Groupoid Representations for Generalized Quotients
Part 2: Morita Equivalence, Foliations, and Orbifolds

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Effective Descent and Groupoids

If we write \( \text{Sh}(Y) \simeq \text{Sh}(X) \times_{\mathcal{E}} \text{Sh}(X) \) and

\[
\begin{array}{c}
\text{Sh}(Y) \xrightarrow{t} \text{Sh}(X) \\
\downarrow^{s} \quad \simeq \quad \downarrow^{\pi_X} \\
\text{Sh}(X) \xrightarrow{\pi_X} \mathcal{E}
\end{array}
\]

and we write \( u \) for \( \delta \), the truncated simplicial topos becomes

\[
Y \times_{s,X,t} Y \xrightarrow{\pi_1} Y \xrightarrow{s} X \leftarrow u X
\]

\( i.e. \), a localic groupoid.

[Butz-Moerdijk] If \( \mathcal{E} \) has enough points, we can get a topological groupoid with the same properties.
Description of $\mathcal{E}$ in terms of $\mathcal{G}$

Let $\mathcal{G}$ be a topological groupoid with source and target maps open or proper surjections.

Definition

- A **$\mathcal{G}$-space** is a space $p: X \rightarrow G_0$ with a left action by $G_1$,

  \[ \alpha: G_1 \times_{s, G_0, p} X \rightarrow X, \quad \alpha(g, x) = g \cdot x, \]

  such that

  - $p(g \cdot x) = t(g)$;
  - $g_1 \cdot (g_2 \cdot x) = (g_1g_2) \cdot x$ (where $g_1g_2$ is composition in $\mathcal{G}$) (cocycle condition);
  - $u(p(x)) \cdot x = x$ (unit condition).

- An **(equivariant) $\mathcal{G}$-sheaf** is a local homeomorphism $p: X \rightarrow G_0$ with a left $G_1$-action.

Remark

We could also have given $\alpha: G_1 \times_{s, G_0, p} X \sim X \times_{p, G_0, t} G_1$. 
The Topos Sh(\mathcal{G})

- A morphism \( \varphi : E \to E' \) between \( \mathcal{G} \)-sheaves is a morphism of spaces over \( G_0 \),

\[
\begin{array}{ccc}
E & \xrightarrow{\varphi} & E' \\
\downarrow p & & \downarrow p' \\
G_0 & & G_0
\end{array}
\]

that respects the \( \mathcal{G} \)-action, \( \varphi(g \cdot x) = g \cdot \varphi(x) \).

- The category of \( \mathcal{G} \)-sheaves forms a Grothendieck topos \( \text{Sh}(\mathcal{G}) \). We also write \( \text{Sh}_G(X) \) for \( \text{Sh}(G \ltimes X) \).
Etale Complete Groupoids

Which groupoids can be obtained in this fashion?

- The source and target maps must be open or closed surjections.
- There is geometric morphism $\pi_{G_0} : \text{Sh}(G_0) \to \text{Sh}(G)$ with $\pi^*_{G_0} : \text{Sh}_G(X) \to \text{Sh}(G_0)$ the forgetful functor (that forgets the action).
- The groupoid $G$ is **étale complete** if the following square of toposes is a pullback:

\[
\begin{array}{ccc}
\text{Sh}(G_1) & \xrightarrow{t} & \text{Sh}(G_0) \\
\downarrow s & & \downarrow \pi_{G_0} \\
\text{Sh}(G_0) & \xrightarrow{\pi_{G_0}} & \text{Sh}(G)
\end{array}
\]
Etale Groupoids

- A groupoid is étale if both source and target maps are local homeomorphisms.
- For an étale groupoid, \( t: G_1 \to G_0 \) is an element of \( \text{Sh}(\mathcal{G}) \).
- We have that \( \text{Sh}(\mathcal{G})/(G_1 \to G_0) \cong \text{Sh}(G_0) \).
- Every étale groupoid is étale complete, that is, the following square is a weak pullback:

\[
\begin{array}{ccc}
\text{Sh}(G_0)/\pi^*_G(G_1) & \cong & \text{Sh}(G_1) \\
\downarrow s & & \downarrow t \\
\text{Sh}(G_0) & \cong & \text{Sh}(\mathcal{G})/G_1 \\
\downarrow \pi_G & & \downarrow \pi_G \\
\text{Sh}(G_0) & \xrightarrow{\pi_G} & \text{Sh}(\mathcal{G})
\end{array}
\]
A Site for $\text{Sh}(\mathcal{G})$

- A **bisection** of $\mathcal{G}$ consists of an open subset $U \subseteq G_0$ with a section $\sigma: U \rightarrow G_1$ of $s$ such that $V = t \circ \sigma(U) \subseteq G_0$ is open and $t \circ \sigma: U \sim V$.

- The **objects** of $C(\mathcal{G})$ are the domains of all possible bisections of $\mathcal{G}$.

- An **arrow** $(U, \sigma): U \rightarrow U'$ is a bisection $\sigma: U \rightarrow G_1$ such that $t \circ \sigma(U) \subseteq U'$.

- A family of arrows $(U_i, \sigma_i): U_i \rightarrow U$ is a **covering** if

$$\bigcup_{i \in I} t \circ \sigma_i(U_i) = U.$$
Etendues

Definition
A Grothendieck topos $\mathcal{E}$ is an étendue if it contains an epimorphism $U \rightarrow 1$, such that $\mathcal{E}/U$ is a topos of sheaves on a topological space, i.e., $\mathcal{E}/U \simeq \text{Sh}(X)$.

Remarks
- In our example above, we had
  $$(U \rightarrow 1) = (G_1 \xrightarrow{t} G_0).$$

- We can also describe étendues as toposes with a site with monic maps.
The 2-Category of Topological Groupoids

• A homomorphism \( \varphi: \mathcal{G} \rightarrow \mathcal{H} \) of topological groupoids is an internal functor in the category of topological spaces. It consists of continuous maps

\[
\varphi_0: G_0 \rightarrow H_0, \quad \varphi_1: G_1 \rightarrow H_1,
\]

which commute with all the structure maps.

• A 2-cell \( \alpha: \varphi \Rightarrow \psi: \mathcal{G} \Rightarrow \mathcal{H} \) is represented by a continuous map

\[
\alpha: G_0 \rightarrow H_1, \text{ such that } s \circ \alpha = \varphi \text{ and } t \circ \alpha = \psi
\]

and

\[
m(\psi_1(g), \alpha(s(g))) = m(\alpha(t(g)), \varphi_1(g)).
\]
Essential Equivalences of Groupoids

Definition
A morphism \( \varphi : \mathcal{G} \to \mathcal{H} \) of topological groupoids is an **essential equivalence** when it satisfies the following two conditions:

- \( \varphi \) is **essentially surjective on objects** in the sense that \( t \circ \pi_2 \) is an open surjection:

\[
\begin{array}{ccc}
G_0 \times_{H_0} H_1 & \overset{\pi_2}{\longrightarrow} & H_1 \\
\downarrow \pi_1 & & \downarrow s \\
G_0 & \overset{\varphi_0}{\longrightarrow} & H_0
\end{array}
\]

- \( F \) is **fully faithful** in the sense that the following diagram is a pullback:

\[
\begin{array}{ccc}
G_1 & \overset{\varphi_1}{\longrightarrow} & H_1 \\
(s,t) \downarrow & & \downarrow (s,t) \\
G_0 \times G_0 & \overset{\varphi_0 \times \varphi_0}{\longrightarrow} & H_0 \times H_0
\end{array}
\]
Equivalences of Toposes

Proposition
An essential equivalence $\varphi: \mathcal{G} \rightarrow \mathcal{H}$ of groupoids gives rise to an equivalence of categories

$$\varphi: \text{Sh}(\mathcal{G}) \sim \text{Sh}(\mathcal{H}).$$

Remark
This gives us one of the implications for our proposed Morita equivalence theorem.
Fibered Products

The 2-category of topological groupoids has both strong and weak fibered products. Let \( \varphi : G \to K \) and \( \psi : H \to K \).

- The strong fibered product \( G \times_K H \) has space of objects \( G_0 \times_{K_0} H_0 \) and space of arrows \( G_1 \times_{K_1} H_1 \).
- The weak fibered product \( G \times_K H \) has space of objects

\[
G_0 \times_{K_0} K_1 \times_{K_0} H_0 = \{(x, k, y) \mid \varphi(x) \xrightarrow{k} \psi(y)\}
\]

and the elements of the space of arrows with source \((x, k, y)\) and target \((x', k', y')\) are determined by pairs of arrows \( g \in G_1 \) and \( h \in H_1 \) such that \( k' \varphi(g) = \psi(h)k \) in \( K \).
Properties of Essential Equivalences

- The weak pullback of an essential equivalence along an arbitrary morphism of topological groupoids is again an essential equivalence.
- If \( \theta \) is an essential equivalence and there is a 2-cell \( \alpha : \theta \circ \varphi \Rightarrow \theta \circ \psi \) then there is a 2-cell \( \alpha' : \varphi \Rightarrow \psi \) such that \( \theta \circ \alpha' = \alpha \).
- The class of essential equivalences is closed under 2-isomorphisms.
- The class of essential equivalences admits a right calculus of fractions.
Localic Groupoid Representations

• (Moerdijk, 1988) For open étale complete localic groupoids, there is an equivalence of categories with isomorphism classes of maps:

\[ [\text{OEC-LocGrpds}][W^{-1}] \cong [\text{Toposes}] \]

• Essential step in the proof: show that for each geometric morphism \( \varphi : \text{Sh}(\mathcal{G}) \rightarrow \text{Sh}(\mathcal{H}) \) there exist an essential equivalence \( w : \mathcal{K} \rightarrow \mathcal{G} \) and a groupoid homomorphism \( f : \mathcal{G} \rightarrow \mathcal{H} \) such that \( \text{Sh}(f) \cong \varphi \circ \text{Sh}(w) \).

\[
\begin{array}{ccc}
\text{Sh}(K_0) & \xrightarrow{w_0} & \text{Sh}(H_0) \\
\downarrow \pi_{H_0} & & \downarrow \pi_{H_0} \\
\text{Sh}(G_0) & \xrightarrow{\varphi} & \text{Sh}(\mathcal{H}) \\
\end{array}
\]

\[
\begin{array}{ccc}
K_1 & \xrightarrow{w_1} & G_1 \\
\downarrow (s,t) & & \downarrow (s,t) \\
K_0 \times K_0 & \xrightarrow{w_0 \times w_0} & G_0 \times G_0 \\
\end{array}
\]
Comments

- Morphisms between open étale complete localic groupoids should be equivalence classes of spans

\[ \mathcal{G} \xleftarrow{w} \mathcal{K} \xrightarrow{f} \mathcal{H} \]

where \( w \) is an essential equivalence.

- For open étale complete localic groupoids \( \mathcal{G} \) and \( \mathcal{H} \), \( \text{Sh}(\mathcal{G}) \cong \text{Sh}(\mathcal{H}) \) if and only if there exists a localic groupoid \( \mathcal{K} \) with essential equivalences

\[ \mathcal{G} \xleftarrow{\varphi} \mathcal{K} \xrightarrow{\psi} \mathcal{H} \]

- In this case we call the two groupoids \( \mathcal{G} \) and \( \mathcal{H} \) Morita equivalent.
We have some issues left:

- This result is about categories, not about 2-categories.
- Spacial groupoids and spacial toposes?
Theorem

There is an equivalence of bicategories

\[ \text{SpEtendues}_{2-\text{iso}} \cong \text{EtaleGrpd}[W^{-1}], \]

But we can do a bit better...
• **Proposition**

  An étale complete open topological groupoid is Morita equivalent to an étale groupoid if and only if all its isotropy groups are discrete.

  • We will call such groupoids topological foliation groupoids, and denote their category by $\text{TopFolGrpd}$.

• **Theorem**

  There is an equivalence of bicategories

  \[
  \text{SpEtendues}_{2-\text{iso}} \cong \text{TopFolGrpd}[W^{-1}].
  \]
Consequences

- For any two topological foliation groupoids \( G \) and \( H \), \( \text{Sh}(G) \simeq \text{Sh}(H) \) if and only if there is a third such groupoid \( K \) with essential equivalences

\[
G \xleftarrow{\varphi} K \xrightarrow{\psi} H. 
\]

- In this case we will call \( G \) and \( H \) **Morita equivalent**.

- A geometric morphism \( G \to H \) corresponds to a span of groupoid homomorphisms

\[
G \xleftarrow{\psi} K \xrightarrow{\varphi} H,
\]

where \( \psi \) is an essential equivalence. We call such a span a **generalized morphism**.

- Generalized morphisms are composed using chosen weak pullbacks and the identities are represented by spans of identity arrows.
A 2-cell \((\psi, \varphi) \Rightarrow (\psi', \varphi')\) is an equivalence class of diagrams of the form

\[
\begin{array}{c}
\text{G} \\
\downarrow \alpha \\
\text{L} \\
\downarrow \theta \\
\text{K} \\
\uparrow \theta' \\
\text{K'} \\
\downarrow \psi' \\
\end{array}
\quad \begin{array}{c}
\text{K} \\
\uparrow \theta \\
\text{L} \\
\downarrow \beta \\
\text{H} \\
\end{array}
\]

where \(\theta\) and \(\theta'\) are essential equivalences.
Further Comments

- Any homotopy invariants defined for geometric objects represented by these topological groupoids needs to be invariant under Morita equivalence.

- There is a different description of the arrows in this bicategory in terms of groupoid bibundles (Hilsum-Skandalis maps) with isomorphisms of bibundles as 2-cells. This fits in the rest of our theory as expressed in the following theorem:

**Theorem**

*For topological foliation groupoids $\mathcal{G}$ and $\mathcal{H}$ the following are equivalent:*

1. $\text{Sh}(\mathcal{G}) \cong \text{Sh}(\mathcal{H})$;
2. $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent;
3. there is a left principal $\mathcal{G}$-, right principal $\mathcal{H}$-bibundle $\mathcal{G} \to \mathcal{H}$.
Lie Groupoids

Definition

- A **Lie groupoid** \( \mathcal{G} \) is a diagram

\[
G_1 \times_{G_0} G_1 \xrightarrow{m} G_1 \xrightarrow{i} G_1 \xleftarrow{u} G_1 \xrightarrow{s} G_0,
\]

in the category of manifolds and smooth maps.

- The source and target maps need to be **surjective submersions**.

- Note that the source and target maps are effective descent morphisms.

- \( G_0 \) needs to be Hausdorff.

- We call \( \mathcal{G} \) Hausdorff if \( G_1 \) is.
The 2-Category of Lie Groupoids

- A homomorphism $\varphi : G \to H$ between Lie groupoids is an internal functor, that is, it is given smooth maps $\varphi_0 : G_0 \to H_0$ and $\varphi_1 : G_1 \to H_1$, which commute with all the structure maps.

- A 2-cell $\alpha : \varphi \Rightarrow \psi : G \Rightarrow H$ is an internal natural transformation, that is, it is given by a smooth map $\alpha : G_0 \to H_1$ such that $s \circ \alpha = \varphi_0$, $t \circ \alpha = \psi_0$, and the following square commutes for each $g \in G_1$,

$$
\begin{array}{c}
\varphi_0(s(g)) \xrightarrow{\alpha(s(g))} \psi_0(s(g)) \\
\downarrow \varphi_1(g) \quad \downarrow \psi_1(g) \\
\varphi_0(t(g)) \xrightarrow{\alpha(t(g))} \psi_0(t(g))
\end{array}
$$
Examples

- For a manifold $M$ the fundamental groupoid of homotopy classes of paths in $M$ is a Lie groupoid.
- For a manifold $M$ with atlas $\mathcal{U}$ there is a groupoid homomorphism
  \[
  \mathcal{G}(\mathcal{U}) \to M.
  \]
- For any manifold $M$, the space $\Gamma(M)$ is the groupoid of germs of diffeomorphisms between open subsets of $M$.
- For an étale Lie groupoid (i.e., a Lie groupoid with étale source and target maps) there is a homomorphism $\mathcal{G} \to \Gamma(G_0)$. If this homomorphism is injective, we call $\mathcal{G}$ effective. Note that $G \ltimes M$ is effective as a groupoid precisely when the action of $G$ on $M$ is effective.
Gauge Groupoids and Bisections, I

- For a Lie group $G$ and a principal $G$-bundle $\pi: P \to M$, the associated **gauge groupoid** $\text{Gauge}(P)$ has
  - space of objects $M$;
  - space of arrows $(P \times P)/G$, where $G$ acts diagonally.

Source and target maps are the projections composed with $\pi$.

- Moreover, the quotient map of the action by $G$ induces a homomorphism $\text{Pair}(P) \to \text{Gauge}(P)$.

- A **global bisection** $\sigma: G_0 \to G_1$ of a Lie groupoid $\mathcal{G}$ is a section of $s$, such that $t \circ \sigma: G_0 \to G_0$ is a diffeomorphism.

- The set of global bisections forms a group, this is called the **gauge group** of $\mathcal{G}$. The product of two sections $\sigma$ and $\sigma'$ is defined by

$$\sigma' \sigma(x) = \sigma'(t(\sigma(x))) \sigma(x).$$
Gauge Groupoids and Bisections, II

• Consider the pullback

\[
P \times P \xrightarrow{\pi_1} (P \times P)/G = \text{Gauge}(P) \xrightarrow{s} M
\]

A section \(\sigma\) of \(s\) corresponds to a map \((1, \sigma): P \rightarrow P \times P\), where \(\sigma\) is a \(G\)-equivariant diffeomorphism of \(P\).

• The gauge group of \(\text{Gauge}(P)\) is isomorphic to \(\text{Diff}_G(P)\).

• A **local bisection** \((U, \sigma)\) of a Lie groupoid \(\mathcal{G}\) consists of
  
  • \(U \subseteq G_0\) open;
  • \(\sigma\) is a section of \(s\) on \(U\), such that
  • \(t \circ \sigma\) is an open embedding.

The germs of such bisections form the manifold of arrows of the groupoid \(\text{Bis}(\mathcal{G})\) over \(G_0\). \(\text{Bis}(\mathcal{G})\) is an étale groupoid with a surjective groupoid homomorphism \(\text{Bis}(\mathcal{G}) \twoheadrightarrow \mathcal{G}\).
Essential Equivalences

Definition

An **essential equivalence** $\varphi : G \to H$ between Lie groupoids is a homomorphism with the following properties.

1. It is **essentially surjective**, *i.e.*, the map

   \[ t \circ \pi_2 : G_0 \times_{H_0} H_1 \to H_0 \]

   from the manifold $G_0 \times_{H_0} H_1 = \{(x, h) | \phi_0(x) = t(h)\}$ is a surjective submersion.

2. It is **fully faithful**, *i.e.*, the diagram

   \[
   \begin{array}{ccc}
   G_1 & \xrightarrow{\varphi_1} & H_1 \\
   (s,t) \downarrow & & \downarrow (s,t) \\
   G_0 \times G_0 & \xrightarrow{\varphi_0 \times \varphi_0} & H_0 \times H_0
   \end{array}
   \]

   is a pullback of manifolds.
Examples

1. For a manifold $M$ the homomorphism $\text{Pair}(M) \to 1$, is an essential equivalence.

2. A surjective submersion $p: N \to M$ between manifolds induces a weak equivalence $\text{Ker}(p) \to M$.

3. In particular, take $N = \bigsqcup_i U_i$ for an open cover $\{U_i\}$ of $M$, and $p$ is the evident map. Then $\text{Ker}(p)$ is the atlas groupoid, and we see that $\mathcal{G}(U) \to M$ is an essential equivalence.

4. A Lie groupoid $\mathcal{G}$ is transitive if the map $(s, t): G_1 \to G_0 \times G_0$ is a surjective submersion. For any object $x$ of a transitive Lie groupoid $\mathcal{G}$, the inclusion $G_x \to \mathcal{G}$ is an essential equivalence. In particular, for a principal $G$-bundle $P \to M$ this yields an essential equivalence $G \to \text{Gauge}(P)$. 
Morita Equivalence

- Essential equivalences of Lie groupoids are stable under weak pullbacks.
- Two Lie groupoids $G$ and $H$ are called **Morita equivalent** if there exists a third Lie groupoid $K$ with essential equivalences

$$G \xleftarrow{\varphi} K \xrightarrow{\psi} H.$$
Smooth Structure for Toposes

- The category \( \text{Sh}(\mathcal{G}) \) for a Lie groupoid is defined in exactly the same way as if \( \mathcal{G} \) were just a topological groupoid.
- For each manifold \( M \), there is a **structure sheaf** \( \mathcal{O}_M \) of germs of smooth functions on \( M \).
- The structure sheaf \( \mathcal{O}_{G_0} \) can be made into a \( \mathcal{G} \)-sheaf where the action is by composition with germs of bisections.
- The sheaf \( \mathcal{O}_\mathcal{G} \) is a sheaf of rings, so \( \text{Sh}(\mathcal{G}, \mathcal{O}_\mathcal{G}) \) a ringed topos.
- A morphism of ringed toposes \((\varphi, f) : (\mathcal{E}, R) \to (\mathcal{E}', R')\) consists of a geometric morphism \( \varphi : \mathcal{E} \to \mathcal{E}' \) together with an arrow \( f : \varphi^* R' \to R \) in \( \mathcal{E} \).
Proposition

For a Lie groupoid $\mathcal{G}$ the following are equivalent:

1. The source and target maps are local diffeomorphisms (étale maps).
2. $\dim \mathcal{G}_0 = \dim \mathcal{G}_1$.

Theorem

For a smooth groupoid $\mathcal{G}$, the following are equivalent:

- $\mathcal{G}$ is Morita equivalent to a smooth étale groupoid;
- All isotropy Lie groups of $\mathcal{G}$ are discrete.
- $\mathcal{G}$ is a foliation groupoid.
Smooth Etendues

Definition
A ringed Grothendieck topos \((\mathcal{E}, R)\) is a smooth étendue if it contains an epimorphism \(U \rightarrow 1\), such that \(\mathcal{E}/U\) is a topos of sheaves on a Hausdorff manifold, i.e., \(\mathcal{E}/U \simeq \text{Sh}(M)\), and \(\pi^*_M(R) \cong \mathcal{O}_M\).

Theorem
There are equivalences of bicategories
\[
\text{SmoothEtendues} \simeq \text{Etale Lie Grpds}[W^{-1}],\n\text{Smooth Etendues} \simeq \text{Foliation Grpds}[W^{-1}]\]
Definition of a Foliation, 1

- Let $M$ be a manifold of dimension $n$. A **foliation** on $M$ is given by a **foliation atlas** of codimension $q$, with $0 \leq q \leq n$, with **foliation charts**

$$
(\varphi_i : U_i \rightarrow \mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q)_{i \in I}
$$

and change-of-charts diffeomorphisms that are locally of the form

$$
\varphi_{i,j}(x, y) = (g_{i,j}(x, y), h_{i,j}(y)),
$$

with respect to the decomposition $\mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q$. 
Definition of a Foliation, II

- Each chart $U_i$ is divided into plaques, the connected components of $\varphi_i^{-1}(\mathbb{R}^{n-q} \times \{y\})$ for $y \in \mathbb{R}^q$.
- These plaques amalgamate globally into leaves. These leaves are $(n - q)$-dimensional manifolds that are injectively immersed into $M$.
- Two foliation atlases define the same foliation if they give the same decomposition of $M$ into leaves. A **foliated manifold** $(M, \mathcal{F})$ is a manifold with a maximal foliation atlas $\mathcal{F}$.
- The space of leaves is the quotient space $M/\mathcal{F}$.
- The dimension of $\mathcal{F}$ is $n - q$.
- A map $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ between foliated manifolds is a smooth map $f : M \to M'$ which sends leaves in $\mathcal{F}$ to leaves in $\mathcal{F}'$. 
Examples

- **(Trivial Foliation)** $\mathbb{R}^n \to \mathbb{R}^{n-q} \times \mathbb{R}^q$ gives the trivial foliation on $\mathbb{R}^n$.

- **(Submersions)** A submersion $f: M \to N$ induces a foliation $\mathcal{F}(f)$ on $M$ whose leaves are the connected components of the fibers $f^{-1}(x)$ of $f$.

- Foliations derived from submersions are called **simple foliations**. The foliation is **strictly simple** if the fibers are all connected. A foliation is strictly simple precisely when its space of leaves is Hausdorff.
Examples

- **(Kronecker Foliation)** Let $a$ be an irrational real number and define the submersion

$$p: \mathbb{R}^2 \to \mathbb{R} \text{ by } p(x, y) = x - ay.$$ 

This induces a foliation $\mathcal{F}(p)$ of $\mathbb{R}^2$ Consider

$$\pi: \mathbb{R}^2 \to T^2 = S^1 \times S^1, \pi(x, y) = (e^{2\pi ix}, e^{2\pi iy}).$$

Then $\mathcal{F}(p)$ induces a foliation $\mathcal{F}$ on $T^2$ as follows. If $(\varphi, U) \in \mathcal{F}(p)$ is a foliation chart such that $\pi|_U$ is injective, then $\varphi \circ (\pi|_U)^{-1}$ is a foliation charts for $\mathcal{F}$ on $T^2$. Note that each leaf of $\mathcal{F}$ is diffeomorphic to $\mathbb{R}$ and lies dense in $T^2$. 
Holonomy of Paths in Leaves

- For a point $x$ in a leaf $L$, a **transversal** $T$ at $x$ is a submanifold of $M$ of dimension $q$ which contains $x$ and is transversal to the leaves of $\mathcal{F}$.

- For a path $\alpha$ in a leaf $L$ from $x$ to $y$, we may define its **holonomy germ** $\text{hol}(\alpha) = \text{hol}^{S,T}(\alpha)$ with respect to transversals $T$ at $x$ and $S$ at $y$, as the germ of the induced homeomorphism from a neighbourhood of $x$ in $T$ to a neighbourhood of $y$ in $S$. 
Remarks

• If two paths are homotopic in $L$ they give rise to the same holonomy germ.
• For $x = y$, we obtain a group homomorphism

$$\pi_1(L, x) \to \text{Diff}_x(T) \cong \text{Diff}_O(\mathbb{R}^q)$$

We call the image of this homomorphism the **holonomy group** at $x$. 
Foliation Groupoids, I

The **monodromy groupoid** $\text{Mon}(M, \mathcal{F})$ is the groupoid with object space $M$ and the space of arrows is defined as follows:

- If $x, y \in M$ are points in the same leaf $L$, then $\text{Mon}(M, \mathcal{F})(x, y)$ is the space of homotopy classes of paths from $x$ to $y$ in $L$ (with homotopies in $L$);
- If $x, y \in M$ lie in different leaves then $\text{Mon}(M, \mathcal{F})(x, y) = \emptyset$.

**Remark**

Note that $\text{Mon}(M, \mathcal{F})_x = \pi_1(L, x)$ where $x \in L$. 
Foliation Groupoids, II

The **holonomy groupoid** $\text{Hol}(M, \mathcal{F})$ is the groupoid with object space $M$ and the space of arrows is defined as follows:

- If $x, y \in M$ are points in the same leaf $L$, then $\text{Hol}(M, \mathcal{F})(x, y)$ is the space of holonomy classes of paths from $x$ to $y$ in $L$;
- If $x, y \in M$ lie in different leaves then $\text{Hol}(M, \mathcal{F})(x, y) = \emptyset$.

**Remarks**

- Note that $\text{Hol}(M, \mathcal{F})_x$ is the holonomy group at $x$.
- Since homotopic paths have the same germ in the holonomy group, there is a quotient homomorphism

$$\text{Mon}(M, \mathcal{F}) \twoheadrightarrow \text{Hol}(M, \mathcal{F}).$$
Examples

1. For a surjective submersion \( f: M \to N \) with connected fibers, the leaves of \( \mathcal{F}(f) \) have all trivial holonomy and

\[
\text{Hol}(M, \mathcal{F}(f)) = \text{Ker}(f).
\]

If the fibers of \( f \) are also simply connected, then

\[
\text{Mon}(M, \mathcal{F}(f)) = \text{Ker}(f).
\]

2. Let \( \mathcal{F} \) be a foliation on \( M \), whose leaves are invariant under a free properly discontinuous action by a discrete group \( G \). This gives rise to an isomorphism of Lie groupoids

\[
\text{Mon}(M, \mathcal{F})/G \cong \text{Mon}(M/G, \mathcal{F}/G)
\]

and a surjective homomorphism

\[
\text{Hol}(M, \mathcal{F})/G \to \text{Hol}(M/G, \mathcal{F}/G),
\]

which is generally not an isomorphism.
Proposition

Let $\mathcal{F}$ be a foliation of a manifold $M$.

- The orbits of $\text{Mon}(M, \mathcal{F})$ and $\text{Hol}(M, \mathcal{F})$ are the leaves of $\mathcal{F}$.
- The isotropy groups of $\text{Mon}(M, \mathcal{F})$ and $\text{Hol}(M, \mathcal{F})$ are discrete.
- For $x \in L$, the target map of $\text{Mon}(M, \mathcal{F})$ restricts to a universal covering
  \[ t : \text{Mon}(M, \mathcal{F})(x, -) \to L, \]
- $t : \text{Hol}(M, \mathcal{F})(x, -) \to L$ is the covering projection corresponding to the kernel of the holonomy homomorphism $\pi(L, x) \to \text{Hol}(L, x)$. 
Etale Groupoid and Foliations

• Any étale Lie groupoid induces a foliation by its orbits on the space of objects $G_0$.

• For any foliation $\mathcal{F}$ on a manifold $M$, a groupoid $\mathcal{G}$ over $M$ is said to integrate $\mathcal{F}$ if its orbits coincide with the leaves of the foliation. If moreover $s$ has connected fibers, there are local diffeomorphisms

$$\text{Mon}(M, \mathcal{F}) \rightarrow \mathcal{G} \rightarrow \text{Hol}(M, \mathcal{F}).$$
Etale Holonomy and Monodromy Groupoids

- Let $\mathcal{F}$ be a foliation of a manifold $M$. Choose a complete transversal section $T$ of $(M, \mathcal{F})$, i.e., an immersed (not necessarily connected) submanifold of $M$ of dimension $q$, which is transversal to the leaves of $\mathcal{F}$ and intersects each leaf in at least one point.

- Define the Lie groupoid $\text{Mon}_T(M, \mathcal{F})$ over $T$ as the restriction of the (full) monodromy groupoid $\text{Mon}(M, \mathcal{F})$ to $T$.

- $\dim \text{Mon}_T(M, \mathcal{F})_1 = \dim T$.

- The inclusion $\text{Mon}_T(M, \mathcal{F}) \to \text{Mon}(M, \mathcal{F})$ is a weak equivalence.

- Analogously, define the étale holonomy groupoid $\text{Hol}_T(M, \mathcal{F})$ over $T$. 
Examples

• Let $\mathcal{F}$ be the standard foliation of the Möbius band $M$. The étale holonomy groupoid of $(M, \mathcal{F})$ is isomorphic to the translation groupoid $\mathbb{Z}/2 \times (-1, 1)$.

• The étale holonomy groupoid of the Kronecker foliation $\mathcal{F}$ of the torus $T^2$ is $\mathbb{Z} \rtimes S^1$. 
Some References


References - Continued

