Little cubes operad actions and the bar construction of algebras

Benoît Fresse

We consider the classical reduced bar construction of associative algebras $B(A)$. If the product of A is commutative, then $B(A)$ can be equipped with the classical shuffle product, so that $B(A)$ is still a commutative algebra. This assertion can be generalized for algebras which are commutative up to homotopy. Namely, one observes that the bar construction $B(A)$ can be endowed with the structure of an associative algebra, if A is an E_2-algebra, with the structure of an E_2-algebra, if A is an E_3-algebra, ... and with the structure of an E_∞-algebra, if A is an E_∞-algebra. In this talk, we prove that this observation is a consequence of a more precise result on the endomorphism PROP of the bar construction, the structure that parameterizes all natural operations on $B(A)$, for A an E_n-algebra. Explicitly: this PROP is equivalent to the PROP of coassociative-$E_{(n-1)}$-bialgebras.

Our result allow to iterate the bar construction for algebras equipped with an E_n-structure. We prove that the n^{th} iterated bar complex defines Quillen's homology in the category of E_n-algebras.

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Recall: the classical reduced bar construction of augmented associative algebras

Fix a ground field $\mathbf{F}$.

For A an augmented associative algebra the bar construction is the chain complex $B(A)$ such that

$$B_n(A) = \tilde{A}^{\otimes n},$$

where $\tilde{A}$ denotes the augmentation ideal of A, together with the differential

$$\partial : B_n(A) \to B_{n-1}(A)$$

defined by the formula

$$\partial(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} a_1 \otimes \cdots \otimes a_{i-1} a_i \otimes \cdots \otimes a_n.$$

Recall: If the product of A is commutative, then the classical shuffle product of tensors provides $B(A)$ with the structure of a differential commutative algebra. If the product of A is “strictly” not commutative, then $B(A)$ does not carry any natural multiplicative structure.

Purpose: Give insights into the algebraic structure of $B(A)$ for algebras A equipped with a homotopy commutative product.

Results:

1. If A is an E_∞ algebra, then $B(A)$ can still be equipped with the structure of an E_∞ algebra.
2. If A is an E_n algebra, then $B(A)$ can be equipped with the structure of an E_{n-1} algebra.

Remarks:

Statement (1) was obtained by Smirnov, by Justin Smith, . . .

The case $n = 2$ of (2) can be deduced from an explicit construction of Hans Baues.

For the cochain algebra of a space $A = C^*(X)$, such structure results have been obtained by Kadeishvili-Saneblidze and
by Hess-Parent-Scott.

Motivations and implications of (1) are described in a previous work:
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The goal of this talk is to give more details on part (2).

In fact, this assertion is a consequence of a more precise results on the endomorphism prop of the bar construction,
the structure that parametrizes all natural operations on $B(A)$, for A an E_n-algebra.

Namely:
This prop is equivalent to the prop of coassociative and E_{n-1}-multiplicative bialgebras.

§0. Conventions and recalls

- An A_∞ operad denotes a Σ_*-cofibrant operad $\mathcal{K}$ in the category of chain complexes together with an operad equivalence $\mathcal{K} \xrightarrow{\sim} \mathcal{A}$, where $\mathcal{A}$ denotes the operad of associative algebras.

  An A_∞ algebra denotes an algebra over some fixed A_∞ operad $\mathcal{K}$.

- An E_∞ operad denotes a Σ_*-cofibrant operad $\mathcal{E}$ in the category of chain complexes together with an operad equivalence $\mathcal{E} \xrightarrow{\sim} \mathcal{C}$, where $\mathcal{C}$ denotes the operad of associative and commutative algebras.

  An E_∞ algebra denotes an algebra over some fixed E_∞ operad $\mathcal{E}$.

Facts:

- The definition of the bar complex can be extended to algebras over any A_∞ operad $\mathcal{K}$. In this context, we have:

  $$\partial(a_1 \otimes \cdots \otimes a_n) = \sum_{r,i} a_1 \otimes \cdots \otimes \mu_r(a_i, \dots, a_{i+r-1}) \otimes \cdots \otimes a_n,$$

  for a sequence of operations $\mu_r \in \mathcal{K}(r)$.

- Any E_∞ operad $\mathcal{E}$ is endowed with an operad morphism $\mathcal{K} \to \mathcal{E}$, where $\mathcal{K}$ is some A_∞ operad. Consequently, any E_∞ algebra has a bar construction.

§1. Previous results

Theorem 1 (existence): Fix a cofibrant E_∞ operad $\mathcal{E}$.
The bar complex of an $\mathcal{E}$-algebra $B(A)$ can be equipped with the structure of an $\mathcal{E}$-algebra, functorially in A, and so that:
in the case where A is a commutative algebra this $\mathcal{E}$-algebra structure reduces to the classical commutative algebra structure of $B(A)$, the one defined by the shuffle product.

Theorem 2 (uniqueness): The structure defined by theorem 1 is unique up to homotopy. To be precise, let

$$\rho^0, \rho^1 : \mathcal{E} \to \mathsf{End}_{B(A)}$$

denote operad morphisms which provide $B(A)$ with the structure of an $\mathcal{E}$-algebra as in theorem 1.
Then, the algebras $(B(A), \rho_0)$ and $(B(A), \rho_1)$ can be connected by weak-equivalences of $\mathcal{E}$-algebras

$$(B(A), \rho_0) \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B(A), \rho_1),$$

functorially in A.

Theorem 3 (homotopy interpretation):
Let $F_A \xrightarrow{\sim} A$ denote a cofibrant model of a given $\mathcal{E}$-algebra A.
Suppose that the bar construction $B(A)$ is equipped with the structure of an $\mathcal{E}$-algebra as in theorem 1.
There is a weak-equivalence of $\mathcal{E}$-algebras

$$\Sigma F_A \xrightarrow{\sim} B(A),$$

where ΣF_A denotes the suspension of F_A in the model category of $\mathcal{E}$-algebras.

Theorem 4:

We assume that X is a pointed nilpotent space whose homotopy groups are degreewise finitely generated.

We let $R_\infty X$ denote Bousfield-Kan' p completion of X.

We fix a cofibrant model F_X of $C^*(X)$, as in theorem 3. We have

$$H^0(\Sigma^n F_X) = \mathrm{Fun}(\pi_n(R_\infty X)_p^\wedge, \mathbf{F}_p),$$

the module of maps $\alpha : \pi_n(R_\infty X) \to \mathbf{F}_p$ which are continuous in regard to the p-profinite topology and

$$H^*(\Sigma^n F_X) = H^0(\Sigma^n F_X) \otimes H^*(\Omega_0^n R_\infty X, \mathbf{F}_p),$$

where $\Omega_0^n R_\infty X$ denotes the connected component of the base point of $\Omega^n R_\infty X$.

§2. Little cubes operads and the bar construction

Recalls: The little n-cubes operads $\mathcal{D}_n$, defined by Boardmann-Vogt and May, form a nested sequence of topological operads.
We consider the associated chain complexes which form a nested sequence of differential graded operads

$$\begin{aligned} & C_*(\mathcal{D}_1) \subset C_*(\mathcal{D}_2) \subset \dots \\ & \qquad \subset C_*(\mathcal{D}_n) \subset \dots \subset C_*(\mathcal{D}_\infty) = C_*(\mathcal{D}), \end{aligned}$$

such that $C_*(\mathcal{D}_1)$ is an A_∞ operad and $C_*(\mathcal{D}_\infty)$ is an E_∞ operad.

In fact, these chain operads are equipped with a coassociative coproduct

$$\Delta : C_*(\mathcal{D}_n(r)) \to C_*(\mathcal{D}_n(r)) \otimes C_*(\mathcal{D}_n(r))$$

supplied by the Alexander-Whitney diagonal of chain complexes, so that $\mathcal{E}_n = C_*(\mathcal{D}_n)$ defines an operad in the monoidal category of differential graded coassociative coalgebras.

For such operads, called Hopf-operads,
we have a well-defined notion of a coassociative Hopf-algebra:
a Hopf-algebra over a Hopf-operad $\mathcal{P}$ consists of a coalgebra Γ together with a $\mathcal{P}$-algebra structure such that the diagonal $\Delta : \Gamma \to \Gamma \otimes \Gamma$ forms a morphism of $\mathcal{P}$-algebras, where $\mathcal{P}$ operates diagonally on $\Gamma \otimes \Gamma$.

We fix a Hopf-operad $\mathcal{Q}$ together with the following properties:

1. We have a coassociative diagonal

   $$\Delta : \mathcal{Q}(r) \to \mathcal{Q}(r) \otimes \mathcal{Q}(r),$$

   which provides $\mathcal{Q}$ with the structure of a Hopf-operad.

2. The operad $\mathcal{Q}$ has a filtration by sub-Hopf-operads

   $$* = \mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \cdots \subset \mathcal{Q}_n \subset \cdots \subset \mathcal{Q}_\infty = \mathcal{Q}$$

   such that $\mathcal{Q}$ is equivalent as a filtered Hopf-operad to the nested sequence of the chain operads of little cubes.

3. The operad embeddings $\mathcal{Q}_{n-1} \to \mathcal{Q}_n$ are operad cofibrations.

We let Alg_n, respectively BiAlg_n^a, denote the category of augmented algebras, respectively Hopf-algebras, over $\mathcal{Q}_n$. We have the following precise result:

Theorem 5: The bar complex of a $\mathcal{Q}_n$-algebra $B(A)$ can be endowed with the structure of a Hopf $\mathcal{Q}_{n-1}$-algebra, functorially in A, so that the bar construction of A_∞ algebras extends to a tower of functors

$$\begin{tikzcd}
\mathsf{Alg}_1 \arrow[d, "B"] & \cdots \arrow[l] & \mathsf{Alg}_n \arrow[l] \arrow[d, "B"] & \cdots \arrow[l] \\
\mathsf{CoAlg}^a & \cdots \arrow[l] & \mathsf{BiAlg}_{n-1}^a \arrow[l] & \cdots \arrow[l]
\end{tikzcd} .$$

The previous statement gives only an existence assertion. The next theorem implies that the induced algebra structure at $n = \infty$ gives always the right result and does not depend on particular choices up to homotopy equivalence.

Theorem 6:

The $\mathcal{Q}_\infty$-algebra structure deduced from theorem 5 satisfies automatically the requirement of the existence and uniqueness theorem 1. Namely, if A turns out to be a commutative algebra, then this $\mathcal{Q}_\infty$-algebra structure reduces to the classical commutative algebra structure of $B(A)$, the one defined by the shuffle product of tensors.

Definition: Let $\mathcal{P}$ denote an operad equipped with an operad morphism $\mathcal{K} \to \mathcal{P}$, where $\mathcal{K}$ is an A_∞ operad, so that any $\mathcal{P}$-algebra has an associated bar complex.

The endomorphism prop of the bar construction for $\mathcal{P}$-algebras is formed by the differential graded modules $\mathrm{End}_B^{\mathcal{P}}(r,s)$ defined by the collection of all natural transformations

$$\theta_A : B(A)^{\otimes r} \to B(A)^{\otimes s},$$

where A ranges over the category
of $\mathcal{P}$-algebras.

Recall that a prop is a structure which parametrizes operations with r inputs and s outputs $\theta : \Gamma^{\otimes r} \to \Gamma^{\otimes s}$ and which is associated to a category of bialgebras.

The category of Hopf-algebras over a Hopf operad $\mathcal{P}$ is associated to a prop $\mathcal{B}_{\mathcal{P}}$. In fact, an element of $\mathcal{B}_{\mathcal{P}}$ is a formal composite multiple operation

$$\Gamma^{\otimes r} \xrightarrow{\Delta} \Gamma^{\otimes N} \xrightarrow{\sigma} \Gamma^{\otimes N} \xrightarrow{\sqcap} \Gamma^{\otimes s},$$

where

- $\Delta = \Delta_1 \otimes \cdots \otimes \Delta_r \in \mathcal{A}(m_1)^{\vee} \otimes \cdots \otimes \mathcal{A}(m_r)^{\vee}$ denotes an r-fold iterated associative coproduct,

- $\sigma \in \Sigma_N$ denotes a tensor permutation,

- $\sqcap = p_1 \otimes \cdots \otimes p_s \in \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_s)$ denotes an s-fold operadic operation.

The endomorphism prop of the bar construction for $\mathcal{P}$-algebras is the universal prop which operates on the bar construction of a $\mathcal{P}$-algebra A functorially in A. Hence, theorem 6 is a consequence of the following lemma:

Lemma: Let $\mathcal{B}_n^\wedge$ be the prop of connected Hopf $\mathcal{Q}_n$-algebras. Let $\mathrm{End}_B^{Q_n}$ be the endomorphism prop of the bar construction for $\mathcal{Q}_n$-algebras. We have a tower of prop morphisms

$$\begin{tikzcd}
\mathcal{B}_0^\wedge \arrow[r] \arrow[d, "\nabla_1"] & \mathcal{B}_1^\wedge \arrow[r] \arrow[d, "\nabla_2"] & \cdots \arrow[r] & \mathcal{B}_{n-1}^\wedge \arrow[r] \arrow[d, "\nabla_n"] & \cdots \\
\mathrm{End}_B^{Q_1} \arrow[r] & \mathrm{End}_B^{Q_2} \arrow[r] & \cdots & \mathrm{End}_B^{Q_n} \arrow[l] \arrow[r] & \cdots
\end{tikzcd}.$$

The next theorem proves that our construction gives all the structure of the bar construction:

Theorem 7:
In a sequence of prop morphisms as in the previous lemma, the morphism ∇_n defines automatically a weak-equivalence of props

$$\nabla_n : \mathcal{B}^{\wedge}_{n-1} \xrightarrow{\sim} \mathrm{End}^{\mathcal{Q}_n}_B,$$

for $n = 1, 2, \ldots, \infty$.

The main ingredient in the proof of this theorem is supplied by the following lemma:

Lemma: Let $A = \mathcal{Q}_n(V)$ denote the free $\mathcal{Q}_n$-algebra generated by a module V. We have a quasi-isomorphism

$$\mathcal{Q}_{n-1}(\Sigma V) \xrightarrow{\sim} B(\mathcal{Q}_n(V)),$$

where $\mathcal{Q}_{n-1}(\Sigma V)$ is the free $\mathcal{Q}_{n-1}$-algebra generated by the suspension of V.

§3. The iterated bar complex of an E_n-algebra

Our construction allows to define an iterated bar construction $B^n(A) = B(\dots B(B(A))\dots)$ whenever A is equipped with an E_n-algebra structure.

The next theorem provides a homotopical interpretation for the resulting chain complex $B^n(A)$:

Theorem 8: Let A be an algebra over some E_n-operad $\mathcal{E}_n$.
The complex $B^n(A)$ determines the Quillen homology of A in the category of $\mathcal{E}_n$-algebras.
More precisely, the homology of the iterated bar complex agrees with this homology theory $H_*^{E_n}$ up to a shift of degree, so that

$$H_*(B^n(A)) = H_{*-n}^{E_n}(A).$$

The Quillen homology of $\mathcal{E}_n$-algebras $H_*^{E_n}$ is defined by the left derived functor of $\mathsf{Indec}_{\mathcal{E}_n}$, the indecomposable functor on the category of $\mathcal{E}_n$-algebras.

Let us recall that

$$\mathsf{Indec}_{\mathcal{E}_n} : \mathcal{E}_n\,\mathsf{Alg} \to \mathsf{dg}\,\mathbf{F}\,\mathsf{Mod}$$

denotes the left-adjoint of the canonical functor

$$(-)_+ : \mathsf{dg}\,\mathbf{F}\,\mathsf{Mod} \to \mathcal{E}_n\,\mathsf{Alg}$$

which maps a dg-module V to the $\mathcal{E}_n$-algebra $V_+ = \mathbf{F} \oplus V$ equipped with a trivial $\mathcal{E}_n$-algebra structure.

By definition, for a given $\mathcal{E}_n$-algebra A, we have

$$H_*^{E_n}(A) = H_*(\mathsf{Indec}_{\mathcal{E}_n} F_A),$$

for any chosen cofibrant $\mathcal{E}_n$-algebra F_A equivalent to A.

As a corollary, we obtain the following theorem:

Theorem 9: We assume that X is a pointed nilpotent space whose homotopy groups are degreewise finitely generated.
We let $R_\infty X$ denote Bousfield-Kan' p completion of X.
We have

$$H^{En}_{-n}(C^*(X)) = H^0 = \mathrm{Fun}(\pi_n(R_\infty X)^\wedge_p, \mathbf{F}_p),$$

the module of maps $\alpha : \pi_n(R_\infty X) \to \mathbf{F}_p$ which are continuous in regard to the p-profinite topology and

$$H^{En}_{*-n}(C^*(X)) = H^0 \otimes H^{-*}(\Omega^n_0 R_\infty X, \mathbf{F}_p),$$

where $\Omega^n_0 R_\infty X$ denotes the connected component of the base point of $\Omega^n R_\infty X$.

(Compare with results of Smirnov and Po Hu.)

Thank you for your attention!
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