(Co)formality of the little ball operad and the rational homotopy type of spaces of long knots

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(Joint works with V. Turchin, I. Volic, and Greg Arone) Sinha has shown how the space of long knots can be expressed as the totalisation of the cosimplicial space associated to a certain multiplicative operad equivalent to the little ball operad. We show how to use this result to deduce the collapsing of the Vassiliev spectral sequence computing the rational homology of the space of long knots of codimension > 2. This completely determines the rational homotopy type of long knots in codimension > 2. The proof involves an interesting new construction that associates to any morphism of operad a diagram generalizing the cosimplicial object usually associated to a multiplicative operad. We also prove the "coformality" of the little ball operad and deduce the collapsing of a "dual" spectral sequence computing the rational homotopy of long knots. Some results pass in codimension 2.

THE SPACE OF LONG KNOTS

Fix $d \geq 3$

Fix a linear embedding $\varepsilon : \mathbb{R} \hookrightarrow \mathbb{R}^d$
$t \longmapsto (t, 0, \cdots, 0)$

Déf The space of long knots is

$$\boxed{\mathcal{L} = \mathrm{Emb}(\mathbb{R}, \mathbb{R}^d) = \{ f : \mathbb{R} \hookrightarrow \mathbb{R}^d \text{ smooth emb. s.t. } f(t) = \varepsilon(t) \text{ for } |t| \gg 1 \}}$$

Slight variation:

$$\begin{array}{ccccc} \overline{\mathrm{Emb}}(\mathbb{R}, \mathbb{R}^d) & \longrightarrow & \mathrm{Emb}(\mathbb{R}, \mathbb{R}^d) & \overset{j}{\hookrightarrow} & \mathrm{Imm}(\mathbb{R}, \mathbb{R}^d) \\ \| & & \| & & \wr\!\wr \text{ smale} \\ \overline{\mathcal{L}} & & \mathcal{L} & & \Omega S^{d-1} \end{array}$$

$j \simeq * \Longrightarrow \overline{\mathcal{L}} \simeq \mathcal{L} \times \Omega^2 S^{d-1}$.

PROBLEM: How to determine the homotopy type of $\overline{\mathcal{L}}$?

$H_*(\overline{\mathcal{L}}) = ?$ $\quad \pi_*(\overline{\mathcal{L}}) = ?$ $\quad$ e.g. over $\mathbb{Q}$?

VASSILIEV APPROACH :

There is a spectral sequence

$$\underbrace{E^1_{p,q}}_{} \Longrightarrow H_*(\overline{\mathcal{L}}) \quad \text{if } d \geq 4$$

$\hookrightarrow$ combinatorial object well understood ("chord diagrams")

COSIMPLICIAL MODEL FOR $\overline{\mathcal{L}}$

Thm (Sinha)

There exits a cosimplicial space

$$F[\bullet,\mathbb{R}^d] = \left\{ \cdots \; F[2,\mathbb{R}^d] \underset{d^2}{\overset{d^0}{\leftleftarrows}} F[1,\mathbb{R}^d] \underset{d^1}{\overset{d^0}{\leftleftarrows}} F[0,\mathbb{R}^d] \right\}$$

such that if $d \geq 4$, $\overline{\mathcal{L}} \simeq \mathrm{Tot}(F[\bullet,\mathbb{R}^d]) \simeq \operatorname{holim}_{\Delta} F[\bullet,\mathbb{R}^d]$

$F[m,\mathbb{R}^d]$ = a suitable compactification (h.e.) of the configuration space

$$F(m,\mathbb{R}^d) = \{(x_1,\cdots,x_m) \mid x_i \in \mathbb{R}^d,\ x_i \neq x_j\}$$

cofaces d^i. fix an infinitesimal vector $m \neq 0$ in $\mathbb{R}^d$

$$d^i : F[m,\mathbb{R}^d] \longrightarrow F[m+1,\mathbb{R}^d]$$

$$(x_1,\cdots,x_m) \longmapsto (x_1,\cdots,x_i,x_i+m,\ldots,x_m) \qquad 1 \leq i \leq m$$

$$\longmapsto (-m.\infty, x_1,\cdots,x_m) \qquad i=0$$

$$\longmapsto (x_1,\cdots,x_m, m.+\infty) \qquad i=m+1$$

codegeneracies s^j

$$s^j : F[m,\mathbb{R}^d] \longrightarrow F[m-1,\mathbb{R}^d]$$

$$(x_1,\cdots,x_m) \longmapsto (x_1,\ldots,\widehat{x_j},\ldots,x_m)$$

COSIMPLICIAL OBJECT ASSOCIATED TO A MULTIPLICATIVE NON-Σ OPERAD

$\mathcal{O} = \{\mathcal{O}(n)\}_{n \geq 0}$ a non-Σ operad (e.g. in Top)

$\mathcal{A} = \{*\}_{n \geq 0}$ the associative non-Σ operad

<u>Def</u> $\mathcal{O}$ is <u>multiplicative</u> if it is equipped with a morphism $\mu : \mathcal{A} \longrightarrow \mathcal{O}$.

$$\mathcal{O} \text{ multiplicative} \iff \begin{cases} \exists\, m \in \mathcal{O}(2) : & m \circ_1 m = m \circ_2 m \\ \exists\, u \in \mathcal{O}(0) : & m \circ_1 u = id = m \circ_2 u \end{cases}$$

<u>Proposition</u> (Gerstenhaber Kapranov / McClure Smith)

If $\mathcal{O}$ is a multiplicative non-Σ-operad one can define a cosimplicial object

$$\mathcal{O}^\bullet = \left\{ \cdots \;\overleftarrow{\overleftarrow{\overleftarrow{\overleftarrow{}}}}\; \mathcal{O}(2) \;\underset{d^1}{\overset{d^0}{\overleftarrow{\overleftarrow{\overleftarrow{}}}}}\; \mathcal{O}(1) \;\underset{d^1}{\overset{d^0}{\leftleftarrows}}\; \mathcal{O}(0) \right\}$$

where $d^i : \mathcal{O}(n) \longrightarrow \mathcal{O}(n+1)$

$$\alpha \longmapsto \begin{cases} \alpha \circ_i m & \text{if } 1 \leq i \leq n \\ m \circ_2 \alpha & \text{if } i = 0 \\ m \circ_1 \alpha & \text{if } i = n+1 \end{cases}$$

$$s^j : \mathcal{O}(n) \longrightarrow \mathcal{O}(n-1)$$
$$\alpha \longmapsto \alpha \circ_j u$$

$\therefore$ $\mathcal{O}$ an $\mathcal{A}$-bimodule $\Longrightarrow \mathcal{O}^\bullet : \Delta \longrightarrow$ Top

For all $n \geq 0$: $\quad \mathcal{O}^\bullet_{(n)} : \Delta[n] \longrightarrow$ Top

GENERALISATION OF THE GERSTENHABER-KAPRANOV CONSTRUCTION

<u>Thm</u> (L + Turchin + Volić)

- There exits a finite category (poset) of <u>n-fans</u> Φ_n
- If $\mu : \mathcal{B} \longrightarrow \mathcal{O}$ is a morphism of non Σ operads then we can construct a diagram $\hat{\mathcal{O}}_{(n)} : \Phi_n \longrightarrow \mathrm{Top}$ and this construction is natural
- There exits a left cofinal functor $\lambda : \Phi_n \longrightarrow \Delta[n]$
- If $\mu : \mathcal{A} \longrightarrow \mathcal{O}$ is a multiplicative non Σ operad then

$$\begin{tikzcd} \Phi_n \arrow[r, "\lambda"] \arrow[dr, "\hat{\mathcal{O}}_{(n)}"'] & \Delta[n] \arrow[d, "\mathcal{O}^\bullet_{(n)}"] \arrow[dl, phantom, "\simeq"] \\ & \mathrm{Top} \end{tikzcd}$$

and $\mathrm{Tot}^n(\mathcal{O}^\bullet) \simeq \operatorname{holim}_{\Phi_n} \hat{\mathcal{O}}_{(n)}$

- If

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\mu"] \arrow[d, "\simeq"'] & \mathcal{O} \arrow[d, "\simeq"] \\ \mathcal{B}' \arrow[r, "\mu'"] & \mathcal{O}' \end{tikzcd}$$

then $\hat{\mathcal{O}}_{(n)} \xrightarrow{\simeq} \hat{\mathcal{O}}'_{(n)}$

FORMALITY OF THE FULTON-MCPHERSON OPERAD

Thm (Kontsevich)

$\boxed{\begin{array}{l}\text{The Fulton McPherson is formal over } \mathbb{R} \\ \text{i.e. there exists a zigzag of quasi-iso of operads} \\ \qquad C_*(\mathcal{F}_d) \xleftarrow{\simeq} \cdots\cdots \xrightarrow{\simeq} H_*(\mathcal{F}_d)\end{array}}$

Thm (L + Volić)

Let $d \geq 2m+1$ and fix $\varepsilon : \mathbb{R}^m \hookrightarrow \mathbb{R}^d$ a linear embed.
Then $\varepsilon_* : \mathcal{F}_m \longrightarrow \mathcal{F}_d$ is formal:

$$\begin{array}{ccccccc} C_*(\mathcal{F}_m) & \xleftarrow{\simeq} & \cdot & \cdots & \cdot & \xrightarrow{\simeq} & H_*(\mathcal{F}_m) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C_*(\mathcal{F}_d) & \xleftarrow{\simeq} & \cdot & \cdots & \cdot & \xrightarrow{\simeq} & H_*(\mathcal{F}_d) \end{array}$$

Application $m=1$, $d \geq 3$ coeff $= \mathbb{R}$

$$H_*(\overline{\mathcal{L}}) \overset{d\geq 4}{=} H_*(\mathrm{Tot}(\mathcal{K}_d^\bullet))$$

$$\overset{H_*\,\mathrm{BKSS}}{=} H_*\Big(\underbrace{\bigoplus_{p=0}^{\infty} C_*(\mathcal{K}_d^p)}\Big)$$

$\to$ • horiz. diff. from cofac
• vertical diff. internal to $C_*(\mathcal{K}$

$$\overset{\text{formality}}{=} H_*\Big(\underbrace{\bigoplus_{p=0} H_*(\mathcal{K}_d^p)}\Big)$$

$\to$ no vertical differential!

Thm (L + Turchin + Volić)

$\boxed{\text{The Vassiliev s.s. computing } H_*(\overline{\mathcal{L}};\mathbb{R}) \text{ collapses at } E_1}$