Euclidean knots and double loop spaces

Paolo Salvatore

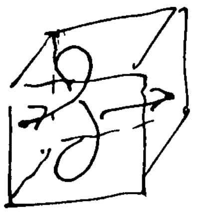
We show that the space of long knots in a euclidean space of dimension larger than three is a double loop space. The proof is based upon the operadic approach of Dev Sinha. We show also that the forgetful map from framed long knots to long knots is not a double loop map, but that there is a double loop map going the other way around.

EUCLIDEAN KNOTS AND DOUBLE LOOP SPACES

(VILLARS 2006)

long knot in $\mathbb{R}^n$: embedding $\mathbb{R} \to \mathbb{R}^n$ agreeing with linear inclusion near infinity.

Model: Emb_n = Embeddings $I \xrightarrow{\gamma} I^n$ fixing $\gamma(0), \gamma(1), \gamma'(0), \gamma'(1)$



Emb_n has a concatenation product $\Longrightarrow$ A_∞-SPACE

For $n > 3$ Emb_n is connected (Whitney) $\Longrightarrow$ $\simeq$ Ω-SPACE

$u : Emb_n \to \Omega S^{n-1}$ unit derivative

$$\begin{tikzcd} \overline{Emb_n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P\Omega S^{n-1} \arrow[d] \\ Emb_n \arrow[r, "u"] & \Omega S^{n-1} \end{tikzcd}$$

__Theorem__ (SINHA) The pullback is $\simeq$ Ω^2-space for $n > 3$ and $u \simeq *$, so that $\overline{Emb_n} \simeq Emb_n \times \Omega^2 S^{n-1}$ (1)

__DEFINITION__: The space of framed knots in $\mathbb{R}^n$ is the pullback

$$\begin{tikzcd} fEmb_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega SO(n) \arrow[d] \\ Emb_n \arrow[r, "u"] & \Omega S^{n-1} \end{tikzcd}$$

ACTUALLY THERE IS PULLBACK

$$\begin{tikzcd} f\mathrm{Emb}_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P SO(n-1) \arrow[d] \\ \mathrm{Emb}_n \arrow[r, "u"'] & \Omega S^{n-1} \arrow[r, "\text{HOLONOMY}"'] & SO(n-1) \end{tikzcd}$$

so that

$$f\mathrm{Emb}_n \simeq \mathrm{Emb}_n \times \Omega SO(n-1) \qquad (2)$$

Theorem (Budney)

$f\mathrm{Emb}_n$ is $\simeq$ Ω^2-space for $n>3$.

QUESTION: CAN WE DECOMPOSE THE Ω^2-STRUCTURES OF (1)(2)?

THEOREM 1 Emb_n is $\simeq$ to a Ω^2-SPACE for $n>3$

THE SPLITTINGS (1) (2) ARE COMPATIBLE WITH Ω-STRUCTURES

BUT NOT Ω^2-STRUCTURES.

THEOREM 2

THE PROJECTIONS $\overline{Emb}_n \to Emb_n$ AND $fEmb_n \to Emb_n$ ARE NOT Ω^2-MAPS FOR n ODD

THEOREM 3

THERE IS A COMMUTATIVE DIAGRAM

$$\begin{tikzcd}
BB\,Emb_n \arrow[r, shift left] \arrow[d, equal] & BB\,fEmb_n \arrow[l, shift left, "\times"] \arrow[r, shift left] & BSO(n-1) \arrow[l, shift left] \\
BB\,Emb_n \arrow[r, shift left] & BB\,\overline{Emb}_n \arrow[l, shift left, "\times"] \arrow[u] \arrow[r, shift left] & S^{n-1} \arrow[l, shift left] \arrow[u]
\end{tikzcd}$$

WITH THE ROWS FIBRATIONS WITH SECTION

OPERADS AND KNOTS

AN OPERAD P WITH MULTIPLICATION IN Top GIVES A COSIMPLICIAL SPACE

$$P(0) \rightleftarrows P(1) \; \substack{\leftleftarrows \\ \rightrightarrows \\ \to} \; P(2) \cdots$$

THEOREM (MCCLURE-SMITH)

THE HOMOTOPY TOTALIZATION $\widetilde{Tot}\, P^* = \operatorname{holim}_{\Delta} P^*$ HAS A NATURAL ACTION OF AN E_2-OPERAD $\widetilde{\mathcal{D}}_2$.

OPERAD $\mathcal{C}_n$ OF LITTLE n-CUBES HAS NO MULTIPLICATION

KONTSEVICH OPERAD $K_n \simeq \mathcal{C}_n$ HAS IT

$$\vartheta_{ij} : F(\mathbb{R}^n, k) \longrightarrow S^{n-1} \qquad (x_1, \dots, x_k) \longmapsto \frac{x_i - x_j}{|x_i - x_j|}$$

THEN $K_n(k) = \overline{\prod \mathrm{Im}\prod(\vartheta_{ij})} \subset (S^{n-1})^{\binom{k}{2}} = B_n(k)$

B_n OPERAD OF FORMAL DIRECTIONS BETWEEN POINTS IN $\mathbb{R}^n$

$K_n \subset B_n$ OPERAD INCLUSION

THEOREM (SINHA)

(1) $\widetilde{\mathrm{Tot}}\, K_n^* \simeq \overline{\mathrm{Emb}_n}$

(2) $\widetilde{\mathrm{Tot}}\, B_n^* \simeq \Omega^2 S^{n-1}$ (SINCE $B_n^* \simeq S^{n-1}$ $^{\Delta^2_*/\partial\Delta^2_*}$)

(3) $\widetilde{\mathrm{Tot}}(K_n^* \rtimes S^{n-1}) \simeq \mathrm{Emb}_n$ WHERE

$K_n^k \rtimes S^{n-1} = K_n(k) \times (S^{n-1})^k$ IS LIKE A CONFIGURATION SPACE DECORATED BY TANGENT DIRECTIONS AND

$$d^i(x; v_1, \dots, v_k) = (x \circ_i v_i; v_1, \dots, v_i, v_i, \dots; v_k) \text{ for } 1 \le i \le k$$

THE EQUIVALENCE IS REALIZED BY MAPS ADJOINT TO

$$\mathrm{Emb}_n \times \Delta^k \longrightarrow K_n(k) \times (S^{n-1})^k$$

$$(\gamma, 0 \le x_1 \le \dots \le x_k \le 1) \longmapsto \left\{ \gamma(x_1), \dots, \gamma(x_k); \frac{\gamma'(x_1)}{|\gamma'(x_1)|}, \dots, \frac{\gamma'(x_k)}{|\gamma'(x_k)|} \right\}$$

PROOF OF THM 1

DEFINE COSIMPLICIAL SPACE $B_n^* \rtimes S^{n-1}$ BY

$$B_n^k \rtimes S^{n-1} = B_n(k) \times (S^{n-1})^k.$$

GET COSIMPLICIAL DIAGRAM

$$\begin{tikzcd}
K_n^* \arrow[r] \arrow[d] & B_n^* = (S^{n-1})^{\Delta^2_*/\partial\Delta^2_*} \arrow[d] \\
K_n^* \rtimes S^{n-1} \arrow[r] \arrow[d] & B_n^* \rtimes S^{n-1} = (S^{n-1})^{\Delta^2_*/(d^0(\Delta^1_*) \cup d^2(\Delta^1_*))} \arrow[d] \\
(S^{n-1})^{\Delta^1_*/\partial\Delta^1_*} \arrow[r, equal] & (S^{n-1})^{\Delta^1_*/\partial\Delta^1_*}
\end{tikzcd}$$

AFTER TOTALIZATION

$$\begin{tikzcd}
\mathrm{Emb}_n \arrow[r] \arrow[d] & \Omega^2 S^{n-1} \arrow[d] \\
\mathrm{Emb}_n \arrow[r] \arrow[d] & P\Omega S^{n-1} \arrow[d] \\
\Omega S^{n-1} \arrow[r, equal] & \Omega S^{n-1}
\end{tikzcd}$$

HOMOTOPY FIBER OF UPPER ROW HAS ACTION OF $\widetilde{D}_2$

(MCCLURE-SMITH)

SEMIDIRECT PRODUCTS OF OPERADS AND GROUPS (S.-Wahl)

G TOP. GROUP, P OPERAD IN G-SPACES, DEFINE

OPERAD $P \rtimes G$

$(P \rtimes G)(n) = P(n) \times G^n$

$$(p; g_1, \dots, g_n) \circ_i (q; h_1, \dots, h_m) = (p \circ_i g_i(q); g_1, \dots, \underbrace{g_i h_1, \dots g_i h_m}, \dots; g_n)$$

<u>EXAMPLE</u>: fD_n FRAMED LITTLE n-DISCS

$fD_n = D_n \rtimes SO(n)$ WITH RESPECT TO ROTATIONS

fK_n FRAMED KONTSEVICH $fK_n = K_n \rtimes SO(n) \simeq fD_n$

HAS MULTIPLICATION

<u>EXAMPLE</u> $Comm \rtimes G := \underline{G}$ HAS MULTIPLICATION

<u>PROPOSITION</u>: $\widetilde{Tot}(fK_n^*) \simeq fEmb_n$

<u>PROOF</u>: $SO(n-1)^k \longrightarrow K_n(k) \times SO(n)^k \longrightarrow K_n(k) \times (S^{n-1})^k$

COMPARE

$$\begin{array}{ccccc} \Omega SO(n-1) & \longrightarrow & fEmb_n & \longrightarrow & Emb_n \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{Tot}\, SO(n-1) & \xrightarrow{\Delta^1/\partial\Delta} & \widetilde{Tot}(fK_n^*) & \longrightarrow & \widetilde{Tot}(K_n^* \rtimes S^{n-1}) \end{array}$$

LEMMA: THERE IS OPERAD MAP $\underline{SO(n-1)} \longrightarrow fB_n = B_n \rtimes SO(n)$

Inducing $\widetilde{Tot}\, \underline{SO(n-1)}^* \simeq \widetilde{Tot}(fB_n^*) \simeq \Omega SO(n-1)$

$$\begin{tikzcd}
SO(n-1)^k \arrow[r] \arrow[rr, bend left] \arrow[dr, bend right] & B_n(k) \times SO(n)^k \arrow[r] \arrow[d] & B_n(k) \times (S^{n-1})^k \arrow[d] \\
 & SO(n)^k \arrow[r] & (S^{n-1})^k
\end{tikzcd}$$

GIVES

$$\begin{tikzcd}
\Omega SO(n-1) \arrow[r, "\simeq"] \arrow[rr, bend left] \arrow[dr, bend right] & \widetilde{Tot}(fB_n^*) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P\Omega S^{n-1} \arrow[d] \\
 & \Omega SO(n) \arrow[r] & \Omega S^{n-1}
\end{tikzcd}$$

PROOF of THM 3

DIAGRAM OF OPERADS

$$\begin{tikzcd}
 & K_n \arrow[r] \arrow[d] & B_n \arrow[d] \\
\underline{SO(n-1)} \arrow[r] \arrow[dr] & fK_n \arrow[r] \arrow[d] & fB_n \arrow[dl] \\
 & \underline{SO(n)} &
\end{tikzcd}$$

GET DOUBLE LOOP MAPS

$$\begin{tikzcd} \Omega SO(n-1) \arrow[r] \arrow[rr, bend left, "1"] \arrow[dr] & f\mathrm{Emb}_n \arrow[r] \arrow[d] & \Omega SO(n-1) \arrow[dl] \\ & \Omega SO(n) & \end{tikzcd}$$

HOMOTOPY FIBERS GIVE DOUBLE LOOP MAPS.

$$\begin{tikzcd} \Omega^2 S^{n-1} \arrow[r] \arrow[rr, bend right, "1"] & \overline{\mathrm{Emb}}_n \arrow[r] & \Omega^2 S^{n-1} \end{tikzcd}$$

PROOF OF THM 2: NEED THEOREM (LAMBRECHTS-TURCHIN-VOLIČ):

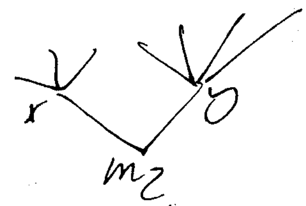
$$H_*(\overline{\mathrm{Emb}}_n;\mathbb{Q}) \cong HH(H_*(K_n;\mathbb{Q}))$$

HOCHSCHILD HOMOLOGY OF OPERAD OF n-algebras
WITH DEGREE 0 PRODUCT $*$, DEGREE $n-1$ BRACKET $[\,,\,]$
'POISSON' RELATIONS.

HOCHSCHILD COMPLEX OF OPERAD $\mathcal{O}$ WITH MULTIPLICATION IN VECTOR SPACES

$$HC(\mathcal{O}) = \bigoplus s^{-k}\mathcal{O}(k)$$

PRODUCT: $x \cdot y = (m_2 \circ_1 x) \circ_2 y$



BRACKET: $\{x,y\} = \sum_i \pm x \circ_i y + \sum_j \pm y \circ_j x$

DIFFERENTIAL: $D(x) = \{x, m_2\}$

THM (GERSTENHABER) $HH(\mathcal{O}) = H(HC(\mathcal{O}), D)$ is GERSTENHABER ALGEBRA.

$=$ e_2-ALGEBRA

ALSO $H_*(\overline{\mathrm{Emb}}_n; \mathbb{Q})$ GERSTENHABER ALGEBRA BY

$\cdot$ = PONTRJAGIN PRODUCT

$\{,\}$ = BROWDER OPERATION

STRUCTURES COMPATIBLE BY FORMALITY.

$HH(H(K_n))$ STUDIED BY TURCHIN

HE STUDIES ALSO $E_2 = E_\infty$- TERMS OF BOUSFIELD SPECTRAL SEQUENCE FOR Emb_n AND MAPS INDUCED BY $\overline{Emb}_n \xrightarrow{r} Emb_n$

<u>PROOF OF THM 2:</u> $\ker(r_*)$ IS NOT A GERSTENHABER IDEAL

$[x_1, x_2]$ REPRESENTS $y \in H_{n-3}(\overline{Emb}_n)$ FIRST NON TRIVIAL CLASS COMING FROM $\Omega^2 S^{n-1}$ FACTOR.

$[x_1, x_3][x_2, x_4]$ REPRESENTS $u \in H_{2n-6}(\overline{Emb}_n)$ AND $r_*(u) \in H_{2n-6}(Emb) \cong \mathbb{Q}$ FIRST NON TRIVIAL CLASS

BUT $r_* \{y, u\}$ GENERATES $H_{3n-8}(Emb_n; \mathbb{Q}) \cong \mathbb{Q}$

$$\begin{tikzcd}
\overline{Emb}'_n \arrow[r, "i"] \arrow[d, phantom, "\cong" rotate=90] & fEmb_n \arrow[r] \arrow[d] & Emb_n \\
\Omega^2 S^{n-1} \times Emb_n \arrow[r] & \Omega SO(n-1) \times Emb &
\end{tikzcd}$$

and $i_*(y) \neq 0$.

DOES NOT WORK FOR n EVEN

$$H_*(\mathrm{Emb}_n) \longrightarrow H(\mathrm{Emb}_n) \quad \text{CORRESPONDS TO}$$

$$HH(H_*(K_n)) \longrightarrow HH(BV_n)$$

BV_n has Δ of DEGREE $n-1$ and $*$

such that $\Delta(x_1, x_2) \pm x_1 \Delta(x_2) \pm \Delta(x_1) x_2 = [x_1, x_2]$

THIS SAYS $p_*(y) = D\Delta$

<u>OBSTRUCTION</u>: FINITE FIELDS - OVER LASHOF OPERATIONS.

<u>EXPECT</u>: $H_*(f\mathrm{Emb}_n) = HH(H_*(fK_n))$

NAMELY $H_*(fK_n) = H_*(SO(n)) \times H_*(K_n)$ (n ODD)

$H_*(fK_n) = H_*(SO(n-1)) \times BV_n$ (n EVEN)

STRING TOPOLOGY AND KNOTS

$\mathrm{Emb}(S^1, S^n)$ = CLOSED KNOTS IN n-SPHERE

FIBRATION $\quad \mathrm{Emb}_n \longrightarrow \mathrm{Emb}(S^1, S^n) \longrightarrow SO(n+1)/SO(n-1)$

THEOREM 4: $\Sigma^{-(2n-1)} \Sigma^\infty \mathrm{Emb}(S^1, S^n)_+$ IS HOMOTOPY E_2-RING SPECTRUM

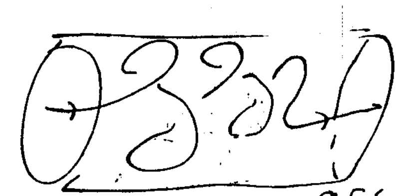
IN PARTICULAR $H_{*+2n-1}(\mathrm{Emb}(S^1, S^n))$ GERSTENHABER ALGEBRA

(ABBASPOUR-CHATAUR-KALLEL)

LEMMA (GRUHER-S.) $F \longrightarrow E \xrightarrow{p} M$ BUNDLE OVER CLOSED MANIFOLD M, FIBERWISE ACTION OF E_2-OPERAD $\Rightarrow$ $E^{p^*(-TM)}$ HOMOTOPY E_2-RING SPECTRUM.

$SO(n-1)$ ACTS ON Emb_n BY ROTATIONS AROUND LONG AXIS.

MODEL $\quad \mathrm{Emb}(I, I \times S^{n-1})$



K_n, B_n OPERADS WITH MULTIPLICATION IN $SO(n-1)$-SPACES

$\mathrm{Tot}(K_n) \longrightarrow \mathrm{Tot}(B_n)$ $\quad \widetilde{D}_2$-MAP IN $SO(n-1)$-SPACES.

SO $\quad \mathrm{Emb}_n \times_{SO(n-1)} SO(n+1) \longrightarrow SO(n+1)/SO(n-1)$ FIBERWISE $\widetilde{D}_2$-ALGEBRA.