Relative homotopy cyclic homology

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This is joint work with David Chataur. In previous work with J.L. Rodriguez we constructed a functorial plus-construction in the category of differential graded algebras over a cofibrant operad. The plus-construction applied to $gl(A)$ in the category of homotopy Lie algebras is of special interest and we defined homotopy cyclic homology groups as the homotopy groups of $gl(A)^+$. Over the rationals they coincide with usual cyclic homology groups by work of M. Livernet. We computed these groups in low degrees over any field.

In this talk I will recall the main results in the absolute case and then focus on a relative version.Given an ideal I in A, the homotopy fiber of the map $gl(A)^+ \to gl(A/I)^+$ yields a long exact sequence in homotopy cyclic homology. We identify the relative groups in low degree, just like relative algebraic K-theory groups were identified by Loday in the late 70's.

§1. Introduction

Just like Quillen introduced the plus construction $X \longrightarrow X^+$ for spaces to define higher algebraic K-theory groups,

$$K_n(R) = \pi_n BGL(R)^+,$$

there is an "operadic way" to perform a plus construction on (differential graded) Lie-algebras (Pirashvili, Livernet) and obtain an "additive K-theory"

$$\pi_n \mathfrak{gl}(R)^+$$

Theorem (Livernet)

When R is a $\mathbb{Q}$-algebra

$$\pi_n\, \mathfrak{gl}R^+ \cong HC_n(R)$$

Aim: Study a relative version, i.e. given an ideal $I \subseteq R$, look at

$$\mathfrak{k}(R,I) = \mathrm{Fib}(\mathfrak{gl}R^+ \longrightarrow \mathfrak{gl}(R/I)^+)$$

and try to compute

$$\pi_n \mathfrak{k}(R,I).$$

§2. Plus construction

(previous work with D. Chataur and J.L. Rodríguez)

We work with $\mathcal{O}$ a (d.g.) cofibrant operad in the sense of Berger–Moerdijk or Spitzweck and with (d.g.) algebras over $\mathcal{O}$. For such objects we have André–Quillen homology, which we denote by $H_*^Q(-)$.

Theorem (CRS)

There exists a universal H_*^Q-acyclic algebra $\mathcal{U}$, so that the corresponding $\mathcal{U}$-nullification

$$X \longrightarrow P_{\mathcal{U}} X$$

yields a functorial plus construction.

Analogy: Berrick–Casacuberta universal acyclic space.

In fact $P_{\mathcal{U}} X \simeq \mathrm{Cof}(\coprod \mathcal{U} \to X)$

<u>Example</u> $\mathcal{O}$ is the Lie-operad, so $\mathcal{O}$-algebras are Lie-algebras.

We must take a cofibrant replacement $\mathcal{L}_\infty$ and write from now on

$$X^+ = P_{\mathcal{U}} X$$

and define

$$HC_n^\infty(R) = \pi_n \, gl(R)^+$$

This is homotopy cyclic homology.

<u>Theorem</u> (CRS)

(i) $\pi_0 \, gl R^+ \cong R/[R,R]$

(ii) $\pi_1 \, gl R^+ \cong Z(st(R))$

(iii) $\pi_2 \, gl R^+ \cong H_2^{\mathbb{Q}}(st(R))$

In other words, for $n \leq 2$

$$HC_n^\infty(R) \cong HC_n(R)$$

see Kassel-Loday for the computations of cyclic homology.

Analogy

$$K_1(R) \cong \pi_1 BGL(R)^+ \cong R/[R,R]$$

$$K_2(R) \cong \pi_2 BSL(R)^+ \cong Z(St(R))$$

$$K_3(R) \cong \pi_3 BSt(R)^+ \cong H_3(St(R);\mathbb{Z})$$

Example of Computation

$$\mathfrak{sl}(R) = \mathrm{Ker}(\mathrm{trace}: \mathfrak{gl}(R) \longrightarrow R/[R,R])$$

$\mathfrak{sl}(R)^+$ is the 0-connected cover of $\mathfrak{gl}(R)^+$

$\mathfrak{st}(R)^+$ is the 1-connected cover.

The central extension

$$Z(\mathfrak{st}(R)) \longrightarrow \mathfrak{st}(R) \longrightarrow \mathfrak{sl}(R)$$

is classified by an element in

$$H^1_Q(\mathfrak{sl}(R); Z(\mathfrak{st}(R))) \cong [\mathfrak{sl}(R), K(Z(\mathfrak{st}(R)),1)]$$

So there is a fibration

$$\begin{tikzcd}
\mathfrak{st}(R) \arrow[r] \arrow[d] & \mathfrak{sl}(R) \arrow[r] \arrow[d] & K(Z(\mathfrak{st}(R)),1) \arrow[d, equal] \\
\mathfrak{st}(R)^+ \arrow[r] & \mathfrak{sl}(R)^+ \arrow[r] & K(Z(\mathfrak{st}(R)),1)
\end{tikzcd}$$

$\mathcal{U}$-local

§ 3. A relative central extension

Let us consider

$$L = \mathrm{Coker}\left(H_2^Q(st(R)) \to H_2^Q(st(R/_I))\right)$$

$$\cong \mathrm{Coker}\left(\pi_2(st(R)^+) \to \pi_2(st(R/_I)^+)\right)$$

$$\cong \mathrm{Coker}\left(HC_2^\infty(R) \to HC_2^\infty(R/_I)\right)$$

and form the Eilenberg-MacLane L_∞-algebra $K(L,2)$. Choose a map

$$k : st(R/_I) \to K(L,2)$$

representing in

$$H^2_Q(st(R/_I);L) \leftarrow H^2_Q(st(R/_I), st(R); L)$$

$$\cong$$

$$H^2_Q(\mathrm{Cof}(st(R) \to st(R/_I)); L)$$

$$\cong$$

$$\mathrm{Hom}(L, L)$$

the image of the identity. Like Loday did in 1978 for algebraic K-theory, construct (homotopy) pull-back squares

$$
\begin{tikzcd}
 & Y \arrow[r] \arrow[d] & \mathcal{Y} \arrow[r] \arrow[d, two heads] & PK(L,2)\simeq * \arrow[d, two heads] \\
st(R) \arrow[r] \arrow[ur, dashed] \arrow[rrr, bend right, "*"'] & st(R/I) \arrow[r] & st(R/I)^+ \arrow[r, "k"] & K(L,2)
\end{tikzcd}
$$

<u>Proposition</u>

$\mathrm{Fib}(st(R) \to Y) \simeq st(R, I)$,

the relative Steinberg Lie algebra.

<u>proof</u>: The relative central extension

$$0 \to L \to \pi_0 F \to st(R) \to st(R/I) \to 0$$

is represented by id_L.

The absolute computations have identified L with

$\mathrm{Coker}(HC_2(R) \to HC_2(R/I))$

so we can identify $F \simeq \pi_0 F$ with Kassel and Loday's universal relative extension. □

The relative Steinberg Lie algebra

Kassel and Loday define

$$D = R \times_{R/I} R$$

and consider the kernels $\mathrm{Ker}(p_i)_*$ of the projections

$$(p_i)_* : st(D) \longrightarrow st(R),$$

and $c(I) = [\mathrm{Ker}(p_1)_*, \mathrm{Ker}(p_2)_*] \subseteq st(D)$

$$gl(R,I) = \mathrm{Ker}((p_1)_* : gl(D) \longrightarrow gl(R))$$
$$= gl(I)$$
$$sl(R,I) = \mathrm{Ker}((p_1)_* : sl(D) \longrightarrow sl(R))$$
$$st(R,I) = \mathrm{Ker}((p_1)_* : st(D)/c(I) \longrightarrow st(R))$$

<u>Theorem</u> (Kassel - Loday)

$$\mathrm{CoKer}(st(R,I) \longrightarrow gl(R,I)) \cong I/[R,I]$$

$$\mathrm{Ker}(st(R,I) \longrightarrow gl(R,I)) \cong HC_1(R,I)$$

§ 4 Playing with fibrations

Lemma

$$\mathrm{Fib}(st(R) \longrightarrow gl(R)) \simeq \mathrm{Fib}(st(R)^{+} \longrightarrow gl(R)^{+}) \qquad \square$$

Consider now the map

$$Y \longrightarrow st(R/I) \longrightarrow gl(R/I)$$

and let

$$W = \mathrm{Fib}(Y \longrightarrow gl(R/I))$$

Lemma

$\mathcal{Y} \simeq Y^{+}$ and

$W \simeq \mathrm{Fib}(\mathcal{Y} \longrightarrow gl(R/I)^{+})$

proof: The first claim comes from fiberwise plus-construction

$$\begin{array}{ccccc} Y & \longrightarrow & st(R/I) & \longrightarrow & K(L,2) \\ \downarrow & & \downarrow & & \| \\ \mathcal{Y} & \longrightarrow & st(R/I)^{+} & \longrightarrow & K(L,2) \end{array}$$

The second comes from looking at the homotopy pull-backs

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & st(R/I) \arrow[r] \arrow[d] & gl(R/I) \arrow[d] \\ \mathcal{Y} \arrow[r] & st(R/I)^+ \arrow[r] & gl(R/I)^+ \end{tikzcd}$$ $\square$

Consider now the diagram of fibrations:

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[d] \\ st(R,I) \arrow[r] \arrow[d] & st(R) \arrow[r] \arrow[d] & Y \arrow[d] \\ gl(R,I) \arrow[r] & gl(R) \arrow[r] & gl(R/I) \end{tikzcd}$$

<u>Note</u>: The L_∞-algebra U has only two non-trivial homotopy groups:

$\pi_{-1} U \cong I/[R,I]$

$\pi_0 U \cong HC_1(R,I)$

§5. First relative homotopy cyclic homology

Apply plus construction to the "discrete" situation and get

$$\begin{tikzcd}
\mathcal{U} \arrow[r] \arrow[d] & \mathcal{V} \arrow[r] \arrow[d] & \mathcal{W} \arrow[d] \\
\mathcal{E} \arrow[r] \arrow[d] & st(R)^+ \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\
k(R, I) \arrow[r] & gl(R)^+ \arrow[r] & gl(R/_I)^+
\end{tikzcd}$$

<u>Proposition</u>

$\mathcal{E}$ is 1-connected.

<u>proof</u>: $\mathcal{Y} = \mathrm{Fib}(st(R/_I)^+ \longrightarrow K(L,2))$

and
$$\begin{array}{ccc} \pi_2\, st(R/_I)^+ & \longrightarrow & \pi_2\, K(L,2) \\ \cong & & \cong \\ HC_2(R/_I) & \twoheadrightarrow & L \end{array}$$

tells us ① $\mathcal{Y}$ is 1-connected

② $\pi_2 \mathcal{Y} \cong \mathrm{Im}(HC_2(R) \to HC_2(R/_I)$

We consider now

$$\mathcal{E} = \mathrm{Fib}(\underbrace{st(R)^+}_{\text{1-connected}} \longrightarrow \underbrace{\mathcal{Y}})$$

The proposition follows since

$$\begin{array}{ccc} \pi_2\, st(R)^+ & \longrightarrow & \pi_2\, \mathcal{Y} \\ \cong & & \cong \\ HC_2(R) & \twoheadrightarrow & \mathrm{Im}(HC_2(R) \to HC_2(R/I)) \end{array}$$

$\square$

Theorem

① $HC_0^\infty(R,I) \cong HC_0(R,I) \cong I/[R,I]$

② $HC_1^\infty(R,I) \cong HC_1(R,I)$.

proof: Look at the long exact sequence in homotopy of

$$\mathcal{U} \longrightarrow \mathcal{E} \longrightarrow k(R,I). \qquad \square$$