On the combinatorial structure of E_n-operads

Clemens Berger

We describe a Quillen adjunction between the category of reduced n-operads and the category of reduced symmetric operads. For $n = 1$, this is the well-known adjunction between non-symmetric and symmetric operads. For higher n, reduced n-operads may be considered as operads endowed with partial symmetry. The main theorem states that the total left derived functor of this adjunction takes the terminal reduced n-operad to an E_n-operad.

The talk is based on joint work in progress with Ieke Moerdijk and Michael Batanin.

On the combinatorial structure of E_n-operads

ALPINE WORKSHOP ON OPERADS
Villars, 6-9 March 2006.

§1. E_n-operads

§2. Δ and Ω

§3. Iterated wreath product over Δ

§4. n-operads versus E_n-operads.

joint work with
M. Batanin and I. Moerdijk.

$\boxed{\text{§1. } E_n\text{-operads}}$

$(X,*)$ based space

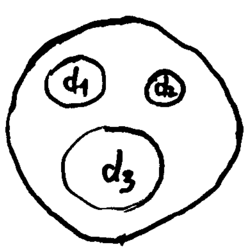
$\Omega^n X = \mathrm{Top}_*(S^n, X)$ n-fold loop space

$(\Omega^n X)^k = \mathrm{Top}_*(\underbrace{S^n\vee\cdots\vee S^n}_{k}, X)$

Operad-action $\mathcal{O}_n(k)\times(\Omega^n X)^k \longrightarrow \Omega^n X$

for $\mathcal{O}_n(k) = \mathrm{Top}_*(S^n, \underbrace{S^n\vee\cdots\vee S^n}_{k})$

$\mathcal{D}_n(k)\subset \mathcal{O}_n(k)$ suboperad of "little n-disks"



$\ni \mathcal{D}_2(3) \hookrightarrow \mathrm{Top}_*(S^2, \underbrace{S^2\vee S^2\vee S^2})$

$S^2 = D^2/\partial D^2 \twoheadrightarrow D^2/D^2-(d_1\cup d_2\cup d_3) = S^2\vee S^2\vee S^2$

$\mathcal{D}_\infty(k) = \lim\limits_{n\to\infty} \mathcal{D}_n(k)$

$\boxed{\text{Thm}}$ (Boardman-Vogt, May, Segal) Let $1\le n\le\infty$.
Each n-fold loop space is canonically a $\mathcal{D}_n$-algebra.
Conversely, each connected $\mathcal{D}_n$-algebra is weakly equivalent to an n-fold loop space.

$\mathcal{D}_\infty \xrightarrow{\sim} Com$ $\quad\quad$ $\left.\begin{array}{l}\mathcal{D}_\infty(k) \xrightarrow{\sim} * \\ \mathfrak{S}_k\text{-action free}\end{array}\right\}$ E_∞-operad

$\mathcal{D}_1 \xrightarrow{\sim} Ass$ $\quad\quad$ $\left.\begin{array}{l}\mathcal{D}_1(k) \xrightarrow{\sim} \mathfrak{S}_k \\ \mathfrak{S}_k\text{-action free}\end{array}\right\}$ E_1-operad

$\boxed{\text{Pbm}}$ Intrinsic characterization of E_n-operads for $1 < n < \infty$

$\rightsquigarrow$ "combinatorial structure" of an n-fold loop space.

$\boxed{\text{Rmk 1}}$ For $n=2$, Fiedorowicz has obtained a characterization (replace symmetric groups by braid groups).

$\boxed{\text{Rmk 2}}$ E_1-operad = symmetrization of a non-symmetric "A_∞-operad"

Com = terminal symmetric operad

Ass = terminal non-symmetric operad

E_∞ = "flat" resolution of Com.

A_∞ = "flat" resolution of Ass.

§2. Δ and Γ

$\boxed{\text{Def.}}$ $\mathrm{Ob}\,\Delta = \{\text{finite non-empty ordinals}\}$
$\mathrm{Mor}\,\Delta = \{\text{order-preserving maps}\}$

$[n] = \{0<1<2<\cdots<n\}$, $n \geq 0$.

$\mathrm{Ob}\,\Gamma = \{\text{finite sets}\}$ $\quad \underline{n} = \{1,2,\ldots,n\}$
$\underline{0} = \emptyset$

$\mathrm{Mor}\,\Gamma$: $\Gamma(\underline{m},\underline{n}) = \{m\text{-tuples of pairwise disjoint subsets of } \underline{n}\}$

$$\varphi : \underline{m} \longrightarrow \underline{n}$$
$$i \longmapsto \varphi(i) \subseteq \underline{n} \qquad \boxed{i \neq j \Rightarrow \varphi(i) \cap \varphi(j) = \emptyset}$$

$$\gamma : \Delta \longrightarrow \Gamma$$
$$(\varphi : [m] \to [n]) \longmapsto \left(\gamma(\varphi) : \begin{array}{l} \underline{m} \longrightarrow \underline{n} \\ i \longmapsto]\varphi(i-1), \varphi(i)] \end{array}\right)$$

$\boxed{\text{Def.}}$ A Δ-space $X : \Delta^{op} \to \mathrm{Top}$ reduced iff $X([0]) = *$
A Γ-space $X : \Gamma^{op} \to \mathrm{Top}$ reduced iff $X(\underline{0}) = *$

$\boxed{\text{Thm}}$ (Segal, Bousfield-Friedlander, Anderson)
Each 1-fold (infinite) loop space can be modelled by a "cofibrant-fibrant" reduced Δ- (Γ-) space, and conversely: each "cofibrant-fibrant" reduced Δ- (Γ-) space gives rise to a 1-fold (infinite) loop space.

$\boxed{\text{Proof-sketch 1}}$ (May-Thomason, Vogt-Schwänzl)

E reduced symmetric operad $(E(0)=*)$

$\rightsquigarrow$ topologically enriched category over Γ:
$\Gamma_E \xrightarrow{\pi} \Gamma$ with $\mathrm{Ob}\,\Gamma_E = \mathrm{Ob}\,\Gamma$.

$$\Gamma_E(\underline{m},\underline{n}) = \coprod_{\varphi\in\Gamma(\underline{m},\underline{n})} E(\#\varphi(1))\times\cdots\times E(\#\varphi(m))$$

operad structure of $E \leftrightarrow$ enriched category structure of Γ_E

For an E_∞-operad E, one has

$$(E\text{-algebras}) \xleftarrow{\sim} (\text{red. } \Gamma_E\text{-spaces}) \xrightarrow{\sim} (\text{red. } \Gamma\text{-spaces})$$

uses some "cofibrancy"-condition on E (free Σ_k-actions)

uses $E(k)\sim *$

Similarly, for an A_∞-operad A, one has

$$(A\text{-algebras}) \xleftarrow{\sim} (\text{red. } \Delta_A\text{-spaces}) \xrightarrow{\sim} (\text{red. } \Delta\text{-spaces})$$

where $\Delta_A \xrightarrow{\pi} \Delta$ and $\mathrm{Ob}\,\Delta_A = \mathrm{Ob}\,\Delta$

$$\Delta_A([m],[n]) = \coprod_{\varphi\in\Delta([m],[n])} A(\#]\varphi(0),\varphi(1)])\times\cdots\times A(\#]\varphi(m-1),\varphi(m)])$$

$\boxed{\text{Rmk}}$ pullback diagram of top. enriched categories:

$$\begin{array}{ccc} \Delta_{\gamma^*E} & \longrightarrow & \Gamma_E \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \Delta & \xrightarrow{\gamma} & \Gamma \end{array}$$

where γ^*: (symm. operads) $\to$ (non-symm. operads) forgetful functor

γ^* has a left adjoint $\gamma_!$

$\boxed{\text{Pbm}}$ Characterize those categories over Γ (resp. Δ) which correspond to symmetric (resp. non-symmetric) operads:

$$id_\Gamma : \Gamma \to \Gamma \quad \text{corresponds to Com}$$
$$id_\Delta : \Delta \to \Delta \quad \text{corresponds to Ass}$$

Factorization systems : $\Gamma = (\Gamma_{cov}, \Gamma_0)$
$\Delta = (\Delta_{cov}, \Delta_0)$

$\Gamma_0 = \{ \varphi \in \Gamma \mid \# \varphi(i) = 1 \ \forall i \} \cong \{\text{injections}\}$

$\Gamma_{cov} = \{ \varphi \in \Gamma \mid \bigcup_{i \in \text{dom}(\varphi)} \varphi(i) = \text{codom}(\varphi) \}$

$\Delta_0 = \{ \varphi \in \Delta \mid \varphi(i+1) = \varphi(i) + 1 \ \forall i \} \cong \{\text{graph-inclusions}\}$

$\Delta_{cov} = \{ \varphi \in \Delta \mid \varphi \text{ endpoint-preserving} \}$

$\Gamma_0 \cap \Gamma_{cov} = \{\text{bijections}\}$ $\quad \Delta_0 \cap \Delta_{cov} = \{\text{identities}\}$

(reduced Γ-space X sth. $X/_{\Gamma_0}$ sheaf) $\equiv$ abelian top. monoid

(reduced Δ-space X sth. $X/_{\Delta_0}$ sheaf) $\equiv$ top. monoid

Let $\overline{\Pi}$ be a top. enriched category together with

(i) $\Pi_0 \to \overline{\Pi} \xrightarrow{\pi} \Pi$ (bijective on objects);

(ii) $\overline{\Pi} = (\overline{\Pi}_{\infty}, \Pi_0)$ π-compatible;

(iii) the representable presheaves on $\overline{\Pi}$ are Π_0-sheaves;

then. $\overline{\Pi} = \Pi_E$ for a __symmetric__ operad E.

Similarly $\Delta_0 \to \overline{\Delta} \xrightarrow{\pi} \Delta$ satisfying (i), (ii), (iii) comes from a __non-symmetric__ operad A: $\overline{\Delta} = \Delta_A$.

__Corollary.__ There exists a __simplicial resolution functor__ for each $\Pi_E \xrightarrow{\pi} \Pi$ (resp. $\Delta_A \xrightarrow{\pi} \Delta$), denoted $\Pi_0 \to W_{\bullet}\Pi_E \xrightarrow{\pi} \Pi$ (resp. $\Delta_0 \to W_{\bullet}\Delta_A \xrightarrow{\pi} \Delta$)

__Theorem__ (I. Moerdijk & C.B)

$|W_{\bullet}\Pi_E| \cong \Pi_{WE}$, where WE is __Boardman-Vogt's W-construction__.

$|W_{\bullet}\Delta_A| \cong \Delta_{WA}$

__Theorem__ (I.M & C.B.)

For each Σ-cofibrant symmetric operad E, the Boardman-Vogt-construction WE is a __cofibrant resolution__ of E.

③ Iterated wreath-product over Δ

$\mathcal{A}$ small category $\rightsquigarrow$ $\Delta\int\mathcal{A}$ small category

$\boxed{\text{Def}}$ $Ob(\Delta\int\mathcal{A}) = \coprod_{k\geq 0} (Ob\mathcal{A})^k$

$Mor(\Delta\int\mathcal{A})$: $(A_1, \ldots, A_k) \xrightarrow{(\varphi; \varphi_1, \ldots, \varphi_k)} (B_1, \ldots, B_m)$

where $\varphi : [k] \to [m]$ in Δ

and $\varphi_i : \mathcal{A}[A_i] \longrightarrow \mathcal{A}[B_{\varphi(i-1)+1}] \times \cdots \times \mathcal{A}[B_{\varphi(i)}]$

$\varphi_i \leftrightarrow (\varphi_i^j : A_i \longrightarrow B_j)_{j \in]\varphi(i-1), \varphi(i)]}$

$Ob(\Gamma\int\mathcal{A}) = Ob(\Delta\int\mathcal{A})$

$Mor(\Gamma\int\mathcal{A})$: $(A_1, \ldots, A_k) \xrightarrow{(\varphi; \varphi_1, \ldots, \varphi_k)} (B_1, \ldots, B_m)$

where $\varphi : \underline{k} \longrightarrow \underline{m}$ in Γ

and $\varphi_i : \mathcal{A}[A_i] \longrightarrow \prod_{j \in \varphi(i)} \mathcal{A}[B_j]$

$\boxed{\text{Rmk}}$ For any $F : \mathcal{A} \to \mathcal{B}$, one has a functor

$\delta\int F : \Delta\int\mathcal{A} \xrightarrow{\delta\int\mathcal{A}} \Gamma\int\mathcal{A} \xrightarrow{\Gamma\int F} \Gamma\int\mathcal{B}$

$\boxed{\text{Rmk}}$ One has a canonical functor

$\alpha : \Gamma\int\Gamma \longrightarrow \Gamma$

$(\underline{k_1}, \ldots, \underline{k_m}) \longmapsto (\underline{k_1 + \cdots + k_m})$

$\boxed{\text{Def}}$ $\Theta_1 = \Delta \qquad \gamma_1 = \gamma : \Delta \to \Gamma$

$\Theta_n = \Delta \int \Theta_{n-1}$

$$\begin{tikzcd} \Theta_n = \Delta\int\Theta_{n-1} \arrow[r, "\gamma\int\gamma_{n-1}"] \arrow[dr, "\gamma_n"'] & \Gamma\int\Gamma \arrow[d, "\alpha"] \\ & \Gamma \end{tikzcd}$$

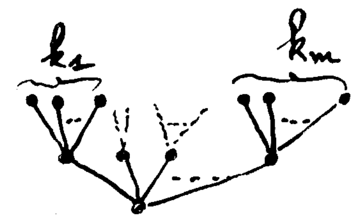
$\boxed{\text{Rmk}}$ The categories Θ_n^{op} have been first defined (in a completely different way) by Joyal (97).

$\boxed{\text{Prop.}}$ Suppose $\mathcal{A}$ admits a flat realization
$$|-|_{\mathcal{A}} : \mathcal{A} \to \mathrm{Top}$$
Then $\Delta\int\mathcal{A}$ admits a canonical reduced flat realization $|-|_{\Delta\int A} : \Delta\int\mathcal{A} \to \mathrm{Top}$.

In particular, for each $n \geq 1$, there is a flat realization $|-|_{\Theta_n} : \Theta_n \to \mathrm{Top}$.

<u>Tree-description of the objects of Θ_n.</u>

$\Theta_1 = \Delta \qquad [m] \longleftrightarrow$  $\qquad [0] \longleftrightarrow \bullet$

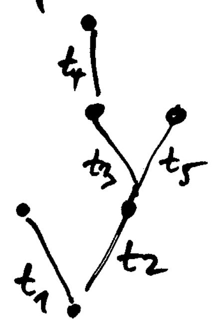
$\Theta_2 = \Delta\int\Delta \qquad ([k_1], \dots, [k_m]) \longleftrightarrow$ 

$\vdots$

$\mathrm{Ob}\,\Theta_n = \{$ planar level-trees of height $\leq n \}$

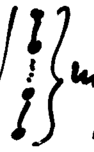
J

Description of $|-|_{\Theta_n} : \Theta_n \to \mathrm{Top}$



realises to $\{(t_i)\in[-1,1]^5$ s.th. $t_1 \le t_2$
$t_3 \le t_5$
$t_2^2+t_3^2+t_4^2 \le 1$
$t_2^2+t_5^2 \le 1\}$

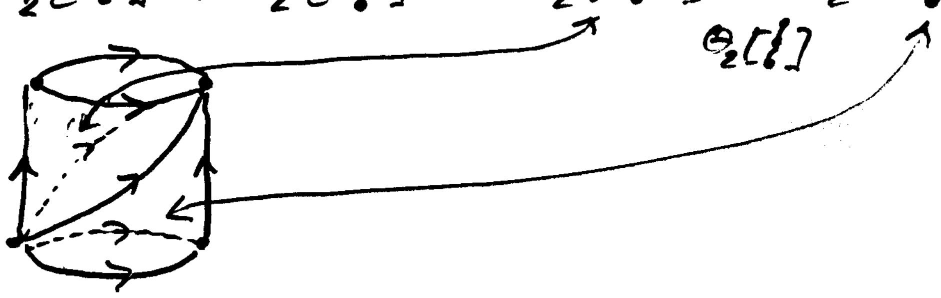
convex subset of $[-1,1]^{\#}$.

In particular $|$  $| \cong \Delta_m$ $\quad |$  $| \cong B^m$

Important consequence of flatness: shuffle-formula

$$\Theta_n[S]\times\Theta_n[T] = \bigcup_{U\in \mathrm{shuff}(S,T)} \Theta_n[U]$$

Example: $\Theta_2[$  $]\times\Theta_2[$  $] = \Theta_2[$  $] \cup_{\Theta_2[\,]} \Theta_2[$  $]$



$\boxed{n=1}$ $\quad \Delta[m]\times\Delta[n] = \bigcup_{\frac{(m+n)!}{m!\,n!}} \Delta[m+n]$

Proposition: Let $A : \Gamma^{op} \to \mathrm{Top}$; $\gamma_n : \Theta_n \to \Gamma$
Then $|\gamma_n^* A|_{\Theta_n}$ is homeomorphic to the n-th space of the Segal-spectrum of A.

§4. n-operads versus E_n-operads

Introduced by Batanin 96.
(definition of weak n-categories)

$$(\text{n-operads}) \longleftrightarrow \left(\begin{array}{c}\text{special}\\ \text{monads on } Sets^{\Theta_n^{op}}\end{array}\right)$$

terminal n-operad ω_n has strict n-categories as algebras.

Def. An n-operad A is a monad on $Sets^{\Theta_n^{op}}$ equipped with a cartesian nat. transformation of monads to ω_n.

Prop. Let $U_n : nCat \rightleftarrows Sets^{\Theta_n^{op}} : F_n$ the free-forgetful adjunction. Then, for any two level-trees $S, T \in Ob\,\Theta_n$, one has

$$\Theta_n(S,T) \cong nCat(F_n(S_*), F_n(T_*))$$

Thm. There is a one-to-one correspondence between n-operads A and certain categories Θ_n^A over Θ_n with $Ob\,\Theta_n^A = Ob\,\Theta_n$.

Corollary. There is an Quillen adjunction

$$f_n^* : \left(\begin{array}{c}\text{reduced}\\ \text{symm. operads}\end{array}\right) \rightleftarrows \left(\begin{array}{c}\text{reduced}\\ \text{n-operads}\end{array}\right) : (f_n)_!$$

Thm. (Batanin) The left derived functor $L(f_n)_!$ maps the terminal reduced n-operad to a cofibrant E_n-operad.

$$\begin{array}{l} \Theta_{n,0} \\ \downarrow \\ \Theta_n^A \\ \downarrow \pi \\ \Theta_n \end{array}$$

Θ_n^A corresponds to an <u>n-operad</u> A

family $(A(S))_{S \text{ level-tree of height} \leq n}$

A <u>reduced</u> iff $A(S) = *$ for S s.th. $ht(S) < n$

- existence of <u>unit</u> $\in A(\,\vdots\,)$

- unitary associative <u>compositions</u>

$$A(S) \times A(T_1) \times \cdots \times A(T_k) \longrightarrow A(T)$$

for each cover $S \to T$ in Θ_n.

$\boxed{n=1}$ $\Theta_1 = \Delta$

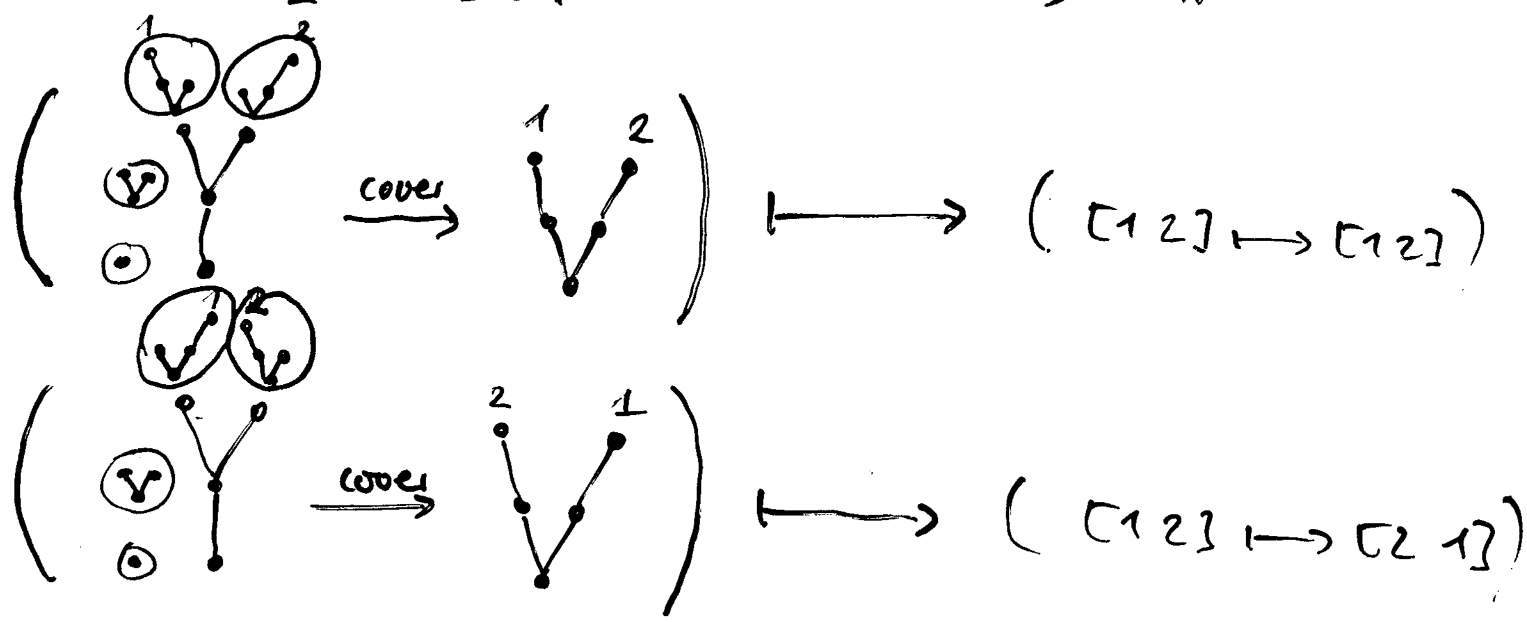
$$A([m]) \times A([k_1]) \times \cdots \times A([k_m]) \longrightarrow A([k_1 + \cdots + k_m])$$

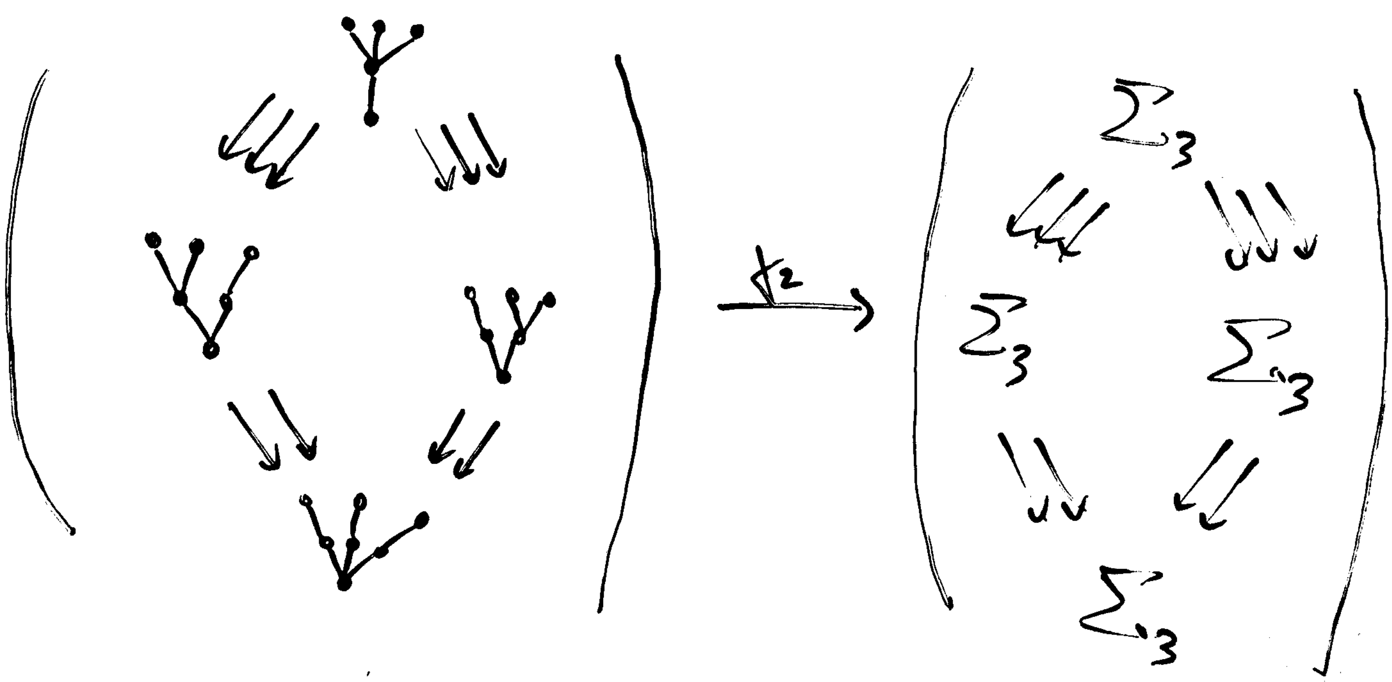
for each cover $[m] \to [k_1 + \cdots + k_m]$

$$\begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto k_1 \\ 2 \mapsto k_1 + k_2 \\ \vdots \\ m \mapsto k_1 + \cdots + k_m \end{array}$$

$A \leadsto$ non-symmetric operad

$\boxed{n=2}$ $\Theta_2 = \Delta S \Delta \xrightarrow{\gamma_2} K$





The Grothendieck construction of this functor Cat $\to$ Sets gives the Fox-Neuwirth model of $F(\mathbb{R}^2, 3)$.
This shows that $(\gamma_2 !)(W\Theta_2)$ has the homotopy type of an E_2-operad.