Applications of 2-categorical algebra to the theory of operads

Mark Weber

With new, more combinatorially intricate notions of "operad" arising recently in the algebraic approaches to higher dimensional algebra, it has become desirable to clarify how these new notions relate to the usual concept. In this talk it is explained how to use parts of 2-dimensional category theory, in particular a 2-categorical version of topos theory and the formal theory of monads, to give both a unified treatment of the theory of operads and a formalism within which the new and established notions of operad can be made to interact.

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1. Introduction

These are notes corresponding to the talk I gave at the Villars operad conference (March 6-9 2006).

The work that I tried to describe in this talk is aimed at simplifying and extending the operadic notions of [**Bat98**], [**Bat02**] and [**Bat03**] so that weak higher dimensional monoidal categories may be described and illuminated by the resulting theory. See [**BD95**] for some discussion on why one would want to consider such categorical structures. Particularly in [**Bat02**] and [**Bat03**] one finds a key point of contact between this area and the classical theory of loop spaces, and it is the further development of this thread of research which was the subject of Clemens Berger's talk at this conference.

The technology that I use in my work comes from 2-dimensional category theory. Papers that are fundamental background for my approach are [**SW78**], [**Str74**], [**Str80**], [**Str72**] and [**BKP89**]. The first three could loosely be referred to as 2-dimensional topos theory, and [**Weba**] (to appear soon) develops some aspects of this theory a little further with applications to higher category theory in mind. The papers [**Str72**] and [**BKP89**] are concerned with monad theory.

2. Monoidal algebras for a 2-monad and a general operad notion

In this section the general notion of [**Webc**] is described.

We will illustrate our general operad notion by writing down the well-known definition of "operad in a symmetric monoidal category $\mathcal{V}$" in the style of the abstract approach.

Let CAT be the 2-category of categories and let S be the 2-monad on CAT whose strict algebras are symmetric strict monoidal categories. This is the basic data we use for our description. In general one has a 2-category $\mathcal{K}$ with cartesian products and a 2-monad S on $\mathcal{K}$.

So in our example SX is the category whose objects are finite sequences of objects from X, and arrows are permutations labelled by the arrows of X. In particular $S1$ is a skeleton of the groupoid of finite sets. Regard $\mathcal{V} \in$ CAT. The act of taking n-fold tensor products $X_1 \otimes ... \otimes X_n$ in $\mathcal{V}$ for all n can be bundled up as a functor

$$a : S\mathcal{V} \to \mathcal{V}$$

and some natural isomorphisms that exhibit $\mathcal{V}$ as a pseudo S-algebra. On the other hand specifying $\mathcal{V}$'s monoidal structure in terms of a binary tensor product and a unit amounts to maps

$$1 \xrightarrow{i} \mathcal{V} \xleftarrow{m} \mathcal{V} \times \mathcal{V}$$

and some natural isomorphisms that exhibit this data as a pseudo monoid. Note that this pseudo monoid data lifts to the 2-category of pseudo S-algebras, thus $\mathcal{V}$ has the structure of a "monoidal pseudo S-algebra".

Now an operad in $\mathcal{V}$ consists of a functor $p : S1 \to \mathcal{V}$, together with natural transformations ι and σ

$$\begin{array}{ccc} 1 & \xrightarrow{\eta_1} & S1 \\ & {\scriptstyle i}\searrow \;\; \overset{\iota}{\Rightarrow} \;\; \swarrow{\scriptstyle p} & \\ & \mathcal{V} & \end{array} \qquad \begin{array}{ccc} S^2 1 & \xrightarrow{\mu_1} & S1 \\ & {\scriptstyle pS(!)\otimes aS(p)}\searrow \;\; \overset{\sigma}{\Rightarrow} \;\; \swarrow{\scriptstyle p} & \\ & \mathcal{V} & \end{array}$$

such that

$$\begin{array}{ccc} i \otimes p & \xrightarrow{\iota ! \otimes \mathrm{id}} & pS(!)\eta_{S1} \otimes p \\ & {\scriptstyle \lambda}\searrow & \downarrow{\scriptstyle \sigma\eta_{S1}} \\ & & p \end{array} \qquad \begin{array}{ccc} p \otimes aS(i) & \xrightarrow{\mathrm{id}\otimes aS(\iota)} & p \otimes aS(p)S(\eta_1) \\ {\scriptstyle \mathrm{id}\otimes\bar{i}}\downarrow & & \downarrow{\scriptstyle \sigma S(\eta_1)} \\ p \otimes i & \xrightarrow[\rho]{} & p \end{array}$$

commute in $\mathrm{CAT}(S1, \mathcal{V})$ and

$$\begin{array}{ccc} pS(!) \otimes aS(pS(!) \otimes aS(p)) & \xrightarrow{\mathrm{id}\otimes \overline{m}} & pS(!) \otimes (aS(p)S^2(!) \otimes aS(a)S^2(p)) \\ {\scriptstyle \mathrm{id}\otimes aS(\sigma)}\downarrow & & \downarrow{\scriptstyle \beta} \\ pS(!) \otimes aS(p)S(\mu_1) & & (pS(!) \otimes aS(p)S^2(!)) \otimes aS(a)S^2(p) \\ {\scriptstyle \sigma S(\mu_1)}\downarrow & & \downarrow{\scriptstyle \sigma S^2(!) \otimes \alpha S^2(p)} \\ p\mu_1 S(\mu_1) & \xleftarrow[\sigma\mu_{S1}]{} & pS(!)\mu_{S1} \otimes aS(p)\mu_{S1} \end{array}$$

commutes in $\mathrm{CAT}(S^3 1, \mathcal{V})$.

In the evident way such an operad p acts on the hom-category $\mathrm{CAT}(1, \mathcal{V})$.

Examples involving CAT:

- $S = 1_{\mathrm{CAT}}$: monoidal pseudo algebras are monoidal categories; operads are monoids.
- $S =$ strict monoidal category monad; monoidal pseudo algebras are braided monoidal categories; operads are non-symmetric operads.
- $S =$ braided strict monoidal category monad; monoidal pseudo algebras are symmetric monoidal categories; operads now have equivariant *braid* group actions.

A general class of examples is obtained by starting with a cartesian monad T on $\widehat{\mathbb{C}}$ (where $\mathbb{C}$ is a small category) such that the functor part is parametrically representable in the following sense:

$$\widehat{\mathbb{C}} \underset{T}{\overset{\exists}{\rightleftarrows}} \widehat{\mathbb{C}}/T1 \longrightarrow \widehat{\mathbb{C}} \quad (\bot)$$

When T has this form it is a consequence that one has left adjoints:

$$\widehat{\mathbb{C}}/X \underset{T_X}{\overset{L_{T,X}}{\rightleftarrows}} \widehat{\mathbb{C}}/TX \quad (\bot)$$

for all X. Then regard T as a 2-monad on $\mathrm{CAT}(\widehat{\mathbb{C}})$. For instance if T started life as the strict ω-category monad on $\widehat{\mathbb{G}}$, then T-operads in our sense are higher operads in the sense of Batanin.

In general we have a 2-adjunction

$$\mathrm{CAT} \underset{\mathrm{Sp}_{\mathbb{C}}}{\overset{E}{\rightleftarrows}} \mathrm{CAT}(\widehat{\mathbb{C}}) \quad (\bot)$$

given by

$$E(X) = \mathrm{el}(X^{\mathrm{op}})^{\mathrm{op}} \qquad \mathrm{Sp}_{\mathbb{C}}(Z) = [(\mathbb{C}/-)^{\mathrm{op}}, Z].$$

and $\mathrm{Sp}_{\mathbb{C}}(\mathrm{Set})$ has a canonical monoidal pseudo T-algebra structure. The pseudo monoid part of this is inherited from Set's pseudo monoid structure coming from cartesian products. The T-algebra part is described as follows. The algebra structure

$$a : T\mathrm{Sp}_{\mathbb{C}}(\mathrm{Set}) \to \mathrm{Sp}_{\mathbb{C}}(\mathrm{Set})$$

corresponds by adjointness to a functor

$$ET\mathrm{Sp}_{\mathbb{C}}(\mathrm{Set}) \to \mathrm{Set}.$$

An object of $ET\mathrm{Sp}_{\mathbb{C}}(\mathrm{Set})$ is a map

$$x : C \to T\mathrm{Sp}_{\mathbb{C}}(\mathrm{Set})$$

which by parametric representability amounts to a map

$$A \to \mathrm{Sp}_{\mathbb{C}}(\mathrm{Set})$$

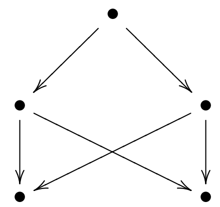
in fact A will be small and discrete as an object of $\mathrm{CAT}(\widehat{\mathbb{C}})$. By adjointness this last map amounts to

$$\mathrm{el}(A)^{\mathrm{op}} \to \mathrm{Set}$$

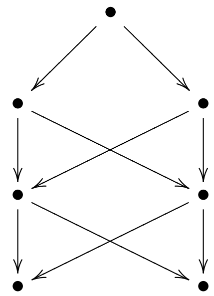
and $a(x)$ is taken to be the limit of this functor.

In the case $\mathbb{C} = \mathbb{G}$ and T is the strict ω-category monad this general construction unpacks as follows. An n-cell of $\mathrm{Sp}_{\mathbb{G}}(\mathrm{Set})$ is an n-span of sets. For instance

2-spans and 3-spans are diagrams of sets and functions with shape



and



respectively. The pseudo monoid part of $\mathrm{Sp}_{\mathbb{G}}(\mathrm{Set})$'s monoidal pseudo T-algebra structure is obtained by componentwise product, and the T-algebra part is obtained using pullbacks and generalises span composition.

In general we have

$$\begin{array}{rcl} \mathrm{CAT}(\widehat{\mathbb{C}})(T1, \mathrm{Sp}_{\mathbb{C}}(\mathrm{Set})) & \cong & \mathrm{CAT}(ET1, \mathrm{Set}) \\ & = & \mathrm{CAT}(\mathrm{el}(T1)^{\mathrm{op}}, \mathrm{Set}) \\ & \simeq & \widehat{\mathbb{C}}/T1 \end{array}$$

and in fact a cartesian operad for $T : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ in the sense of Eugenia's talk, is a T-operad in $\mathrm{Sp}_{\mathbb{C}}(\mathrm{Set})$ in our sense. Note also that

$$\mathrm{CAT}(\widehat{\mathbb{C}})(1, \mathrm{Sp}_{\mathbb{C}}(\mathrm{Set})) \cong \mathrm{CAT}(E1, \mathrm{Set}) = \widehat{\mathbb{C}}$$

However there are other examples we would like to consider. For instance when $\mathcal{V}$ is a symmetric monoidal category, the n-globular category $\Sigma^n(\mathcal{V})$:

$$1 \leftleftarrows \cdots \leftleftarrows 1 \leftleftarrows \mathcal{V}$$

has a monoidal T-algebra structure where T is obtained as above from the strict n-category monad on $\widehat{\mathbb{G}}_{\leq n}$. So you can consider n-operads within symmetric monoidal categories!

One can also blend together the presheaf examples with the CAT examples. Let T be a monad on $\widehat{\mathbb{C}}$ as above with the property that its functor part preserves coproducts (this is true of the strict ω-category monad for example). Let S be a cartesian 2-monad on CAT (all our examples were cartesian). Regard S as a 2-monad on $\mathrm{CAT}(\widehat{\mathbb{C}})$ by applying it pointwise. Under these general conditions we have

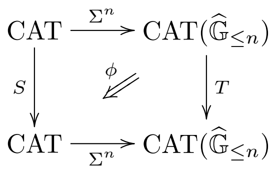
THEOREM 2.1. *There is a distributive law of 2-monads $TS \rightarrow ST$ and the resulting monad structure on ST is cartesian.*

Thus one can consider operads for ST. In particular this gives us braided and symmetric analogues of Batanin operads.

3. Relating different operad notions via monad functors

In this section the functoriality of the above general operad notion is described.

The example of $\Sigma^n\mathcal{V}$ discussed above can be derived formally. Let S be the symmetric monoidal category 2-monad on CAT and T be the strict n-category monad viewed as a 2-monad on $\mathrm{CAT}(\widehat{\mathbb{G}}_{\leq n})$. There is a monad functor:



because

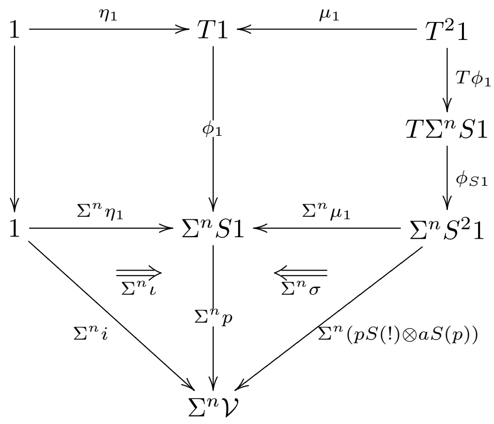
$$\Sigma^n : \mathrm{CAT} \to \mathrm{CAT}(\widehat{\mathbb{G}}_{\leq n})$$

lifts through the forgetful 2-functors to a 2-functor

$$\Sigma^n : S\text{-}\mathrm{Alg}_s \to T\text{-}\mathrm{Alg}_s$$

and in fact Σ^n induces a canonical 2-functor between 2-categories of monoidal pseudo algebras.

An S-operad (p, ι, σ) in $\mathcal{V}$ gives rise to a T-operad $\mathrm{Des}_\phi(p, \iota, \sigma)$ as follows:



If we denote a for the S-algebra structure on $\mathcal{V}$, then induced T-algebra structure on $\Sigma^n\mathcal{V}$ is $a' = \Sigma^n(a)\phi_{\mathcal{V}}$. The right hand composite one-cell in the above diagram is equal to

$$\Sigma^n(p)\phi_1 T(!) \otimes a'T(\Sigma^n(p)\phi_1)$$

so the definition of the operad $\mathrm{Des}_\phi(p, \iota, \sigma)$ parses. The axioms for $\mathrm{Des}_\phi(p, \iota, \sigma)$ follow easily from those of (p, ι, σ) and the monad functor axioms on ϕ. In fact this construction provides a functor

$$\mathrm{Des}_{\phi,\mathcal{V}} : \mathrm{Op}(S, \mathcal{V}) \to \mathrm{Op}(T, \Sigma^n\mathcal{V})$$

This formal construction works for any monad functor in place of (Σ, ϕ) whose 2-functor part preserves finite products. For another example replace T by the monoid monad on CAT, Σ by the identity on CAT, and take ϕ to be induced by the forgetful functor from monoidal categories to symmetric monoidal categories. Then Des_ϕ is the forgetful functor from the category of operads in $\mathcal{V}$ to the category of non-symmetric operads in $\mathcal{V}$ (ie it forgets the symmetric group actions).

For both of the examples discussed here we have:

THEOREM 3.1. *(Batanin) If $\mathcal{V}$ is symmetric monoidal closed and cocomplete; then $\mathrm{Des}_{\phi,\mathcal{V}}$ has a left adjoint denoted EH_ϕ.*

In the case where $\mathcal{V}$ is *cartesian* closed there is a map

$$j : \Sigma^n \mathcal{V} \to \mathrm{Sp}_{\mathbb{G}_{\leq n}} \mathcal{V}$$

which picks out terminal k-spans when $k < n$ and sends $V \in \mathcal{V}$ to the n-span with V in the top position (and necessarily terminal in the lower levels). This map is clearly a morphism of monoidal pseudo T-algebras and so induces

$$j_! : \mathrm{Op}(T, \Sigma^n \mathcal{V}) \to \mathrm{Op}(T, \mathrm{Sp}_{\mathbb{G}_{\leq n}} \mathcal{V})$$

given by composition with j. Now as we shall see in the next section $\Sigma^n \mathcal{V}$ is cocomplete *as a globular category* and this enables us to construct $j^! \dashv j_!$ using left kan extensions in the 2-category $\mathrm{CAT}(\widehat{\mathbb{G}}_{\leq n})$. Thus we have:

$$\mathrm{Op}(S, \mathcal{V}) \underset{\mathrm{Des}_\phi}{\overset{\mathrm{EH}_\phi}{\rightleftarrows}} \mathrm{Op}(T, \Sigma^n \mathcal{V}) \underset{j_!}{\overset{j^!}{\rightleftarrows}} \mathrm{Op}(T, \mathrm{Sp}_{\mathbb{G}_{\leq n}} \mathcal{V})$$

THEOREM 3.2. *(Batanin) Let p be a T-operad in* $\mathrm{Sp}_{\mathbb{G}_{\leq n}} \mathcal{V}$ *where* $\mathcal{V}$ *is a cocomplete cartesian closed category. Then there is an equivalence between the category of* $\mathrm{EH}_\phi j^!(p)$*-algebras and the category of* p*-algebras with one* k*-cell for* $k < n$*.*

For instance this theorem provides an answer to the question: "what is a weak n-category with one k-cell for $k < n$?" by providing an operad in Set whose algebras are such things. Clearly it would be desirable to extend this result to one that unpacks weak n-categories with one k-cell for $k < r \leq n$, but this time the answer will be a higher operad.

As before let S be the symmetric monoidal category 2-monad on CAT, now let T be the strict ω-category monad viewed as a 2-monad on $\mathrm{CAT}(\widehat{\mathbb{G}})$, and fix a cocomplete cartesian closed category $\mathcal{V}$. A pseudo ST-algebra amounts to a symmetric pseudo-monoid in the 2-category Ps-T-Alg and clearly $\mathrm{Sp}_{\mathbb{G}} \mathcal{V}$ has such a structure: the symmetric pseudo monoid part is just componentwise cartesian product of n spans. Moreover a monoidal pseudo ST-algebra is just an ST-algebra by a version of the Eckmann-Hilton argument (applied to pseudo monoids in a 2-category with cartesian products). From these facts it follows that

$$\Sigma^r : \mathrm{CAT}(\widehat{\mathbb{G}}) \to \mathrm{CAT}(\widehat{\mathbb{G}})$$

lifts an endofunctor of strict ST-algebras and thus corresponds to a monad functor

$$\begin{array}{ccc} \mathrm{CAT}(\widehat{\mathbb{G}}) & \xrightarrow{\Sigma^r} & \mathrm{CAT}(\widehat{\mathbb{G}}) \\ {\scriptstyle ST}\downarrow & \overset{\phi}{\swarrow} & \downarrow{\scriptstyle ST} \\ \mathrm{CAT}(\widehat{\mathbb{G}}) & \xrightarrow[\Sigma^r]{} & \mathrm{CAT}(\widehat{\mathbb{G}}) \end{array}$$

Moreover there is a globular functor

$$j : \Sigma^r \mathrm{Sp}_{\mathbb{G}} \mathcal{V} \to \mathrm{Sp}_{\mathbb{G}} \mathcal{V}$$

which sends an $n - r$ span X in $\mathcal{V}$ to the n-span obtained by putting X in the top $(n-r)$ positions and with the bottom r positions terminal. As before this is a map of monoidal pseudo algebras. Thus we have Des_ϕ and $j_!$ as above. Describing left adjoints to $j_!$ and Des_ϕ is what remains to be done.

The quest for such descriptions has led to:

(1) a deeper study of some of the yoneda structures [**SW78**] that arise in our 2-categories culiminating in the notion of a *2-topos* described in [**Weba**].
(2) isolating conditions on the 2-monads we use culminating in the notion of *analytic 2-functor*.
(3) isolating conditions on the monoidal pseudo algebras we consider, culminating in the notion of *distributive* monoidal pseudo algebra. In [**Webb**] (also to appear soon) the composition tensor product of collections in such an algebra is developed at this generality.

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