The associative operad and the weak order on the symmetric group

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The associative operad is a certain algebraic structure on the sequence of group algebras of the symmetric groups. The weak order is a partial order on the symmetric group. There is a natural linear basis of each symmetric group algebra, related to the group basis by Möbius inversion for the weak order. We describe the operad structure on this second basis: the surprising result is that each operadic composition is a sum over an interval of the weak order. The Lie operad, a suboperad of the associative operad, contains some idempotent such as the Dynkin’s idempotent. As a corollary to our results, we derive a simple explicit expression for Dynkin’s idempotent in terms of the second basis.

There are combinatorial procedures for constructing a planar binary tree from a permutation, and a composition from a planar binary tree. These define set-theoretic quotients of each symmetric group algebra. They are operad quotients of the associative operad and we can decline the results of the first section to these contexts.

The talk is based on joint work with M. Aguiar.
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Contents

1. The operad $A_s$
2. The weak order on the symmetric groups
3. The main theorem
4. Application to Dynkin’s idempotents
5. Link with binary planar trees and vertices of the hypercubes
6. Questions
11. Non-$\Sigma$-operad

$K$ is a field.

**Notation.** A *non-$\Sigma$-operad* $\mathcal{P}$ satisfies $\mathcal{P}(0) = 0$, $\mathcal{P}(1) = K$. Compositions

$$
\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n + m - 1)
$$

satisfy

$$
(\mu \circ_i \nu) \circ_{j+i-1} \eta = \mu \circ_i (\nu \circ_j \eta),
$$

$$
(\mu \circ_i \nu) \circ_{j+m-1} \eta = (\mu \circ_j \eta) \circ_i \nu,
$$

$$
\mu \circ_i 1 = \mu \quad \text{and} \quad 1 \circ_1 \mu = \mu.
$$

**Example.** $\mathcal{P}(n) = S_n$. Write $\sigma = (\sigma_1, \ldots, \sigma_n)$ with $\sigma_i = \sigma(i)$.

$$
B_i(\sigma, \tau) := (a_1, \ldots, a_{i-1}, \underbrace{b_1, \ldots, b_m}_{\sigma}, a_{i+1}, \ldots, a_n)
$$

where

$$
a_j := \begin{cases} 
\sigma_j & \text{if } \sigma_j < \sigma_i \\
\sigma_j + m - 1 & \text{if } \sigma_j > \sigma_i 
\end{cases}
$$

$$
b_k := \tau_k + \sigma_i - 1.
$$
12. Example

\[ \sigma = (2, 3, 1, 4), \ \tau = (2, 3, 1), \ \text{and} \ i = 2 \ \text{then} \]
\[ B_2(\sigma, \tau) = (2, 4, 5, 3, 1, 6). \]

In terms of matrices

\[ \sigma = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tau = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \]

\[ B_2(\sigma, \tau) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]
13. $\Sigma$-operads

**Definition.** A $\Sigma$-operad $\mathcal{P}$ is a non-$\Sigma$-operad together with a right $S_n$-action on $\mathcal{P}(n)$ such that

$$(\mu \cdot \sigma) \circ i (\nu \cdot \tau) = (\mu \circ_{\sigma(i)} \nu) \cdot B_i(\sigma, \tau)$$

There is a pair of adjoint functors

$$S : \text{non-}\Sigma\text{-operad} \leftrightarrow \Sigma\text{-operads} : F$$

where $F$ is the forgetful functor and $S\mathcal{P}(n) = \mathcal{P}(n) \otimes KS_n$. 
14. The associative operad

$(F_\sigma)_{\sigma \in S_n}$ is a basis of the vector space $\mathcal{A}_s(n) = KS_n$.

$\mathcal{A}_s$ is the unique $\Sigma$-operad such that

\[
F_{1_n} \circ_i F_{1_m} = F_{1_{n+m-1}}
\]

\[
F_\sigma = F_{1_n} \cdot \sigma
\]

Then

\[
F_\sigma \circ_i F_\tau = F_{B_i(\sigma,\tau)}
\]

In the sequel $\mathcal{A}_s$ is considered as a non-$\Sigma$-operad and “operads” mean non-$\Sigma$-operad.
21. The left weak Bruhat order on $S_n$

**Definition.** The *inversion set* of $\sigma$ is

$$\text{Inv}(\sigma) := \{(i, j) \in [n] \times [n] \mid i < j \text{ and } \sigma_i > \sigma_j\}.$$ 

The *left weak Bruhat order* on $S_n$ is given by

$$\sigma \leq \tau \iff \text{Inv}(\sigma) \subseteq \text{Inv}(\tau).$$

**Example.**

```
\begin{tikzpicture}
    \node (321) at (0,2) {321};
    \node (312) at (-1,-1) {312};
    \node (213) at (-1,1) {213};
    \node (231) at (1,1) {231};
    \node (132) at (1,-1) {132};
    \node (123) at (0,-2) {123};
    \draw[->] (321) -- (312);
    \draw[->] (321) -- (213);
    \draw[->] (312) -- (213);
    \draw[->] (312) -- (231);
    \draw[->] (213) -- (231);
    \draw[->] (213) -- (132);
    \draw[->] (231) -- (132);
    \draw[->] (231) -- (123);
    \draw[->] (132) -- (123);
    \draw[->] (132) -- (321);
    \draw[->] (123) -- (321);
\end{tikzpicture}
```
22. The monomial basis

**Definition.** \((F_\sigma)_{\sigma \in S_n}\) is the *fundamental basis* of \(KS_n\).

\[
F_\sigma = \sum_{\sigma \leq \tau} M_\tau
\]

defines the *monomial basis* \((M_\tau)_{\tau \in S_n}\) of \(KS_n\). By Möbius inversion

\[
M_\sigma = \sum_{\sigma \leq \tau} \mu(\sigma, \tau)F_\tau
\]

**Example.**

\[
M_{(1,2)} = F_{(1,2)} - F_{(2,1)}
\]

\[
M_{(2,1)} = F_{(2,1)}
\]
31. Main theorem

Compare

\[ F_\sigma \circ_i F_\tau = F_{B_i(\sigma,\tau)} \]

with

**Theorem.**

\[ M_\sigma \circ_i M_\tau = \sum_{B_i(\sigma,\tau) \leq \rho \leq T_i(\sigma,\tau)} M_\rho \]
32. Fondamental lemma

The relation

\[ M_\sigma \circ_i M_\tau = \sum_{B_i(\sigma, \tau) \leq \rho \leq T_i(\sigma, \tau)} M_\rho \]

can be written

\[ M_\sigma \circ_i M_\tau = \sum_{P_i(\rho) = (\sigma, \tau)} M_\rho \]

Lemma. The maps

\[
\begin{array}{ccc}
S_n \times S_m & \xrightarrow{T_i} & S_{n+m-1} \\
& \xrightarrow{P_i} & \xrightarrow{B_i} \\
S_n \times S_m & \xrightarrow{P_i} & S_{n+m-1}
\end{array}
\]

satisfy the following properties:

(i) \( P_i \) and \( B_i \) are order-preserving.
(ii) \( P_i \circ B_i = Id = P_i \circ T_i \).
(iii) \( B_i \circ P_i \leq Id \leq T_i \circ P_i \).

The fiber of \( P_i \) is precisely the interval \([B_i, T_i]\)
33. Construction of $T_i$ and $P_i$

Example. $\sigma = (6, 2, 3, 4, 1, 5), \tau = (2, 1, 3)$

$$B_4(\sigma, \tau) = (8, 2, 3, 5, 4, 6, 1, 7)$$
$$T_4(\sigma, \tau) = (8, a_1, a_2, b_1, 2, 7, 1, a_3)$$
$$T_4(\sigma, \tau) = (8, 5, 6, 4, 2, 7, 1, 3)$$

Example. Let $n = 6, m = 3, i = 4$ and

$$\rho = (8, 7, 4, \underbrace{2, 6, 3, 1, 5}_{m=3})$$

$$P'_4(\rho) = P'_4(8, 7, 4, \underbrace{2, 6, 3, 1, 5}_{m=3}) = (8, 7, 2, \underbrace{3, 5, 4, 1, 6}_{m=3})$$

$$P'_4(\rho) = B_4(\text{st}(8, 7, 2, 3, 1, 6), \text{st}(3, 5, 4))$$

$$P_4(\rho) = ((6, 5, 2, 3, 1, 4), (1, 3, 2))$$
34. Picture

The fibers of $P_1 : S_4 \rightarrow S_3 \times S_2$
35. Proof of the theorem

Write

\[ M_{\sigma \tilde{\delta}_i} M_\tau = \sum_{P_i(\rho) = (\sigma, \tau)} M_\rho \]

Since

\[ F_\sigma = \sum_{\sigma \leq \sigma'} M_{\sigma'} \]

one gets

\[ F_\sigma \tilde{\delta}_i F_\tau = \sum_{(\sigma, \tau) \leq (\sigma', \tau')} M_{\sigma'} \tilde{\delta}_i M_{\tau'} \]

\[ = \sum_{(\sigma, \tau) \leq P_i(\rho)} M_\rho \]

\[ = \sum_{B_i(\sigma, \tau) \leq \rho} M_\rho \]

\[ = F_{B_i(\sigma, \tau)} = F_\sigma \circ_i F_\tau \]
41. Application: the coradical filtration

\( H = \oplus K S_n \) is endowed with a coproduct

\[
\Delta(F_\sigma) = \sum_{i=1}^{n-1} F_{\text{st}(\sigma_1,\ldots,\sigma_i)} \otimes F_{\text{st}(\sigma_{i+1},\ldots,\sigma_n)}
\]

**Definition.** \( H^k = \ker(\Delta^{k+1} : H \to H \otimes^{k+2}) \).

\( H^0 \) is the space of *primitive elements*.

**Theorem.** The sequence \((H^k_n)\) is a filtration of the operad \( \mathcal{A}_\Sigma \):

\[
H^k_n \circ_i H^h_m \subset H^{k+h}_{n+m-1}
\]

**Corollary.** The space of primitives \( H^0 \) is a sub (non-\( \Sigma \)) operad of \( \mathcal{A}_\Sigma \).
42. Idea of the proof

**Definition.** A *global descent* $p$ of $\sigma \in S_n$ is an integer such that

$$\forall i \leq p, \forall j > p, \quad \sigma_i > \sigma_j$$

For instance $(3, 5, 4, 1, 2)$ has for global descent set $\{3\}$.

**Proposition.** [A-S] $H_j^k$ is generated by the $M_{\sigma}$’s such that $\sigma$ has at most $k$ global descents.

If $\sigma$ (resp. $\tau$) has at most $k$ (resp. $h$) global descents then the permutations in $P_i^{-1}(\sigma, \tau)$ have at most $k + h$ global descents.
43. Application: Dynkin’s idempotents

The sub-$\Sigma$-operad $\mathcal{L}ie$ of $As$ is generated by
$M_{(1,2)} = F_{(1,2)} - F_{(2,1)}$.

**Fact.** $\mathcal{L}ie$ is a sub (non-$\Sigma$)-operad of $H^0$.

**Definition.**
$\theta_n(v_1, \ldots, v_n) = [[[\ldots [v_1, v_2], \ldots], v_n]$
Dynkin’s idempotent: $\frac{\theta_n}{n}$.

**Theorem.**

$$\theta_n = \sum_{\sigma \in S_n, \sigma(1)=1} M_{\sigma}$$

Introduce the twisted Lie bracket on $KS_n$:
$$\{F_\sigma, F_\tau\} = F_{\sigma \begin{bmatrix} 0 & 0 \\ 0 & \tau \end{bmatrix}} - F_{\begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}}$$
and prove $\theta_n = \{\theta_{n-1}, \theta_1\}$. A closed formula for $\{M_\sigma, M_\tau\}$
yields the result.
51. Link with hypercubes

**Operad structure.** The surjective map

\[
\text{Des} : \quad KS_n \to KQ_n = K\{-1, 1\}^{n-1}
\]

\[
F_\sigma \mapsto F_{\text{Des}(\sigma)}
\]

defined by

\[
\text{Des}(\sigma) = (\epsilon_1, \ldots, \epsilon_{n-1})
\]

\[
\epsilon_i = \begin{cases} 
1 & \text{if } i \in \text{Des}(\sigma) \\
-1 & \text{if not}
\end{cases}
\]

induces a non-$\Sigma$-operad structure on $Q = (KQ_n)_n$ given by

\[
F_{\epsilon \circ_i} F_{\delta} = F(\epsilon_1, \ldots, \epsilon_{i-1}, \delta_1, \ldots, \delta_{m-1}, \epsilon_i \ldots, \epsilon_{n-1})
\]

**Algebra structure.** Algebras over $SQ$ are 2-associative algebras $(A, \cdot, *)$ such that

\[
(a \cdot b) * c = a \cdot (b * c)
\]

\[
(a \ast b) \cdot c = a \ast (b \cdot c)
\]

compare with Richter, Pirashvili, Chapoton.
52. Order structure on hypercubes

**Order structure.** The surjective map $\text{Des}$ induces a weak order on $Q_n$ given by

$$\epsilon \leq \eta \iff \forall i, \; \epsilon_i \leq \eta_i$$

**Monomial basis.**

$$F_\epsilon = \sum_{\epsilon \leq \eta} M_\eta$$

**Theorem.** The map

$$Q \to Q$$

$$F_\epsilon \mapsto M_\epsilon$$

is an operad automorphism.
53. Link with planar binary trees

The surjective map $\text{Des} : KS_n \rightarrow KQ_n$ factors through $KY_n$

$$ S_n \xrightarrow{\lambda} Y_n \xrightarrow{L} Q_n $$

where

$$ \lambda(1_0) = \mid $$

$$ \lambda(1_1) = \bigtriangleup $$

$$ \lambda(a_1, \ldots, a_{i-1}, n, a_i, \ldots, a_n) = \lambda(\text{st}(a_1, \ldots, a_{i-1})) \lor \lambda(\text{st}(a_i, \ldots, a_n)) $$

Example.

$$ \lambda(342651) = $$
54. Operad structure on planar binary trees

The surjective map $\lambda : KS_n \to KY_n$ induces a non-$\Sigma$-operad structure on $\mathcal{Y} = (KY_n)_n$

Example.

$$(1, 3, 2) \circ_2 (1, 2, 3) = (1, 3, 4, 5, 2)$$

![Diagram of trees]

**Algebra structure.** Algebras over $S\mathcal{Y}$ are 2-associative algebras $(A, \cdot, *)$ such that

$$(a \cdot b) * c = a \cdot (b * c)$$

compare with Pirashvili.
55. Monomial basis and planar binary trees

**Order structure.** The surjective map $\lambda$ induces a weak order on $Y_n$ generated by

\[ \bigtriangleup \leq \bigtriangleup \]

**Monomial basis.**

\[ F_t = \sum_{t \leq u} M_u \]

**Theorem.**

\[ M_t \circ_i M_u = \sum_{B_i(t,u) \leq s \leq T_i(t,u)} M_s \]
\[ = \sum_{P_i(s) = (t,u)} M_s \]
61. Questions

- The algebras over $SY$ and $SQ$ are well defined.
  ⇒ What about $SAs$?
- One has a morphism of $\Sigma$-operads

$$SAs \to 2As \to SY \to SQ$$

In $SAs$ there are more operations: for instance $(2, 4, 1, 3)$ is not of the form $B_i(\sigma, \tau)$.

⇒ Question: count the number of permutations $\sigma$ such that $\sigma$ does not contain a subsequence of integers describing an interval.
62. Questions

- Work of Chapoton: there are operad structures on the faces of Stasheff polytopes and hypercubes.
  ⇒ Can we describe one on the faces of the permutohedron? Compatible with the one of Chapoton?
- There’s an order structure (Palacios-Ronco) on the faces of the permutohedron, inducing order structure on Stasheff polytopes and hypercubes.
  ⇒ Is it compatible with the operadic structure?