Manin products and Koszul duality patterns

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Black and white products of quadratic algebras were introduced by Y. Manin in his works on quantum groups and non-commutative geometry. Koszul duality theory for algebras relates these two constructions : the Koszul dual of the product of two algebras is the other product of the Koszul dual algebras. In their seminal paper on Koszul duality for operads, V. Ginzburg and M. Kapranov generalized Manin's constructions to operads. Notice that these generalizations are not straightforward and that these products have not been studied much further. Recently, other Koszul duality theories were proved for colored operad (Van der Laan), dioperads (Gan) and properads-props (BV).

In this talk, we aim to describe first the general framework for Koszul duality theories (for instance, the Koszul dual to consider is a comonoid and not a monoid and its construction only involves universal algebra). Then, we will define the black and white products in any 2-monoidal categories. Hence, we get explicit constructions of these products for operads, regular operads and properads. We will conclude by the proof of a conjecture of M.Aguiar and J.-L. Loday.

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Alpine Operad Workshop

[joint work with M. Aguiar]

Manin products for associative algebras and operads

Associative algebra

Denote by $(k\text{-Mod}, \otimes_k, k)$ the monoidal category of k-modules.

Definition. An *associative algebra* (A, μ, η) is a monoid in $(k\text{-Mod}, \otimes_k, k)$. The product $\mu : A \otimes_k A \xrightarrow{\mu} A$ is associative and $k \xrightarrow{\eta} A$ is the unit of A.

The product $a_1 \dots a_l$ of elements $a_1, \dots, a_l$ of A can be represented by an indexed branch

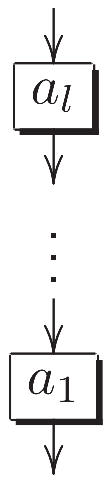


Figure 1: Product of $a_1, \dots, a_l$.

The free algebra on V is given by the tensor module

$$T(V) := \bigoplus_{n\geq 0} V^n.$$

The natural grading corresponds to the number of vertices of the branch.

Manin products for quadratic algebras [M]

Denote by $A(V,R) := T(V)/(R)$ the algebra generated by V and the relations R. When $R \subset V^2$, $A(V,R)$ is called a *quadratic algebra*.

Definition. *Manin's white product* is defined by

$$A(V,R)\circ A(W,S) := A\big(V\otimes W, \tau(R\otimes W^2 + V^2\otimes S)\big)$$

and *Manin's black product* is defined by

$$A(V,R) \bullet A(W,S) := A\big(V\otimes W, \tau(R\otimes S)\big),$$

where τ is the isomorphism

$$V_1\otimes V_2\otimes V_3\otimes V_4 \cong V_1\otimes V_3\otimes V_2\otimes V_4.$$

Definition. When V is finite dimensional, the *Koszul dual algebra* of $A(V,R)$ is the quadratic algebra $A^! := A(V^*, R^\perp)$.

Proposition (M). *One has the relation*

$$(A\circ B)^! = A^! \bullet B^!.$$

Operad

Definition. An $\mathbb{S}$-*module* is a collection $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ of right modules over the symmetric group $\mathbb{S}_n$.

To any $\mathbb{S}$-module M, one associates its *Schur functor* $\widetilde{M}$: k-Mod $\to$ k-Mod defined by

$$\widetilde{M}(V) := \bigoplus_{n\geq 0} M(n) \otimes_{\mathbb{S}_n} V^n.$$

Let M and N be two $\mathbb{S}$-modules.

Proposition. *The composite $\widetilde{M} \circ \widetilde{N}$ of the two Schur functors is the Schur functor of the $\mathbb{S}$-module $M \circ N$ defined by $(M \circ N)(n) :=$*

$$\bigoplus_{k\geq 0} M(k) \otimes_{\mathbb{S}_k} \Big(\bigoplus \mathit{Ind}^{\mathbb{S}_n}_{\mathbb{S}_{i_1}\times\cdots\times\mathbb{S}_{i_k}} \big(N(i_1)\otimes\cdots\otimes N(i_k)\big)\Big),$$

where the second sum is extended to all the k-tuples $(i_1, \ldots, i_k)$ verifying $i_1 + \cdots + i_k = n$.

With this product, the category of $\mathbb{S}$-modules is a monoidal category.

Definition. An *operad* $(\mathcal{P},\ \mu,\ \eta)$ is a monoid in $(\mathbb{S}\text{-Mod}, \circ)$.

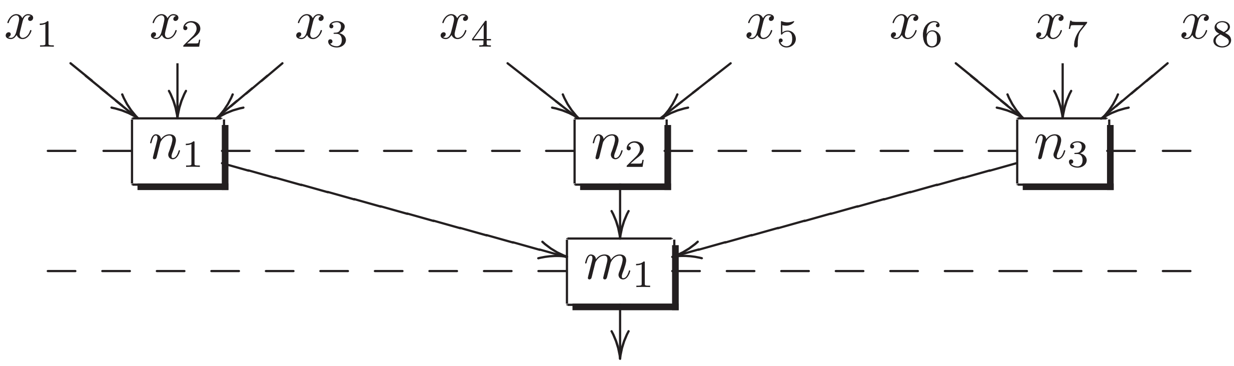


Figure 2: The monoidal product $M \circ N$.

Roughly speaking, the free operad $\mathcal{F}(V)$ is given by trees with vertices indexed by elements of V. The composition product of $\mathcal{F}(V)$ is given by the grafting of trees. This operad is naturally graded by the number of non-trivial vertices.

Definition. The *Hadamard product* of two $\mathbb{S}$-modules M and N is defined by

$$(M \otimes_{\mathsf{H}} N)(n) := M(n) \otimes_k N(n).$$

Manin products for binary quadratic operads [GK]

Denote by $\mathcal{P}(V,R) := \mathcal{F}(V)/(R)$ the operad generated by V and the relations R.
When $V(n)=0$ for $n \neq 2$ the operad $\mathcal{P}(V,R)$ is called *binary*. When $R \subset \mathcal{F}_{(2)}(V)$, $\mathcal{P}(V,R)$ is called a *quadratic operad*.

Definition. *Manin's white product* is defined by

$$A(V,R) \circ A(W,S) :=$$
$$A\big(V \otimes_{\mathsf{H}} W, \Phi^{-1}(R \otimes_{\mathsf{H}} \mathcal{F}(W)(3) + \mathcal{F}(V)(3) \otimes_{\mathsf{H}} S)\big).$$

Manin's black product is defined by

$$\mathcal{P}(V,R) \bullet \mathcal{P}(W,S) := \mathcal{P}\big(V \otimes_{\mathsf{H}} W, \Psi(R \otimes_{\mathsf{H}} S)\big).$$

Definition. When V is finite dimensional, the *Koszul dual operad* of $\mathcal{P}(V,R)$ is the quadratic operad $\mathcal{P}^! := \mathcal{P}(V^*, R^\perp)$.

Proposition (GK). *One has the relation*

$$(\mathcal{P} \circ \mathcal{Q})^! = \mathcal{P}^! \bullet \mathcal{Q}^!.$$

2-monoidal category

Lax monoidal functor

A *lax monoidal functor* [B] is a functor $F\,:\,(\mathcal{A},\boxtimes_{\mathcal{A}},\,I_{\mathcal{A}})\to(\mathcal{B},\boxtimes_{\mathcal{B}},\,I_{\mathcal{B}})$ between two monoidal categories with a natural transformation

$$\phi_{A,\,A'}\;:\;F(A)\boxtimes_{\mathcal{B}}F(A')\to F(A\boxtimes_{\mathcal{A}}A').$$

and a map $\iota\;:\;I_{\mathcal{B}}\to F(I_{\mathcal{A}})$ compatible with the associativity and the units of the monoidal categories :

- <u>Associativity condition</u> :

$$\begin{array}{ccc}
\big(F(A)\boxtimes_{\mathcal{B}}F(A')\big)\boxtimes_{\mathcal{B}}F(A'') & \xrightarrow{\alpha} & F(A)\boxtimes_{\mathcal{B}}\big(F(A')\boxtimes_{\mathcal{B}}F(A'')\big) \\
\downarrow{\scriptstyle \phi\boxtimes id} & & \downarrow{\scriptstyle id\boxtimes\phi} \\
F(A\boxtimes_{\mathcal{A}}A')\boxtimes_{\mathcal{B}}F(A'') & & F(A)\boxtimes_{\mathcal{B}}F(A'\boxtimes_{\mathcal{A}}A'') \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\
F\big((A\boxtimes_{\mathcal{A}}A')\boxtimes_{\mathcal{A}}A''\big) & \xrightarrow{F(\alpha)} & F\big(A\boxtimes_{\mathcal{A}}(A'\boxtimes_{\mathcal{A}}A'')\big)
\end{array}$$

● Unit condition : For every A in $\mathcal{A}$, the following diagram is commutative

$$\begin{array}{ccccc} I_{\mathcal{B}} \boxtimes_{\mathcal{B}} F(A) & \xrightarrow{\iota \boxtimes_{\mathcal{B}} F(A)} & F(I_{\mathcal{A}}) \boxtimes_{\mathcal{B}} F(A) & \xrightarrow{\phi_{I_{\mathcal{A}},A}} & F(I_{\mathcal{A}} \boxtimes_{\mathcal{A}} A) \\ & \searrow^{l^{\mathcal{B}}_{F(A)}} & & & \downarrow F(l^{\mathcal{A}}_{A}) \\ & & & & F(A). \end{array}$$

The same statement holds on the right.

Proposition (B). *Let M be a monoid in $(\mathcal{A}, \boxtimes_{\mathcal{A}})$. The image $F(M)$ of M under a lax monoidal functor F is a monoid in $(\mathcal{B}, \boxtimes_{\mathcal{B}})$.*

(Lax) 2-monoidal category

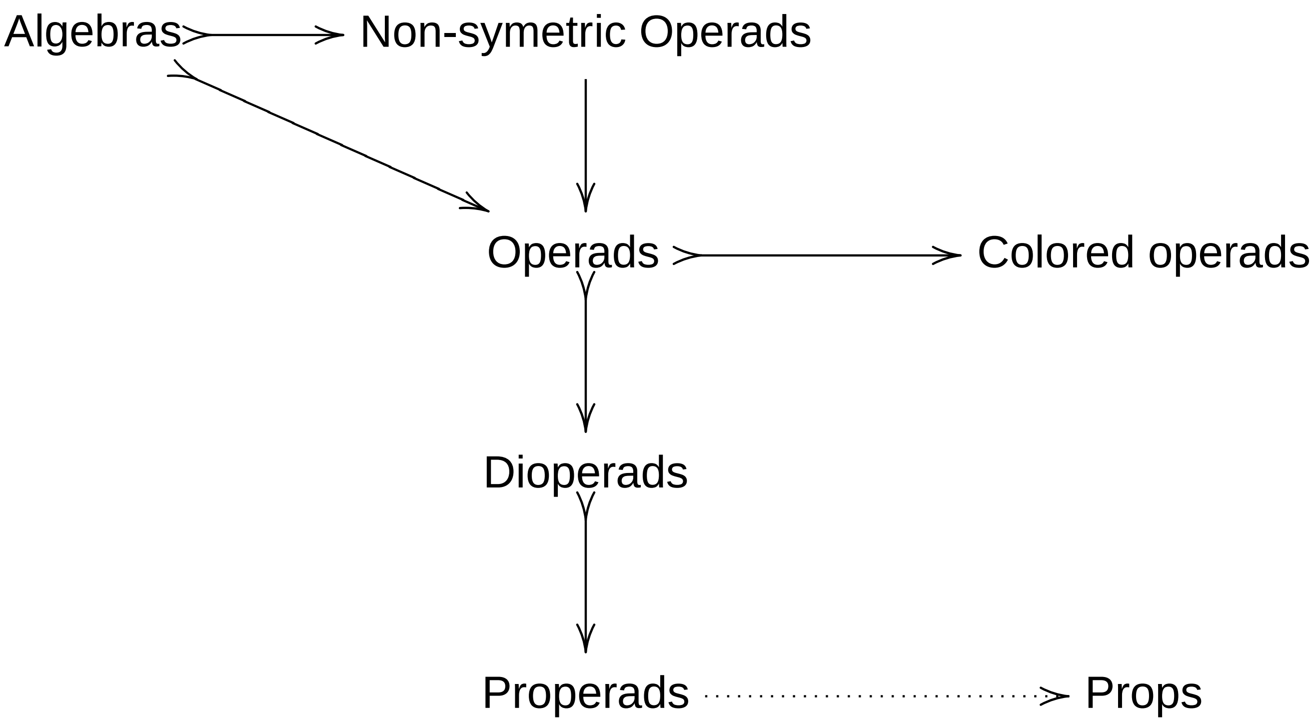
A *2-monoidal category* is a category $(\mathcal{A}, \boxtimes, I, \otimes, K)$, such that $(\mathcal{A}, \boxtimes, I)$ and $(\mathcal{A}, \otimes, K)$ are two monoidal categories and such that the functor $\otimes : \mathcal{A}^2 \to \mathcal{A}$ is a lax monoidal functor. The natural transformation $\phi : (A \otimes B) \boxtimes (A' \otimes B') \to (A \boxtimes A') \otimes (B \boxtimes B')$ is called the *interchange law*.

Proposition. *In a 2-monoidal category, the $\otimes$-product $M \otimes N$ of two monoids M and N for the product $\boxtimes$ is a monoid for $\boxtimes$.*

Examples

The categories $(k\text{-Mod}, \otimes, \otimes)$ and $(\mathbb{S}\text{-Mod}, \circ, I, \otimes, K)$ are 2-monoidal categories, where $I = (k, 0, 0, \ldots)$ and $K = (k, k, k, \ldots)$.
In the first case, the interchange law is given by the twisting isomorphism
$\tau : V_1 \otimes V_2 \otimes V_3 \otimes V_4 \to V_1 \otimes V_3 \otimes V_2 \otimes V_4$.
In the second case, the interchange law map comes from the following map [take coinvariants]

$$
\begin{gathered}
(V \otimes_H V')(l) \otimes_k \big((W \otimes_H W')(i_1) \otimes_k \cdots \otimes_k (W \otimes_H W')(i_l)\big) \\
\downarrow \\
V(l) \otimes_k \big(W(i_1) \otimes_k \cdots \otimes_k W(i_l)\big) \bigotimes_H \\
V'(l) \otimes_k \big(W'(i_1) \otimes_k \cdots \otimes_k W'(i_l)\big) \\
\downarrow \\
\big((V \circ W) \bigotimes_H (V' \circ W')\big)(i_1 + \cdots + i_l).
\end{gathered}
$$



Properad

Definition. An $\mathbb{S}$-*bimodule* is a collection $\{\mathcal{P}(m,n)\}_{m,n\in\mathbb{N}}$ of modules over $\mathbb{S}_m$ on the left and $\mathbb{S}_n$ on the right.

Let M and N be two $\mathbb{S}$-bimodules. One defines a composite product $M \boxtimes N$ which is based on the composite of operations on a <u>connected</u> graph.
The category $(\mathbb{S}\text{-biMod}, \boxtimes)$ is a monoidal category.

Definition. A *properad* $(\mathcal{P},\ \mu,\ \eta)$ is a monoid in $(\mathbb{S}\text{-biMod}, \boxtimes)$.

$$(k\text{-Mod}, \otimes_k) \rightarrowtail (\mathbb{S}\text{-Mod}, \circ) \rightarrowtail (\mathbb{S}\text{-biMod}, \boxtimes)$$

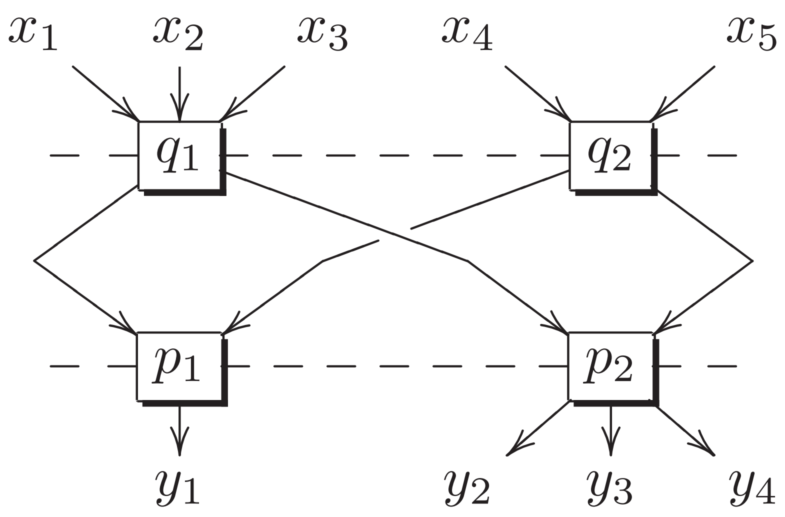
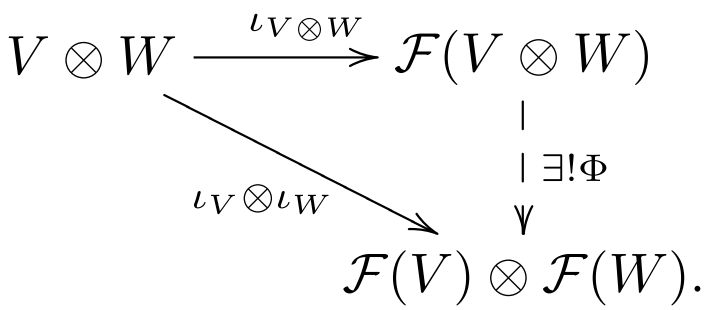


Figure 3: Composition of operations with multiple inputs and multiple outputs.

Associative algebra $\rightarrowtail$ Operad $\rightarrowtail$ Properad

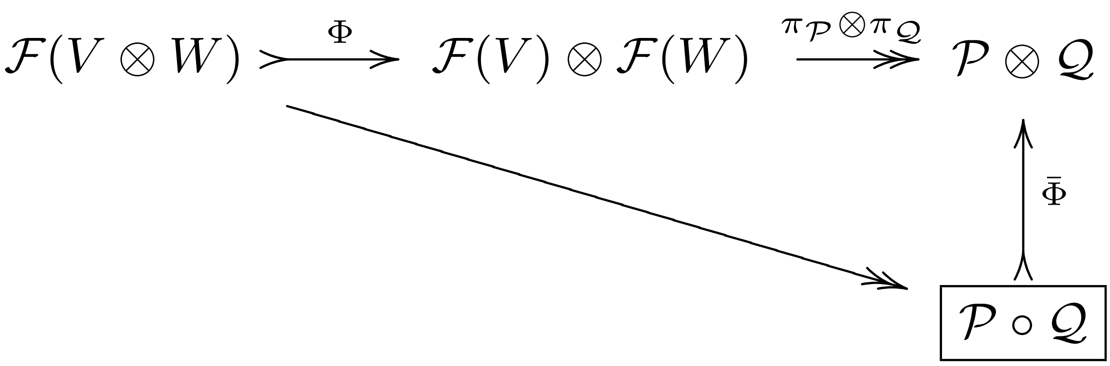
White product

Let $(\mathcal{A}, \boxtimes, \otimes)$ be a 2-monoidal category such that $(\mathcal{A}, \boxtimes)$ has free monoids $\mathcal{F}(V)$ and denote the canonical inclusion by $\iota : V \to \mathcal{F}(V)$.



Proposition. *There exists a natural morphism of monoids* $\Phi : \mathcal{F}(V \otimes W) \to \mathcal{F}(V) \otimes \mathcal{F}(W)$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two properads defined by generators and relations, that is $\mathcal{P} = \mathcal{F}(V)/(R)$ and $\mathcal{Q} = \mathcal{F}(W)/(S)$. Denote by $\pi_{\mathcal{P}} : \mathcal{F}(V) \twoheadrightarrow \mathcal{P}$ and by $\pi_{\mathcal{Q}} : \mathcal{F}(W) \twoheadrightarrow \mathcal{Q}$ the projections.



Since the composite $\pi_{\mathcal{P}} \otimes \pi_{\mathcal{Q}} \circ \Phi$ is a morphism of properads, its kernel is an ideal of $\mathcal{F}(V \otimes W)$.

Definition. The quotient properad

$$\mathcal{P} \circ \mathcal{Q} := \mathcal{F}(V \otimes W)/\text{Ker}(\pi_{\mathcal{P}} \otimes \pi_{\mathcal{Q}} \circ \Phi)$$

is called the *white product of* $\mathcal{P}$ and $\mathcal{Q}$.

Proposition. *The white product of* $\mathcal{P}$ *and* $\mathcal{Q}$ *is equal to*

$$\mathcal{P} \circ \mathcal{Q} := \mathcal{F}(V \otimes W)/\big(\Phi^{-1}(R \otimes \mathcal{F}(W) + \mathcal{F}(V) \otimes S)\big).$$

REMARK. When the composite $\pi_{\mathcal{P}} \otimes \pi_{\mathcal{Q}} \circ \Phi$ is surjective, we have

$$\mathcal{P} \otimes \mathcal{Q} = \mathcal{P} \circ \mathcal{Q}.$$

For example, since τ is an isomorphism, for algebras, we have

$$A(V, R) \otimes A(W, S) = A(V, R) \circ A(W, S).$$

Koszul duality

Ideal notions

Let $(\mathcal{A}, \boxtimes)$ be an abelian monoidal category. Denote by $\mathcal{M}\text{on}_{\mathcal{A}}$ the category of monoids in $\mathcal{A}$. Let A be a monoid in $\mathcal{A}$.

Definition. Let $I \rightarrowtail A \twoheadrightarrow Q$ be an exact sequence in $\mathcal{A}$. The monomorphism $I \rightarrowtail A$ in $\mathcal{M}\text{on}_{\mathcal{A}}$ is an *ideal monomorphism* if $A \twoheadrightarrow Q$ is a morphism in $\mathcal{M}\text{on}_{\mathcal{A}}$.

In this case, we say that I is an *ideal* (subobject) of A and Q is naturally a *quotient* monoid, also denoted A/I.
Let (C, Δ, ϵ) be a comonoid in $\mathcal{A}$, that is C is a monoid in the opposite category $\mathcal{A}^{\text{op}}$. Denote by $\mathcal{C}\text{omon}_{\mathcal{A}}$ the category of comonoids in $\mathcal{A}$.

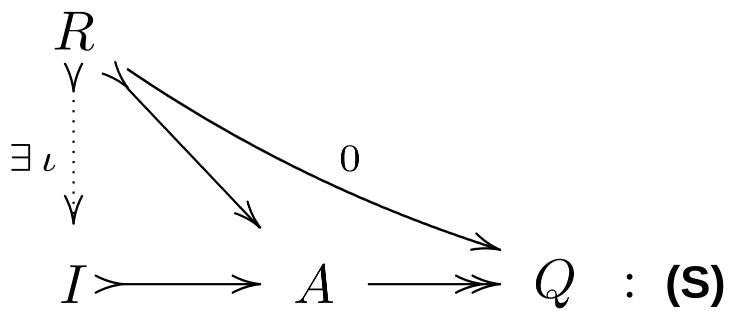
Definition. Let $I \rightarrowtail C \twoheadrightarrow Q$ be an exact sequence in $\mathcal{A}$. The epimorphism $C \twoheadrightarrow Q$ in $\mathcal{C}\text{omon}_{\mathcal{A}}$ is a *coideal epimorphism* if $I \rightarrowtail C$ is a morphism in $\mathcal{C}\text{omon}_{\mathcal{A}}$.

In this case, the subobject $I \rightarrowtail C$ is naturally a *subcomonoid* of C and the quotient Q is called a *coideal quotient*.

REMARK. Different from the notion of coideal that appears in the literature (Hopf algebras, Quantum groups for instance).

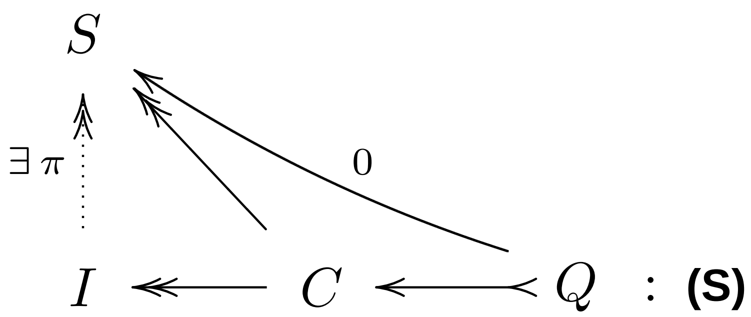
(co)Ideal generated by

When $R \rightarrowtail A$ is a subobject of a monoid A. Consider the category of exact sequences $(\mathbf{S}) \ : I \rightarrowtail A \twoheadrightarrow Q$ such that the composite $R \rightarrowtail A \twoheadrightarrow Q$ is equal to 0.



Definition. When it exists, denote the initial object of this category by $(R) \rightarrowtail A \twoheadrightarrow A/(R)$.
In this case, (R) is called the *ideal generated by* R and $A/(R)$ is the induced *quotient monoid*.

Dually, when $S \twoheadleftarrow C$ be a quotient of a comonoid C. Consider the category of exact sequences $(\mathbf{S}) \ : I \twoheadleftarrow C \leftarrowtail Q$ such that the composite $S \twoheadleftarrow C \leftarrowtail Q$ is equal to 0.



Definition. When it exists, denote the terminal object of this category by $(S) \twoheadleftarrow C \leftarrowtail C(S)$.
In this case, $C(S)$ is called the *subcomonoid of* C *generated by* S and (S) is the induced *coideal quotient*.

Quadratic data and Koszul dual

Let $(V,\, R)$ be a *quadratic data*, that is

$$R \rightarrowtail \mathcal{F}_{(2)}(V) \rightarrowtail \mathcal{F}(V).$$

Definition. The *quadratic properad generated by* V *and* R is the quotient properad of $\mathcal{F}(V)$ by the ideal generated by $R \rightarrowtail \mathcal{F}(V)$. We denote it $\mathcal{P}(V,\, R) = \mathcal{F}(V)/(R)$.

Since the two $\mathbb{S}$-bimodules $\mathcal{F}(V) \cong \mathcal{F}^c(V)$ are isomorphic, we also have

$$\mathcal{F}^c(V) \twoheadrightarrow \mathcal{F}^c_{(2)}(V) \twoheadrightarrow \mathcal{F}^c_{(2)}(V)/R := S.$$

Definition. The *quadratic coproperad generated by* V *and* R is the subcoproperad of $\mathcal{F}^c(V)$ generated by $\mathcal{F}^c(V) \twoheadrightarrow S$. We denote it $\mathcal{C}(V,\, R)$.

The coproperad $\mathcal{C}(V,\, R)$ is called the *Koszul dual coproperad of* $\mathcal{P}(V,\, R)$ and denoted $\mathcal{P}^{¡}$. And the properad $\mathcal{P}(V,\, R)$ is called the *Koszul dual properad of* $\mathcal{C}(V,\, R)$ and denoted $\mathcal{C}^{¡}$

Proposition. *The linear dual of* $\mathcal{C}(V,\, R)$ *is a properad equal to* $\mathcal{P}(V^\vee,\, R^\perp)$.

Bar and cobar constructions

The bar and cobar constructions form a pair of adjoint functors

$\Omega \ : \ \{\text{coaug. dg-coproperads}\} \rightleftharpoons \{\text{aug. dg-properads}\} \ : \ B.$

Theorem. *The unit $\varepsilon \ : \ \Omega(B(\mathcal{P})) \xrightarrow{\backsim} \mathcal{P}$ of the adjunction is a quasi-isomorphism of dg-properads.*

Since $\mathcal{P}(V,\ R)$ is a graded properad, its bar construction is a direct sum of chain complexes

$$B(\mathcal{P}) = \bigoplus_{n \geq 0} B(\mathcal{P})_{(n)}.$$

Each $B(\mathcal{P})_{(n)}$ is finite, more precisely $B_k(\mathcal{P})_{(n)} = 0$ for $k > n$.

Proposition. *We have $H_n(B_\bullet(\mathcal{P})_{(n)}) = \mathcal{C}_{(n)}(V, R)$.*

Therefore $\mathcal{C}(V, R) \rightarrowtail B(\mathcal{P})$ is the top homology of the bar construction of $\mathcal{P}$.
When $\alpha \ : \ \mathcal{C} \to \mathcal{P}$ is a *twisting morphism (cochain)* between a dg-coproperad and a dg-properad, one can define a twisted differential on $\mathcal{C} \boxtimes \mathcal{P}$. This chain is denoted $\mathcal{C} \boxtimes_\alpha \mathcal{P}$.
The map $\alpha \ : \ \mathcal{C}(V, R) \twoheadrightarrow V \rightarrowtail \mathcal{P}(V, R)$ is twisting morphism.

Definition. The chain complex $\mathcal{C} \boxtimes_\alpha \mathcal{P}$ is called the *Koszul complex*.

Theorem (P (70), GK (94), -). *The following assertions are equivalent*

- *The Koszul complex* $\mathcal{C} \boxtimes_\alpha \mathcal{P}$ *is acyclic.*
- *The canonical inclusion* $\mathcal{C} \rightarrowtail B(\mathcal{P})$ *is a quasi-isomorphism.*
- *The canonical projection* $\Omega(\mathcal{C}) \twoheadrightarrow \mathcal{P}$ *is a quasi-isomorphism.*

Definition. In this case, the properad $\mathcal{P}(V, R)$ is called a *Koszul properad*. And $\Omega(\mathcal{C}) \twoheadrightarrow \mathcal{P}$ is the *minimal model* of $\mathcal{P}$.

Black product of comonoids

Colax monoidal functor

A *colax monoidal functor* [B] is a functor $F : (\mathcal{A}, \boxtimes_{\mathcal{A}}, I_{\mathcal{A}}) \to (\mathcal{B}, \boxtimes_{\mathcal{B}}, I_{\mathcal{B}})$ between two monoidal categories with a natural transformation

$$F(A) \boxtimes_{\mathcal{B}} F(A') \leftarrow F(A \boxtimes_{\mathcal{A}} A') \ : \ \psi_{A,\, A'}.$$

and a map $I_{\mathcal{B}} \leftarrow F(I_{\mathcal{A}}) \ : \ \varepsilon$ compatible with the associativity and the units of the monoidal categories :

Proposition (B). *Let C be a comonoid in $(\mathcal{A}, \boxtimes_{\mathcal{A}})$. The image $F(C)$ of C under a colax monoidal functor F is a comonoid in $(\mathcal{B}, \boxtimes_{\mathcal{B}})$.*

Colax 2-monoidal category

A *colax 2-monoidal category* is a category $(\mathcal{A}, \boxtimes, I, \otimes, K)$, such that $(\mathcal{A}, \boxtimes, I)$ and $(\mathcal{A}, \otimes, K)$ are two monoidal categories and such that the functor $\otimes \ : \ \mathcal{A}^2 \to \mathcal{A}$ is a colax monoidal functor.

Proposition. *In a colax 2-monoidal category, the $\otimes$-product $C \otimes D$ of two comonoids C and D for the product $\boxtimes$ is a comonoid for $\boxtimes$.*

REMARK. One can also define the notion of *bilax 2-monoidal category*, that is a lax and colax 2-monoidal category with compatible relations between the two structures.

Black product

Let $(\mathcal{A},\, \boxtimes,\, \otimes)$ be a colax 2-monoidal category such that $(\mathcal{A},\, \boxtimes)$ has free comonoids $\mathcal{F}^c(V)$ and denote the canonical projection by $p_V \ :\ \mathcal{F}^c(V) \twoheadrightarrow V$.

$$\begin{array}{ccc} V \otimes W & \xleftarrow{\ p_{V\otimes W}\ } & \mathcal{F}^c(V \otimes W) \\ & \nwarrow^{p_V \otimes p_W} & \uparrow \exists ! \Psi \\ & & \mathcal{F}^c(V) \otimes \mathcal{F}^c(W). \end{array}$$

Proposition. *There exists a natural morphism of comonoids $\Psi\ :\mathcal{F}^c(V) \otimes \mathcal{F}^c(W) \to \mathcal{F}^c(V \otimes W)$.*

Let $\mathcal{C} = \mathcal{C}(V, R)$ and $\mathcal{D} = \mathcal{C}(W, S)$ be two coproperads defined by generators and relations. Denote by $i_{\mathcal{C}}\ :\ \mathcal{C} \rightarrowtail \mathcal{F}^c(V)$ and by $i_{\mathcal{D}}\ :\ \mathcal{D} \rightarrowtail \mathcal{F}^c(W)$ the natural inclusions.

$$\begin{array}{ccccc} \mathcal{F}^c(V\otimes W) & \xleftarrow{\Psi} & \mathcal{F}^c(V)\otimes\mathcal{F}^c(W) & \xleftarrow{i_{\mathcal{C}}\otimes i_{\mathcal{D}}} & \mathcal{C}\otimes\mathcal{D} \\ & & \nwarrow & & \big\downarrow \bar{\Psi} \\ & & & & \boxed{\mathcal{C}\bullet\mathcal{D}} \end{array}$$

Since the composite $\Psi\circ i_{\mathcal{C}}\otimes i_{\mathcal{D}}$ is a morphism of coproperads, its cokernel is an coideal quotient $\mathcal{F}^c(V\otimes W)$.

Definition. The subcoproperad $\mathcal{C}\bullet\mathcal{D}$ defined by the induced kernel is called the *black product of* $\mathcal{C}$ and $\mathcal{D}$.

Proposition. *We have* $(\mathcal{P}\circ\mathcal{Q})^{i}=\mathcal{P}^{i}\bullet\mathcal{Q}^{i}$

Black and White products for binary quadratic operads

Relation between the white product and the Hadamard product

Let $\mathbb{T}$ denote a binary tree with $n-1$ vertices and the induced *label morphism* by $\mathcal{L}_{\mathbb{T}}^{V} \,:\, V^{\otimes(n-1)} \to \mathcal{F}(V)(n)$.

Proposition. *Let $\mathcal{P}$ be a binary quadratic operad such that for every $n \geq 3$ and every binary tree $\mathbb{T}$ with $n-1$ vertices, the composite*

$$\pi_{\mathcal{P}} \circ \mathcal{L}_{\mathbb{T}}^{V} \,:\, V^{\otimes(n-1)} \to \mathcal{F}(V)(n) \twoheadrightarrow \mathcal{P}(n)$$

is surjective.
For every binary quadratic operad $\mathcal{Q}$, we have

$$\boxed{\mathcal{P} \circ \mathcal{Q} = \mathcal{P} \otimes \mathcal{Q}.}$$

Corollary. *For every binary quadratic operad $\mathcal{Q}$, we have*

- $\mathcal{C}om \circ \mathcal{Q} = \mathcal{C}om \otimes \mathcal{Q} = \mathcal{Q}$.
  The operad $\mathcal{C}om$ is neutral for the white product in the category of binary quadratic operads.
- $\mathcal{P}erm \circ \mathcal{Q} = \mathcal{P}erm \otimes \mathcal{Q}$
- $\mathcal{C}om\mathcal{T}rias \circ \mathcal{Q} = \mathcal{C}om\mathcal{T}rias \otimes \mathcal{Q}$

A $\mathcal{P}erm$-algebra, is a vector space equipped with a binary operation such that

$$(x * y) * z = x * (y * z) = x * (z * y).$$

Theorem. *We have* $\mathcal{P}erm \circ \mathcal{A}s = \mathcal{D}ias$*.*

Some black and white products

Under finite dimensional assumptions, the linear dual of an operad is cooperad. Define the black product of two operads $\mathcal{P}$ and $\mathcal{Q}$ by the formula

$$\mathcal{P} \bullet \mathcal{Q} := (\mathcal{P}^\vee \bullet \mathcal{Q}^\vee)^\vee.$$

We define a similar map

$$\Psi \ : \ \mathcal{F}(V) \otimes \mathcal{F}(W) \to \mathcal{F}(V \otimes W \otimes \text{sgn})$$

Proposition. *The black product of* $\mathcal{P}$ *and* $\mathcal{Q}$ *is equal to*

$$\mathcal{P} \bullet \mathcal{Q} := \mathcal{F}(V \otimes W \otimes \textit{sgn})/\big(\Psi(R \otimes S)\big).$$

It verifies $(\mathcal{P} \circ \mathcal{Q})^! = \mathcal{P}^! \bullet \mathcal{Q}^!$*.*

Corollary. *We have* $\mathcal{P}relie \bullet \mathcal{A}s = \mathcal{D}end$*.*

Theorem. *For any binary quadratic operad* $\mathcal{P}$*, the operad* $\mathcal{P} \bullet \mathcal{P}^!$ *is a Hopf operad.*

Theorem. *We have*

$$\mathcal{P}relie \bullet \mathcal{C}om = \mathcal{Z}inb \quad \text{and} \quad \mathcal{P}erm \circ \mathcal{L}ie = \mathcal{L}eib.$$

We adopt the following convention. Denote by $v_1, \ldots, v_{12}$ the 12 elements of $\mathcal{F}(V)(3)$.

1	$\mu \circ_{I} \mu \leftrightarrow (xy)z$	**5**	$\mu \circ_{III} \mu \leftrightarrow (zx)y$	**9**	$\mu \circ_{II} \mu \leftrightarrow (yz)x$
2	$\mu' \circ_{II} \mu \leftrightarrow x(yz)$	**6**	$\mu' \circ_{I} \mu \leftrightarrow z(xy)$	**10**	$\mu' \circ_{III} \mu \leftrightarrow y(zx)$
3	$\mu' \circ_{II} \mu' \leftrightarrow x(zy)$	**7**	$\mu' \circ_{I} \mu' \leftrightarrow z(yx)$	**11**	$\mu' \circ_{III} \mu' \leftrightarrow y(xz)$
4	$\mu \circ_{III} \mu' \leftrightarrow (xz)y$	**8**	$\mu \circ_{II} \mu' \leftrightarrow (zy)x$	**12**	$\mu \circ_{I} \mu' \leftrightarrow (yx)z$

This labelling corresponds to the labelling of the permutoassociahedron.

Theorem. *An algebra over the operad $\mathcal{P}relie \bullet \mathcal{P}erm$ is a dendriform algebra such that the two operations $\prec$ and $\succ$ verify the two extra relations*

$$x \prec (y \prec z) + x \prec (y \succ z) = x \prec (z \prec y) + x \prec (z \succ y)$$
$$x \succ (y \prec z) = x \succ (z \succ y).$$

Using the notation $x * y := x \prec y + x \succ y$, we sum up the 5 relations of a $\mathcal{P}relie \bullet \mathcal{P}erm$-algebra by

$$\left\{ \begin{array}{l} (x \prec y) \prec z = x \prec (y * z) \\ (x \succ y) \prec z = x \succ (y \prec z) \\ (x * y) \succ z = x \succ (y \succ z) \\ x \prec (y * z) = x \prec (z * y) \\ x \succ (y \prec z) = x \succ (z \succ y). \end{array} \right.$$

All this results show that $\mathcal{P}relie \bullet -$ is an operation that "splits the associativity".

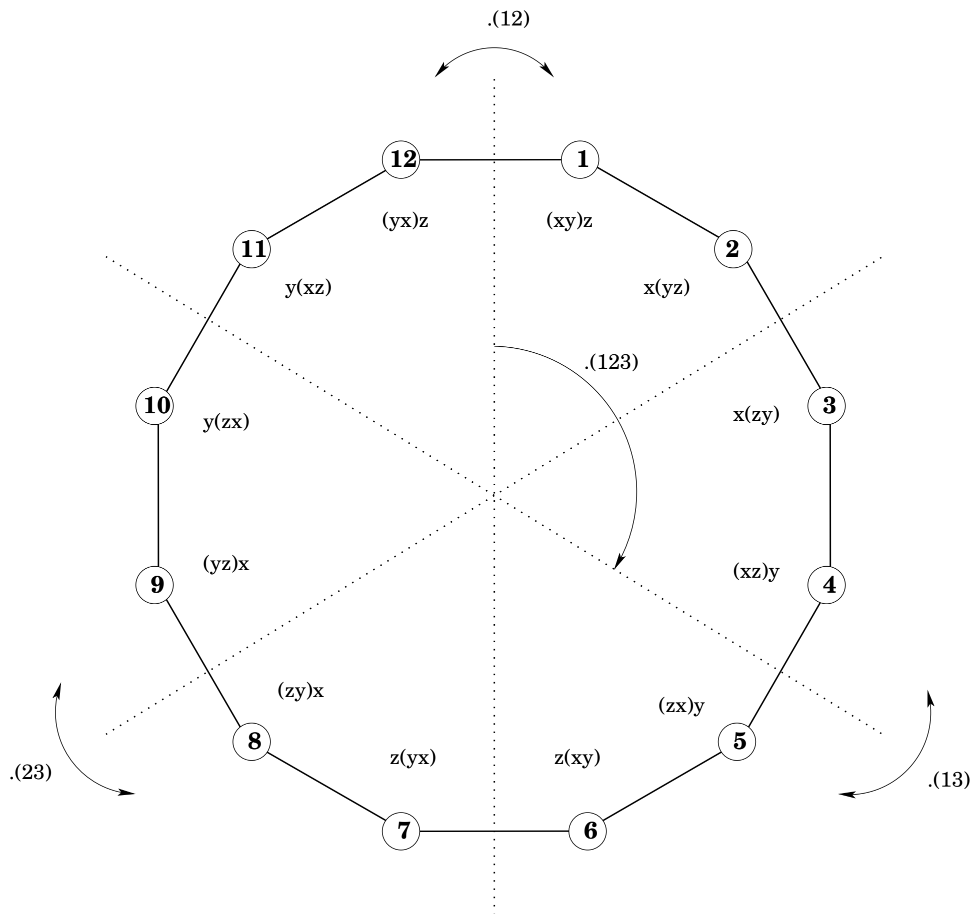


Figure 4: The permutoassociahedron

Relation with unital operations

When an operad $\mathcal{P}$ has a Rota-Baxter operator, that is a unital operator verifying $\rho(a) * \rho(b) = \rho(\rho(a) * b) + \rho(a * \rho(b))$, one can also define two new generating operations : $\rho(a) * b$ and $a * \rho(b)$ that "split the relations".
There are relations between

- Rota-Baxter operators and $\mathcal{P}relie \bullet -$,
- Averaging operators and $\mathcal{P}erm \bullet -$,
- differential operators and $\mathcal{L}eib \bullet -$.

Black and White square products for regular operads

Non-symetric and regular operads

A *non-symmetric operad* is, roughly speaking, an operad without the actions of the symmetric groups. The underlying category is a bilax 2-monoidal category. Denote by $\mathcal{P} \circ \mathcal{Q}$ the white product of two non-symmetric operads.
To a non-symmetric operad $\{\mathcal{P}'_n\}_n$, one can associate an operad $\mathcal{P}(n) := \mathcal{P}'_n \otimes_k k[\mathbb{S}_n] =: \Sigma\mathcal{P}'_n$. Such an operad

is called a *regular operad*.

$$\text{Non-symmetric Operads} \ \underset{\Sigma}{\overset{U}{\leftrightarrows}} \ \text{Regular Operads.}$$

Black and white square products

Definition (White square-product). The *white square-product* of two binary quadratic regular operads $\mathcal{P}$ and $\mathcal{Q}$ is defined by the following formula

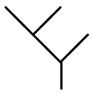
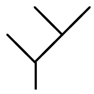
$$\mathcal{P} \,\square\, \mathcal{Q} := \Sigma(U(\mathcal{P}) \circ U(\mathcal{Q})).$$

REMARK. The white circle and the white square products are not equal

$$\mathcal{P} \,\square\, \mathcal{Q} \neq \mathcal{P} \circ \mathcal{Q}.$$

One can also define a *black square product* which verifies

$$\mathcal{P} \blacksquare \mathcal{Q} = (\mathcal{P}^! \,\square\, \mathcal{Q}^!)^!.$$

Proposition. *If you take*  *and*  *for basis of* $\mathcal{F}(V')(3)$*, the black square product* $\mathcal{P} \blacksquare \mathcal{Q}$ *is equal to the one of [EFG].*

The operad $\mathcal{Q}uad$

The operad $\mathcal{Q}uad$ was defined by M. Aguiar and J.-L. Loday as a split of an associative product $\star$ into four

products $\nearrow$, $\searrow$, $\swarrow$ and $\nwarrow$, that is
$\star = \nearrow + \searrow + \swarrow + \nwarrow$.

Proposition (EFG). *The operad $\mathcal{Q}uad$ is equal to*

$$\mathcal{Q}uad = \mathcal{D}end \blacksquare \mathcal{D}end.$$

Proposition. *The Koszul dual operad of $\mathcal{Q}uad$ is the operad*

$$\mathcal{Q}uad^! = \mathcal{D}ias \square \mathcal{D}ias = \mathcal{P}erm \circ \mathcal{D}ias = \mathcal{P}erm^{\otimes 2} \otimes \mathcal{A}s.$$

Theorem. *The operads $\mathcal{Q}uad^!$ and $\mathcal{Q}uad$ are Koszul operads.*

PROOF. The operad $\mathcal{Q}uad^!$ is the algebraic operad associated to a set operad. Consider the related partition posets $\{\Pi_{\mathcal{Q}uad^!}(n)\}_n$. They are totally semi-modular. Therefore, they are Cohen-Macaulay, which is equivalent to the Koszulity of $\mathcal{Q}uad^!$. $\square$

THANK YOU !