Y := (Y(t), t \geq 0)$ is the availability (breakdown/repair) process on the state space $\mathcal{P}(J)$. $Y$ is a Markov chain.

From this description it is easy to see that the joint process $Z := (Z(t), t \geq 0)$ with $Z(t) := (Y(t), X(t)), t \geq 0$, is a strong Markov chain on the state space $E = \mathcal{P}(J) \times \mathbb{N}^J$. States of $Z$ are $(I; n_1, n_2, \ldots, n_J) \in \mathcal{P}(J) \times \mathbb{N}^J$ with the meaning: $I$ is the set of nodes under repair. The numbers $n_j \in \mathbb{N}$ indicate for nodes $j \in J\setminus I$, which work in up status, that there are $n_j$ customers present; for nodes $j \in I$ in down status the numbers $n_j \in \mathbb{N}$ indicate that there are $n_j$ customers waiting for the return of the repaired server at node $j$.

For these general models with breakdowns and repairs and with the above rerouting principles it was shown in Sauer and Daduna (2003) that on the state space $E$ the steady state distribution for $Z$ is of product form: if for all $j \in J$ we have $\eta_j < \mu_j$ then $Z$ is ergodic and the stationary distribution is given for all $(I; n_1, n_2, \ldots, n_J) \in E$ by:

$$
\pi(I; n_1, n_2, \ldots, n_J) = \pi_Y(I) \pi_0(n_1, n_2, \ldots, n_J), \quad \text{where}
$$

$$
\pi_0(n_1, \ldots, n_J) = \prod_{j=1}^{J} \left( 1 - \frac{\eta_j}{\mu_j} \right)^{n_j}, \quad \pi_Y(I) = \left( \sum_{K \subseteq J} \frac{a(K)}{b(K)} \right)^{-1} \frac{a(I)}{b(I)} \quad \text{for} \ I \subseteq J.
$$

We see that in equilibrium the queue length process $X$ and the breakdown/repair process $Y$ decouple and behave at a fixed time instant as if they were independent.

Furthermore, the marginal steady state $\pi_0(n_1, \ldots, n_J)$ of the non Markovian process $X$ always coincides with the equilibrium distribution in a standard (completely reliable) open exponential network, Jackson (1957). Nevertheless, the behavior of the queueing process $X$ is completely different in all cases (standard Jackson, rerouting with blocking, stalling, skipping). Moreover, it is possible to vary the breakdown/repair intensities without changing the stationary distribution of $Z$. Thus, the network’s behavior over time presents itself for comparison in dependence orderings.

We first consider the breakdown/repair process $Y$ on its own. It is easy to see that $Y$ is reversible and has the up-and-down property. We obtain as direct consequence from Corollary 2.6 that the breakdown/repair process $Y$ is monotone in lag:
**Corollary 3.3** Consider in the degradable exponential network the breakdown/repair process \( Y = (Y(t) : t \geq 0) \) on its state space \( \mathcal{P}(J) \). We assume \( Y \) to be \( \subset \)-monotone. Then for all \( 0 < s < t \) we have: \( (Y_0,Y_t) <_{\text{idif}(\subset^2)} (Y_0,Y_s) \).

The queue length process in a standard Jackson network is not up-and-down. This follows from the structure of its infinitesimal generator. Thus, with our dependence order theory developed so far we are not able to derive here covariance inequalities for the queue length processes in two standard networks. However, we show that we are able to compare the queue length processes of two degradable networks. This is surprising since these processes are not even Markovian.

We consider two degradable networks as just described which differ only in the breakdown/repair behavior. In the first model breakdowns and repairs are controlled by the Markov chain \( Y \) on \( \mathcal{P}(J) \), and we denote the network process by \( Z = (Y,X) \), \( X \) is the queue length process of the model. In the second model breakdowns and repairs are controlled by the Markov chain \( \hat{Y}^c \) which originates from speeding down the process \( Y \) as described in Section 2.2, thus \( Q^{\hat{Y}^c} = cQ^Y \). It follows that in the second model changes in the availability status take a longer time than in the first model. We denote the network process by \( \hat{Z} = (\hat{Y}^c, \hat{X}) \). Note that \( Z \) and \( \hat{Z} \) have the same stationary distribution given by (3). Clearly, the behavior of \( \hat{X} \) is influenced by \( \hat{Y}^c \). However, a priori, we cannot say that \( \hat{X} \) is more internal dependent than \( X \). If you think e.g. of rerouting in terms of **skipping** then there are times where customers in the second network can move faster through the network and leave it earlier as in the first network.

Theorem 2.5 yields the following comparison result for the two breakdown/repair processes working in the environments of the degradable networks with different speeds:

**Corollary 3.4** Consider two degradable exponential networks which differ only in the behavior of their breakdown/repair processes \( Y = (Y(t) : t \geq 0) \) and \( \hat{Y}^c = (\hat{Y}^c(t) : t \geq 0) \) on the state space \( \mathcal{P}(J) \). We assume \( Y \) and \( \hat{Y}^c \) to be \( \subset \)-monotone. Then for all \( t \geq 0 \) : \( (Y_0,Y_t) <_{\text{idif}(\subset^2)} (\hat{Y}^c_0,\hat{Y}^c_t) \).

In order to derive the comparison result for the queue length processes we introduce a
partial order on the state space $E = \mathcal{P}(J) \times \mathbb{N}^J$ of the degradable networks:

**Definition 3.5** On the space $E = \mathcal{P}(J) \times \mathbb{N}^J$ we define the partial order $(\subset_e \times \leq^J)$:

$$(I_1, n_1) (\subset_e \times \leq^J) (I_2, n_2) \text{ if and only if } I_1 \subset_e I_2 \text{ and } n_1 \leq^J n_2$$

where $I_1 \subset_e I_2$ if and only if $I_1 = I_2$.

**Theorem 3.6** Assume that two stationary degradable exponential networks have the same arrival intensity $\lambda$, the same service intensities $\mu_i$, $i \in J$, the same routing matrix $R_0 = (r(i, j) : i, j \in J_0)$ and the same rerouting strategy in case of a breakdown. They differ in the breakdown/repair behavior. It is in the first model (described by $Z = (Y, X)$) determined by the Markov chain $Y$ and in the second model (described by $\tilde{Z} = (\tilde{Y}^c, \tilde{X})$) determined by the Markov chain $\tilde{Y}^c$. We assume that $Y$ and the time-reversal $\tilde{Y}^c$ are stochastically monotone. Then we have for the network processes

$$Z \leq_{\text{idif}} (\subset_e \times \leq^J)^{(\infty)} \tilde{Z},$$

i.e., for all $n \geq 2$ and all $t_1 < \cdots < t_n$, and all functions $f$ with isotone differences on $(E^n, (\subset_e \times \leq^J)^n)$ we have $E\left(f(Z_{t_1}, \ldots, Z_{t_n})\right) \leq E\left(f(\tilde{Z}_{t_1}, \ldots, \tilde{Z}_{t_n})\right)$.

**Proof:** Theorem 3.6 is proved in analogy to Theorem 3.2 in Daduna et al. (2006). For a more detailed proof we refer to Section 5.3.2 in Sauer (2006). The main arguments:

1. The describing Markov process $Z = (Y, X)$ of a degradable network as introduced above is $(\subset_e \times \leq^J)$-monotone.

2. If we have for all $h \in \mathcal{L}_{\text{idif}}(\mathcal{P}(J)^2, \leq^2)$

$$\sum_{I_1 \in E} \pi(I_1) \sum_{I_2 \in E} q^Y(I_1, I_2) h(I_1, I_2) \leq \sum_{I_1 \in E} \pi(I_1) \sum_{I_2 \in E} q^{\tilde{Y}^c}(I_1, I_2) h(I_1, I_2), \quad (4)$$

then it follows for all $f \in \mathcal{L}_{\text{idif}}(E, (\subset \times \leq^J)^2)$ :

$$\sum_{(I_1, n_1) \in E} \pi(I_1, n_1) \sum_{(I_2, n_2) \in E} q^Z(I_1, n_1; I_2, n_2) f(I_1, n_1, I_2, n_2) \leq \sum_{(I_1, n_1) \in E} \pi(I_1, n_1) \sum_{(I_2, n_2) \in E} q^{\tilde{Z}}(I_1, n_1; I_2, n_2) f(I_1, n_1, I_2, n_2). \quad (5)$$
In view of Theorem 2.2 in Daduna et al. (2006) the inequalities are equivalent to isotone differences ordering between \( Y \) and \( \bar{Y}^c \), (4), and \( Z = (Y, X) \) and \( \bar{Z} = (\bar{Y}^c, \bar{X}) \), (5). □

The comparison result for the queue length processes directly follows:

**Corollary 3.7** Under assumptions of Theorem 3.6 we have for the queue length processes \( X \) and \( \bar{X} : X <_{\text{idif}(t)} \bar{X} \), i.e. for all \( t_1 \leq \cdots \leq t_n \) and for all functions \( g \) with isotone differences on \( ((\mathbb{N}^J)^n, (\leq J)^n) : \mathbb{E}\left(g\left(X_{t_1}, \ldots, X_{t_n}\right)\right) \leq \mathbb{E}\left(g\left(\bar{X}_{t_1}, \ldots, \bar{X}_{t_n}\right)\right) \).

**Remark 3.8** Thus, the process \( \bar{X} \) is more internal dependent than \( X \). This means that the internal dependencies in the non-Markovian queue length process \( X \) in a degradable network decrease if we speed up the environmental process which determines the changes in the availability status and governs the breakdowns and repairs of the nodes. However, note that the above result does not imply dependence in lag for the queue length process \( X \), i.e., we do not have \( (X_0, X_t) <_{\text{idif}} (X_0, X_s) \) for \( 0 < s < t \), since the transition kernels of \( X \) and \( \bar{X} \) are not related by \( P_{t}^{\bar{X}} = P_{ct}^{X} \) for \( t \geq 0 \).

**Remark 3.9** The network process \( Z = (Y, X) \) is not monotone with respect to the order \( (\subset \times \leq J) \) on its state space \( E \) as explicated in Section 3.2.3 in Daduna et al. (2006). Thus, we are not able to derive covariance inequalities for the processes \( Z \) and \( \bar{Z} \) like \( \mathbb{E}\left(f\left(Z_{t_1}, \ldots, Z_{t_n}\right)\right) \leq \mathbb{E}\left(f\left(\bar{Z}_{t_1}, \ldots, \bar{Z}_{t_n}\right)\right) \) for all functions \( f \) with isotone differences on \( (E^n, (\subset \times \leq J)^n) \).

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**References**


