

On the combinatorial structure of E_n -operads

Clemens Berger

We describe a Quillen adjunction between the category of reduced n -operads and the category of reduced symmetric operads. For $n = 1$, this is the well-known adjunction between non-symmetric and symmetric operads. For higher n , reduced n -operads may be considered as operads endowed with partial symmetry. The main theorem states that the total left derived functor of this adjunction takes the terminal reduced n -operad to an E_n -operad.

The talk is based on joint work in progress with Ieke Moerdijk and Michael Batanin.

Operads and multicategories

Eugenia Cheng

Multicategories are a generalisation of operads in a categorical setting. In this talk I will give an introduction to and overview of some of the types of multicategory that have been defined. Where the morphisms of a category have one input and one output, the morphisms of a multicategory have multiple inputs and one output, so can be used as a formalism for operations of arities other than 1, such as composition. A further generalisation allows the inputs to have some sort of “shape” given by a cartesian monad. This leads to operads and multicategories that give a natural formalism for describing more complicated forms of composition.

The iterated bar construction and little cubes homology

Benoît Fresse

We consider the classical reduced bar construction of associative algebras $B(A)$. If the product of A is commutative, then $B(A)$ can be equipped with the classical shuffle product, so that $B(A)$ is still a commutative algebra. This assertion can be generalized for algebras which are commutative up to homotopy. Namely, one observes that the bar construction $B(A)$ can be endowed with the structure of an associative algebra, if A is an E_2 -algebra, with the structure of an E_2 algebra, if A is an E_3 -algebra, ... and with the structure of an E_∞ -algebra, if A is an E_∞ -algebra. In this talk, we prove that this observation is a consequence of a more precise results on the endomorphism prop of the bar construction, the structure that parametrizes all natural operations on $B(A)$, for A an E_n -algebra. Explicitly: this prop is equivalent to the prop of coassociative- $E_{(n-1)}$ -bialgebras.

Our results allow to iterate the bar construction for algebras equipped with an E_n -structure. We prove that the n^{th} iterated bar complex defines Quillen's homology in the category of E_n -algebras

An operad coming from representation theory

André Henriques

Representations of quantum groups form a braided monoidal category. And the structure of such categories is controlled by the braid operad, also called E_2 operad. We discuss a similar story where quantum group representations are replaced by "crystals", and where the braid groups are replaced by the fundamental groups of some compact real algebraic varieties.

Given a semi-simple Lie algebra g , a g -crystal is a combinatorial object that resembles very much a representation of g . In particular, the isomorphism classes of g -crystals are in bijection with isomorphism classes of g -representations. Just like g -reps, g -crystals form a monoidal category, and the Grothendieck construction applied to these two categories gives isomorphic rings.

But unlike g -reps, g -crystals do not form a symmetric monoidal category: the commutor $\sigma_{A,B} : A \otimes B \rightarrow B \otimes A$ doesn't satisfy the required axioms. It satisfies $\sigma \circ \sigma = 1$, but fails the hexagon axiom.

Instead, crystals form a coboundary category (terminology introduced by Drinfeld). The structure of coboundary categories is controlled by the following topological operad: its n -th space is the set of real points of the operad of stable curves of genus zero with $n+1$ marked points. The n th space in this operad is denoted $\bar{M}_{0,n+1}(\mathbb{R})$, it is a $K(\pi, 1)$, and the corresponding group can be given by generators and relations.

If V is a quantum group representation, the braid group B_n acts on $V^{\otimes n}$. Similarly we have an action of $\pi_1(\bar{M}_{0,n+1}(\mathbb{R}))$ on iterated tensor products of crystals.

(Co)formality of the little ball operad and the rational homotopy type of spaces of long knots

Pascal Lambrechts

(Joint works with V. Turchin, I. Volic, and Greg Arone) Sinha has shown how the space of long knots can be expressed as the totalisation of the cosimplicial space associated to a certain multiplicative operad equivalent to the little ball operad. We show how to use this result to deduce the collapsing of the Vassiliev spectral sequence computing the rational homology of the space of long knots of codimension > 2 . This completely determines the rational homotopy type of long knots in codimension > 2 . The proof involves an interesting new construction that associates to any morphism of operad a diagram generalizing the cosimplicial object usually associated to a multiplicative operad. We also prove the "coformality" of the little ball operad and deduce the collapsing of a "dual" spectral sequence computing the rational homotopy of long knots. Some results pass in codimension 2.

The associative operad and the weak order on the symmetric group

Muriel Livernet

The associative operad is a certain algebraic structure on the sequence of group algebras of the symmetric groups. The weak order is a partial order on the symmetric group. There is a natural linear basis of each symmetric group algebra, related to the group basis by Möbius inversion for the weak order. We describe the operad structure on this second basis: the surprising result is that each operadic composition is a sum over an interval of the weak order. The Lie operad, a suboperad of the associative operad, contains some idempotent such as the Dynkin's idempotent. As a corollary to our results, we derive a simple explicit expression for Dynkin's idempotent in terms of the second basis.

There are combinatorial procedures for constructing a planar binary tree from a permutation, and a composition from a planar binary tree. These define set-theoretic quotients of each symmetric group algebra. They are operad quotients of the associative operad and we can decline the results of the first section to these contexts.

The talk is based on joint work with M. Aguiar.

Cohomology operations and the Deligne conjecture

Martin Markl

The aim of the talk, which raises more questions than it answers, will be to study natural operations acting on the cohomology of various types of algebras. It will contain a lot of very surprising partial results and examples.

An old-fashioned elementary talk

Peter May

I will explain an easy conceptual theorem that every expert on operads should know, but that does not seem to be in the literature in its general form. Let L be the cosimplicial chain complex given by the simplicial chains of the standard simplices. The Eilenberg-Zilber operad EZ is the endomorphism operad of L . Let F be a cosimplicial commutative DGA. Then the cochain complex $Hom(L, F)$ is an EZ -algebra. That's the "theorem".

Working $mod p$, this gives Steenrod operations on $H^*(Hom(L, F))$. Working rationally, $Hom(L, F)$ is equivalent to a commutative DGA. As I will explain, the theorem specializes to give the action of EZ on the singular cochains of spaces, on the Čech (or Godement) cochains of sheaves, and on the cobar construction of cocommutative Hopf algebras. The ideas illustrate a general methodology, no easy variant of which can work to recover Voevodsky's Steenrod operations in motivic cohomology.

Euclidean knots and double loop spaces

Paolo Salvatore

We show that the space of long knots in a euclidean space of dimension larger than three is a double loop space. The proof is based upon the operadic approach of Dev Sinha. We show also that the forgetful map from framed long knots to long knots is not a double loop map, but that there is a double loop map going the other way around.

Relative homotopy cyclic homology

Jérôme Scherer

This is joint work with David Chataur. In previous work with J.L. Rodriguez we constructed a functorial plus-construction in the category of differential graded algebras over a cofibrant operad. The plus-construction applied to $gl(A)$ in the category of homotopy Lie algebras is of special interest and we defined homotopy cyclic homology groups as the homotopy groups of $gl(A)^+$. Over the rationals they coincide with usual cyclic homology groups by work of M. Livernet. We computed these groups in low degrees over any field.

In this talk I will recall the main results in the absolute case and then focus on a relative version. Given an ideal I in A , the homotopy fiber of the map $gl(A)^+ \rightarrow gl(A/I)^+$ yields a long exact sequence in homotopy cyclic homology. We identify the relative groups in low degree, just like relative algebraic K -theory groups were identified by Loday in the late 70's.

The duality between Lie and Comm revisited

Dev Sinha

Almost forty years ago, Quillen defined adjoint functors between 1-connected differential graded Lie algebras and cocommutative coalgebras, which established the equivalence of their homotopy categories. This result is now understood as a Koszul duality between the Lie and commutative operads. We use a combinatorially interesting dual to the Lie operad to construct corresponding functors between 1-connected differential graded algebras and Lie coalgebras. One corollary is that “graph homology on the cochains of X computes the linear dual to homotopy of X .” The main application is to defining homotopy periods using cochain data.

Manin products and Koszul duality patterns

Bruno Vallette

Black and white products of quadratic algebras were introduced by Y. Manin in his works on quantum groups and non-commutative geometry. Koszul duality theory for algebras relates these two constructions : the Koszul dual of the product of two algebras is the other product of the Koszul dual algebras. In their seminal paper on Koszul duality for operads, V. Ginzburg and M. Kapranov generalized Manin's constructions to operads. Notice that these generalizations are not straightforward and that these products have not been studied much further. Recently, other Koszul duality theories were proved for colored operad (Van der Laan), dioperads (Gan) and properads-props (BV).

In this talk, we aim to describe first the general framework for Koszul duality theories (for instance, the Koszul dual to consider is a comonoid and not a monoid and its construction only involves universal algebra). Then, we will define the black and white products in any 2-monoidal categories. Hence, we get explicit constructions of these products for operads, regular operads and properads. We will conclude by the proof of a conjecture of M. Aguiar and J.-L. Loday.

Applications of 2-categorical algebra to the theory of operads

Mark Weber

With new, more combinatorially intricate notions of "operad" arising recently in the algebraic approaches to higher dimensional algebra, it has become desirable to clarify how these new notions relate to the usual concept. In this talk it is explained how to use parts of 2-dimensional category theory, in particular a 2-categorical version of topos theory and the formal theory of monads, to give both a unified treatment of the theory of operads and a formalism within which the new and established notions of operad can be made to interact.