Solution devoir 1

P. 135 1.a)

On cherche la solution générale du système $X' = AX$

avec $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

On remarque que si $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix}$

alors $$\begin{pmatrix} x_2' \\ x_1' \\ x_3' \end{pmatrix} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right) \begin{pmatrix} x_2 \\ x_1 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \boxed{x_2(t) = K_2 e^t}$$

$$\begin{pmatrix} x_1' \\ x_3' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$$

les valeurs propres de A sont $\lambda = \pm 1$ et les vecteurs propres associés sont $v_1 = (1,1)$ et $v_{-1} = (-1, 1)$

$$\Rightarrow \begin{pmatrix} x_1(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} K_1 \\ K_3 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} K_1 \\ K_3 \end{pmatrix}$$

②

d'où
$$\boxed{\begin{aligned} x_1(t) &= K_1 \cosh t + K_3 \sinh t \\ x_3(t) &= K_1 \sinh t + K_3 \cosh t \end{aligned}}$$

avec $K_1, K_2, K_3 \in \mathbb{R}$ et $\cosh t = \frac{e^t + e^{-t}}{2}$

$\sinh t = \frac{e^t - e^{-t}}{2}$

P. 136 #6 :

a) On a
$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} a & 0 & b \\ 0 & b & 0 \\ -b & 0 & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

et on cherche la solution générale.

Comme au numéro précédent on remarque que
$$\begin{pmatrix} x_2' \\ x_1' \\ x_3' \end{pmatrix} = \left(\begin{array}{c|cc} b & 0 & 0 \\ \hline 0 & a & b \\ 0 & -b & a \end{array}\right) \begin{pmatrix} x_2 \\ x_1 \\ x_3 \end{pmatrix}$$

d'où $\boxed{x_2(t) = K_2 e^{bt}}$

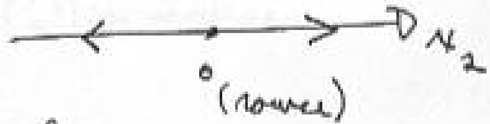
et
$$\begin{pmatrix} x_1(t) \\ x_3(t) \end{pmatrix} = e^{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} t} \begin{pmatrix} K_1 \\ K_3 \end{pmatrix}$$

$$= e^{at} \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix} \begin{pmatrix} K_1 \\ K_3 \end{pmatrix}$$

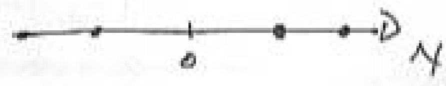
$$\boxed{\begin{aligned} x_1(t) &= e^{at}\left(K_1 \cos bt + K_3 \sin bt\right) \\ x_3(t) &= e^{at}\left(-K_1 \sin bt + K_3 \cos bt\right) \end{aligned}}$$

avec $K_1, K_2, K_3 \in \mathbb{R}$

b) <u>le cas de x_2</u> : si $b>0$ alors le diag. de phase est



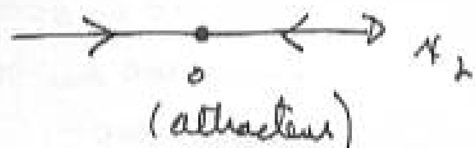
si $b=0$ alors



On calcule
• $\left\|\begin{pmatrix} x_1 \\ x_3 \end{pmatrix}\right\| = e^{at} \left\|\begin{matrix} K_1 \\ K_2 \end{matrix}\right\|$ vu en classe

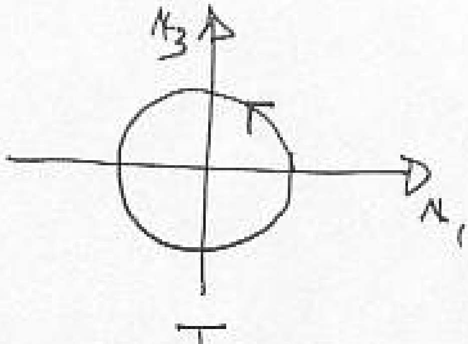
• le signe de b détermine le sens de rotation

si $b<0$ alors



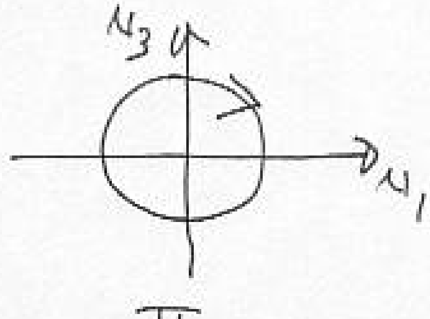
<u>le cas de x_1 et x_3</u> : si <u>$a=0$</u> alors

<u>$b>0$</u>



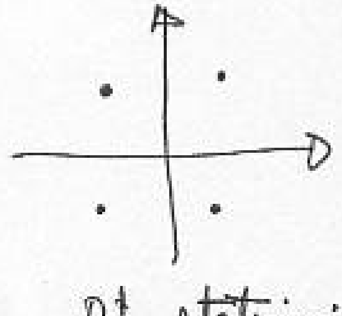
I

<u>$b<0$</u>



II

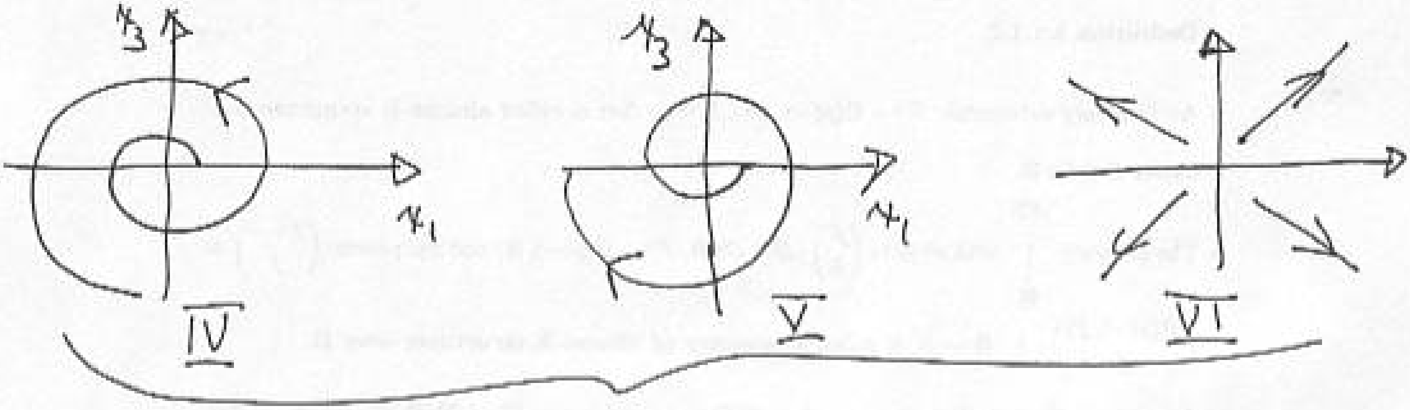
<u>$b=0$</u>



pt. stationnaire
III

si $a>0$ alors

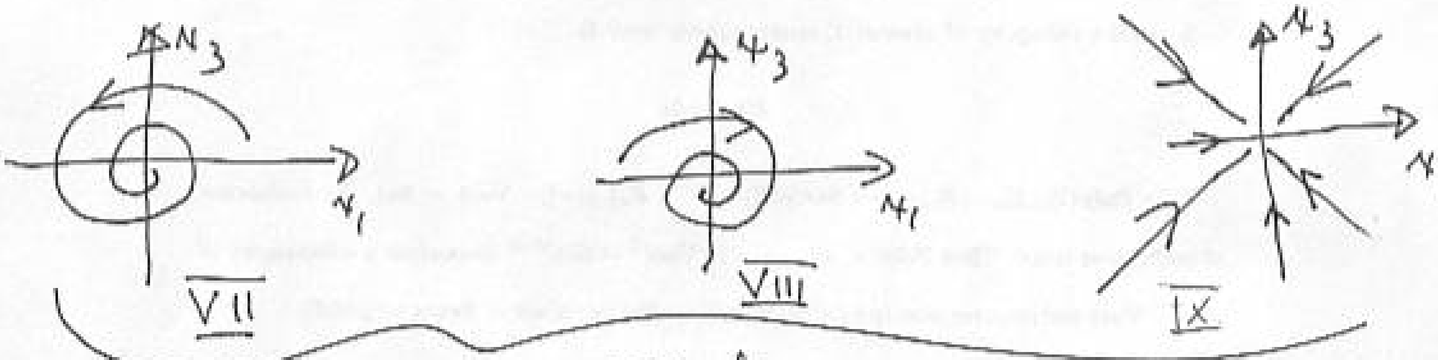
$b>0$ $\qquad$ $b<0$ $\qquad$ $b=0$



source

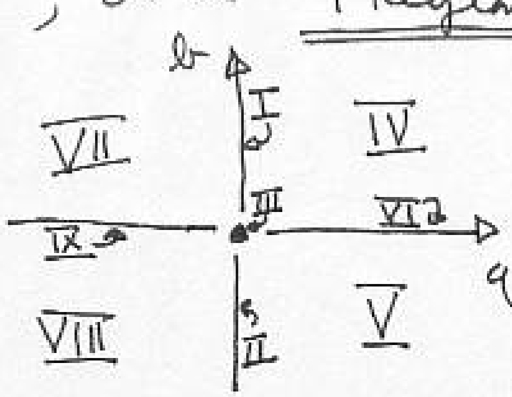
si $a<0$ alors

$b>0$ $\qquad$ $b<0$ $\qquad$ $b=0$



pt attracteur

Dans le plan ab, on a 9 régions différentes



P. 136 #7:

a)

$$x_1' = x_3$$

$$x_3' = x_1'' = -(k_1+k_2)\, x_1 + k_2 x_2$$

$$x_2' = x_4$$

$$x_4' = x_2'' = k_2 x_1 - (k_1+k_2)\, x_2$$

$$\Leftrightarrow \quad x_3' = -A x_1 + k_2 x_2$$

$$x_4' = k_2 x_1 - A x_2$$

avec $A = k_1 + k_2$

$$x_1' = x_3$$

$$x_2' = x_4$$

$$\Leftrightarrow \quad \boxed{\begin{pmatrix} x_3' \\ x_4' \\ x_1' \\ x_2' \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & 0 & -A & k_2 \\ 0 & 0 & k_2 & -A \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \begin{pmatrix} x_3 \\ x_4 \\ x_1 \\ x_2 \end{pmatrix}}$$

b) i) On calcule le polynôme caractéristique

$$P_A(\lambda) = \begin{vmatrix} -\lambda & 0 & -A & k_2 \\ 0 & -\lambda & k_2 & -A \\ 1 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & k_2 & -A \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} + \begin{vmatrix} 0 & -A & k_2 \\ -\lambda & k_2 & -A \\ 1 & 0 & -\lambda \end{vmatrix}$$

$$= -\lambda \left\{ -\lambda^3 + -A\lambda \right\} + A^2 - k_2^2 + \lambda^2 A$$

⑥

$$P_A(\lambda) = \lambda^4 + 2A\lambda^2 + A^2 - k_2^2$$

$$P_A(\lambda) = 0 \iff \lambda^2 = \frac{-2A \pm \sqrt{4A^2 - 4(A^2 - k_2^2)}}{2}$$

$$= \frac{-2A \pm 2k_2}{2} \qquad (k_1, k_2 \geq 0)$$

$$= \begin{cases} -k_1 \\ -(k_1 + 2k_2) \end{cases}$$

$$\iff \lambda = \pm i\sqrt{k_1} = \boxed{\pm i\omega_1}$$

$$\& \quad \pm i\sqrt{k_1 + 2k_2} = \boxed{\pm i\omega_2}$$

ii) On trouve les vecteurs propres

$\underline{\lambda = i\omega_1}$: $\ker \begin{pmatrix} -i\omega_1 & 0 & -A & k_2 \\ 0 & -i\omega_1 & k_2 & -A \\ 1 & 0 & -i\omega_1 & 0 \\ 0 & 1 & 0 & -i\omega_1 \end{pmatrix}$

$\underline{\text{Gauss-Jordan}}$: $\begin{pmatrix} 1 & 0 & -i\omega_1 & 0 \\ 0 & 1 & 0 & -i\omega_1 \\ -i\omega_1 & 0 & -A & k_2 \\ 0 & -i\omega_1 & k_2 & -A \end{pmatrix}$

$$\hookrightarrow \begin{pmatrix} 1 & 0 & -i\omega_1 & 0 \\ 0 & 1 & 0 & -i\omega_1 \\ 0 & 0 & -k_2 & k_2 \\ 0 & 0 & k_2 & -k_2 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 0 & 0 & -i\omega_1 \\ 0 & 1 & 0 & -i\omega_1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \quad \begin{aligned} x_3 &= i\omega_1 x_2 \\ x_4 &= i\omega_1 x_2 \\ x_1 &= x_2 \\ x_2 &= t \end{aligned} \quad \Rightarrow \begin{pmatrix} x_3 \\ x_4 \\ x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} i\omega_1 \\ i\omega_1 \\ 1 \\ 1 \end{pmatrix}$$

$$\ker (A - i\omega_1 \mathbb{1}) = \langle \underbrace{(i\omega_1, i\omega_1, 1, 1)}_{=: v_1} \rangle$$

[illegible] dann,

$$\Rightarrow \ker (A + i\omega_1 \mathbb{1}) = \langle \underbrace{(-i\omega_1, -i\omega_1, 1, 1)}_{\overline{v_1}} \rangle$$

$\underline{\lambda = i\omega_2}$: $$\ker \begin{pmatrix} -i\omega_2 & 0 & -A & h_2 \\ 0 & -i\omega_2 & h_2 & -A \\ 1 & 0 & -i\omega_2 & 0 \\ 0 & 1 & 0 & -i\omega_2 \end{pmatrix}$$

$\underline{\text{Gauss-Jordan}}$: $$\begin{pmatrix} 1 & 0 & -i\omega_2 & 0 \\ 0 & 1 & 0 & -i\omega_2 \\ -i\omega_2 & 0 & -A & h_2 \\ 0 & -i\omega_2 & h_2 & -A \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -i\omega_2 & 0 \\ 0 & 1 & 0 & -i\omega_2 \\ 0 & 0 & h_2 & h_2 \\ 0 & 0 & h_2 & h_2 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & i\omega_2 \\ 0 & 1 & 0 & -i\omega_2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \quad \begin{aligned} x_3 &= -i\omega_2 x_2 \\ x_4 &= i\omega_2 x_2 \\ x_1 &= -x_2 \\ x_2 &= t \end{aligned}$$

$$\begin{pmatrix} x_3 \\ x_4 \\ x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} -i\omega_2 \\ i\omega_2 \\ -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \ker(A - i\omega_2 \mathbb{1}) = \langle \underbrace{(-i\omega_2, i\omega_2, -1, 1)}_{= v_2} \rangle$$

$$\Downarrow$$

$$\ker(A + i\omega_2 \mathbb{1}) = \langle \underbrace{(i\omega_2, -i\omega_2, -1, 1)}_{= \overline{v_2}} \rangle$$

c) la forme canonique de Jordan est :

$$P^{-1}AP = \left(\begin{array}{cc|cc} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{array}\right)$$

Soit $w_1 = \frac{1}{2}(v_1 + \overline{v_1}) = (0, 0, 1, 1)$

$w_2 = \frac{1}{2i}(v_1 - \overline{v_1}) = (\omega_1, \omega_1, 0, 0)$

$w_3 = \frac{1}{2}(v_2 + \overline{v_2}) = (0, 0, -1, 1)$

$w_4 = \frac{1}{2i}(v_2 - \overline{v_2}) = (-\omega_2, \omega_2, 0, 0)$

$$\Rightarrow P = \begin{pmatrix} 0 & \omega_1 & 0 & -\omega_2 \\ 0 & \omega_1 & 0 & \omega_2 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \rightsquigarrow P^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1/\omega_1 & 1/\omega_1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1/\omega_2 & 1/\omega_2 & 0 & 0 \end{pmatrix}$$

∴ La solution générale est

$$\begin{pmatrix} u_3 \\ u_4 \\ u_1 \\ u_2 \end{pmatrix} = P\, e^{\left(\begin{array}{cc|cc} 0 & \omega_1 & & \\ -\omega_1 & 0 & \multicolumn{2}{c}{O} \\ \hline \multicolumn{2}{c|}{O} & 0 & \omega_2 \\ & & -\omega_2 & 0 \end{array}\right) t}\, P^{-1} \begin{pmatrix} K_3 \\ K_4 \\ K_1 \\ K_2 \end{pmatrix}$$

$$= P \begin{pmatrix} \cos\omega_1 t & \sin\omega_1 t & 0 & 0 \\ -\sin\omega_1 t & \cos\omega_1 t & 0 & 0 \\ 0 & 0 & \cos\omega_2 t & \sin\omega_2 t \\ 0 & 0 & -\sin\omega_2 t & \cos\omega_2 t \end{pmatrix} P^{-1} \begin{pmatrix} K_3 \\ K_4 \\ K_1 \\ K_2 \end{pmatrix}$$

$$= \frac{1}{2} \left(\begin{array}{c|c|c|c} \cos\omega_1 t + \cos\omega_2 t & \cos\omega_1 t - \cos\omega_2 t & -(\omega_1 \sin\omega_1 t + \omega_2 \sin\omega_2 t) & -\omega_1 \sin\omega_1 t + \omega_2 \sin\omega_2 t \\ \cos\omega_1 t - \cos\omega_2 t & \cos\omega_1 t + \cos\omega_2 t & -\omega_1 \sin\omega_1 t + \omega_2 \sin\omega_2 t & -(\omega_1 \sin\omega_1 t + \omega_2 \sin\omega_2 t) \\ \frac{\sin\omega_1 t}{\omega_1} + \frac{\sin\omega_2 t}{\omega_2} & \frac{\sin\omega_1 t}{\omega_1} - \frac{\sin\omega_2 t}{\omega_2} & \cos\omega_1 t + \cos\omega_2 t & \cos\omega_1 t - \cos\omega_2 t \\ \frac{\sin\omega_1 t}{\omega_1} - \frac{\sin\omega_2 t}{\omega_2} & \frac{\sin\omega_1 t}{\omega_1} + \frac{\sin\omega_2 t}{\omega_2} & \cos\omega_1 t - \cos\omega_2 t & \cos\omega_1 t + \cos\omega_2 t \end{array}\right) \cdot \begin{pmatrix} K_3 \\ K_4 \\ K_1 \\ K_2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} A \cos\omega_1 t + B \cos\omega_2 t - C \omega_1 \sin\omega_1 t - D \omega_2 \sin\omega_2 t \\ A \cos\omega_1 t - B \cos\omega_2 t - C \omega_1 \sin\omega_1 t + D \omega_2 \sin\omega_2 t \\ \frac{A}{\omega_1} \sin\omega_1 t + \frac{B}{\omega_2} \sin\omega_2 t + C \cos\omega_1 t + D \cos\omega_2 t \\ \frac{A}{\omega_1} \sin\omega_1 t - \frac{B}{\omega_2} \sin\omega_2 t + C \cos\omega_1 t - D \cos\omega_2 t \end{pmatrix}$$

où $A = K_3 + K_4$ $\qquad C = K_1 + K_2$

$B = K_3 - K_4$ $\qquad D = K_1 - K_2$

et donc

$$\left\{\begin{aligned} N_1(t) &= \frac{A}{\omega_1}\sin\omega_1 t + \frac{B}{\omega_2}\sin\omega_2 t + C\cos\omega_1 t + D\cos\omega_2 t \\ N_2(t) &= \frac{A}{\omega_1}\sin\omega_1 t - \frac{B}{\omega_2}\sin\omega_2 t + C\cos\omega_1 t - D\cos\omega_2 t \end{aligned}\right.$$

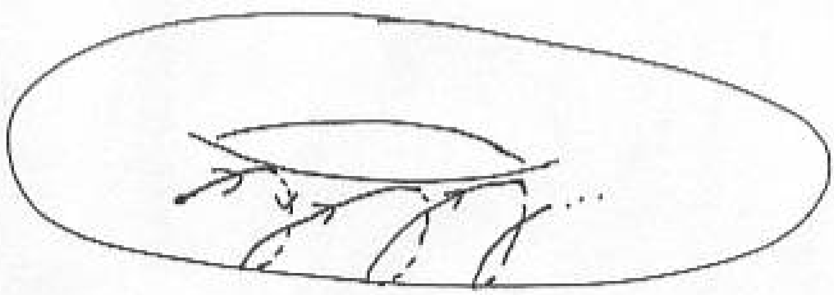
soln générale

$$\left\{\begin{aligned} N_3(t) &= A\cos\omega_1 t + B\cos\omega_2 t - C\omega_1\sin\omega_1 t - D\omega_2\sin\omega_2 t \\ N_4(t) &= A\cos\omega_1 t - B\cos\omega_2 t - C\omega_1\sin\omega_1 t + D\omega_2\sin\omega_2 t \end{aligned}\right.$$

en particulier

$\rightarrow$ sont accessoires

d) J'expliquerai en classe le diagramme suivant



#12 b) $$e^{\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}} = \boxed{e^2 \begin{pmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{pmatrix}}$$

c) $$\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}}_{N}$$

$$DN = ND \Rightarrow e^{\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}} = e^D e^N$$

$$= e^2 e^N$$

mais $N^2 = 0$ $$\Rightarrow e^N = \mathbb{1} + N = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow e^{\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}} = \boxed{e^2 \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}$$

#13) Soit $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ et $B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$

alors $$AB = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = BA$$

$$e^{A+B} = e^{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos 1 & \sin 1 \\ -\sin 1 & \cos 1 \end{pmatrix} \neq$$

mais $$e^A e^B = (\mathbb{1} + A)(\mathbb{1} + B)$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$