# Iterated realizability as a comma construction

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**Abstract**

We show that the 2-category of partial combinatory algebras, as well as various related categories, admit a certain type of lax comma objects. This not only reveals some of the properties of such categories, but it also gives an interpretation of iterated realizability, in the following sense. Let $\phi : A \to B$ be a morphism of PCAs, giving a comma object $A \ltimes_\phi B$. In the realizability topos $\mathbf{RT}(B)$ over $B$, the object $(A, \phi)$ is an internal PCA, so we can construct the realizability topos over $(A, \phi)$. This topos is equivalent to the realizability topos over the comma-PCA $A \ltimes_\phi B$. This result is both an analysis and a generalization of a special case studied by Pitts in the context of the effective monad.

## 1 Introduction

In [Hof05] a framework for studying problems about the 2-category of realizability toposes is introduced. The strategy was to start with a 2-category of so-called basic combinatorial objects, which are to be thought of as "pre-realizability notions", then to show that each such object gives rise to a **Set**-indexed preorder, and then to characterize which of those are in fact triposes. As it turns out, the characterization is formulated in terms of ordered partial combinatory algebras with filters; geometric morphisms (and transformations) between such triposes can then be understood as morphisms (and 2-cells) of these ordered PCAs which satisfy a certain density condition. As a consequence, questions about the category of realizability toposes can be completely reduced to questions about the 2-category in which the ordered PCAs with filters live.

Consequently, it becomes desirable to get an understanding of the latter category. What constructions are possible on (ordered) PCAs? What kind of 2-categorical limits and colimits does this category admit? Are there any interesting representation theorems or structure theorems? There are many elementary questions here, and we are only beginning to answer some of them.

In this paper I hope to provide one small piece of this large puzzle. The inspiration came from trying to understand a result about the effective monad in Andy Pitts' thesis [Pit81]. This result says roughly the following: given a PCA $A$, one can choose a copy of $\mathbb{N}$ inside $A$, such that every partial recursive function is representable in $A$. In the realizability topos $\mathbf{RT}(A)$ over $A$, this means that there is a natural number object. Therefore, one can form the effective topos over $\mathbf{RT}(A)$. The iteration theorem tells us that this resulting topos can be obtained from a **Set**-tripos. What does this tripos look like? Pitts shows that one can construct a new PCA, such that this tripos is the canonical tripos over this new PCA.

This result prompted two questions: first, what is the nature of this (clever, but seemingly ad hoc) construction? And second, does it also work in the more general setting which was advocated in [Hof05]? It turns out that the construction can be viewed as a lax comma construction, and that it in fact already works on the level of basic combinatorial objects. Particularly nice is the fact that all interesting subcategories of the category of BCOs are closed under the construction. In particular, the category of (ordered) PCAs (with filters) is closed under taking lax comma objects, and the theorem proved by Pitts remains true when we replace the choice of numerals (which really is an inclusion $\mathbb{N} \to A$) by an arbitrary morphism of (ordered) PCAs (with filters).

While this interpretation of Pitts' construction was a main motivation for the observations presented here, we have reason to believe that the lax comma construction is of some independent interest. For example, it embodies a closure property which can be expressed neatly in 2-categorical language: the lax comma construction is a family of KZ-monads on lax slices. From the universal property we get other things for free, such as the fact that every morphism can be factored in a certain way; this is of course analoguous to the comprehensive factorization of a functor.

The outline of the paper is as follows. First, we describe the setting in which we work, namely the 2-category of basic combinatorial objects. This is in effect a summary of the work done in [Hof05]. The reader who is not interested in this level of generality can pretend that we are working in the 2-category of PCAs or of ordered PCAs, an exposition of which can be found in [Hof03, Hof03], which, in turn, is both an analysis and a refinement of the work done by Longley on applicative morphisms of PCAs (see [Lon94]).

Next, in section 3, we present the lax comma construction on the category of cartesian BCOs, concentrating on the categorical aspects. The construction preserves many important properties; in particular, it restricts to the subcategory on the ordered PCAs; these examples and special cases are studied in section 4. In section 5 we describe the connection with iterated realizability. The proof of the main theorem is a straightforward adaptation of the special case studied by Pitts; we only outline the key steps. Inspection of the proof then gives some naturality results for free.

## 2 Review of basic combinatorial objects

In this section we set the stage by introducing a convenient 2-category which encompasses a wide range of realizability notions, encoded as certain combinatorial objects. These objects are quite general in nature and comprise both posets (in particular locales) and partial combinatory algebras. The importance of this category lies in the following facts. First, every object $\Sigma$ of the category gives rise to a Set-indexed preorder. Properties of $\Sigma$ can now be related to logical properties of the associated indexed preorder. The main representation theorems characterize when the associated indexed preorder is a tripos (see Theorems 2.1, 2.2). Moreover, it is possible to describe morphisms between triposes/toposes in terms of maps between the underlying combinatorial objects, where the geometric morphisms correspond precisely to maps satisfying a certain density condition. In effect, this shows how the study of the 2-category of realizability toposes reduces to the study of this 2-category of combinatorial objects.

**Notational Convention.** Since we will work throughout the paper with partial functions on posets, some notation will be useful. First of all, when $f : A \rightharpoonup B$ is a partial function, we write $f(x)\downarrow$ for $x \in dom(f)$. For two partial functions $f, g$, we also write $f(x) \simeq g(x)$ for Kleene equality, and this notation is extended to more general expressions.

When $t, s$ are expressions taking values in a poset, then we will write $t \preccurlyeq s$ for the statement that if $s$ is defined then so is $t$, in which case $t \leq s$.

A **basic combinatorial object** (BCO) is a tuple $\Sigma = (\Sigma, \leq, \mathcal{F}_\Sigma)$, where $\leq$ partially orders the set $\Sigma$, and where $\mathcal{F}_\Sigma$ is a set of monotone, partial endofunctions with downwards closed domain on $\Sigma$. We require that:

- $\exists i \in \mathcal{F}_\Sigma \; \forall x \in \Sigma . i(x) \preccurlyeq x$
- $\forall f, g \in \mathcal{F}_\Sigma \; \exists h \in \mathcal{F}_\Sigma \; \forall x . h(x) \preccurlyeq fg(x)$.

We think of the maps of $\mathcal{F}_\Sigma$ as *realizable* or *computable* functions. When there is no chance of confusion, we usually just write $\Sigma$ for a BCO. Keep in mind, however, that it is perfectly possible for a fixed (po)set to have different BCO structures on it (see examples).

The conditions stated are the weakest possible; it can be shown that without loss of generality one may assume that $\mathcal{F}_\Sigma$ contains the identity and is closed under composition, and from now on we will often tacitly do so.

A **morphism of BCOs** $\phi : \Sigma \to \Theta$ is a function between the underlying posets such that

1. $\exists t \in \mathcal{F}_\Theta \; \forall a \leq a' . t(\phi a) \preccurlyeq \phi(a')$
2. $\forall f \in \mathcal{F}_\Sigma \; \exists g \in \mathcal{F}_\Theta \; \forall a \in \Sigma . g\phi(a) \preccurlyeq \phi f(a)$.

Such a morphism $\phi$ should be thought of as a simulation of $\Sigma$ inside $\Theta$, or as an inner model, much in the same vein as one can have a relative interpretation of one theory inside another.

Given two morphisms $\phi, \psi : \Sigma \to \Theta$ we say that $\phi \vdash \psi$ if there exists $g \in \mathcal{F}_\Theta$ with $g(\phi(x)) \preccurlyeq \psi(x)$ for all $x \in \Sigma$. This makes the category **BCO** pre-order enriched. Thinking again of $\phi$ and $\psi$ as inner models, a 2-cell now represents a transformation of inner models. This defines a 2-category which we will denote by **BCO**.

**Finite limits.** The category **BCO** has finite products with all structure coordinatewise. The terminal object is the one-point BCO. This allows us to speak about cartesian objects in the category of BCOs: a BCO $\Sigma$ is said to have **(internal) finite meets** if the diagonal map $\Sigma \to \Sigma \times \Sigma$ has a right adjoint. Similarly, $\Sigma$ has a **top element** if the unique map $\Sigma \to 1$ has a right adjoint. In case a BCO has finite meets and a top element we call it **cartesian**.

It must be stressed that a BCO $\Sigma$ being cartesian does not imply anything about the existence of meets or a top element in the underlying poset of $\Sigma$. Rather, it means that there exists an internal pairing operation and an internal top element.

A morphism $\phi$ of BCOs is said to preserve finite meets if there is a realizable map $g$ with $g(\phi(a) \wedge \phi(b)) \preccurlyeq \phi(a \wedge b)$. This defines a sub-2-category **CartBCO** on the cartesian objects and the finite meet-preserving maps (and all 2-cells between them).

**Downsets.** There is a KZ-monad on the category **BCO**, which takes downward closed subsets in a BCO. Formally, define

$$\mathcal{D}\Sigma = \{\alpha \subseteq \Sigma | a \leq b \in \alpha \Rightarrow a \in \alpha\}.$$

The ordering is given by subset inclusion. A partial function $F : \mathcal{D}\Sigma \to \mathcal{D}\Sigma$ is defined to be realizable when there is $f \in \mathcal{F}_\Sigma$ such that for all $\alpha \in dom(F), a \in \alpha$, $f(a)$ is defined and $f(a) \in F(\alpha)$.

A pseudo algebra for the monad $\mathcal{D}$ should be thought of as a complete BCO, although the underlying poset of a pseudo algebra does not necessarily have suprema. The pseudo algebras, together with pseudo algebra morphisms and all 2-cells, form a 2-category $\mathcal{D} - \mathbf{Alg}$.

**Examples.** Every poset $\Sigma$ can be viewed as a BCO, by letting the only realizable function in $\mathcal{F}_\Sigma$ be the identity function. This extends to an inclusion of 2-categories **Poset** $\hookrightarrow$ **BCO**. This inclusion sends meet-semilattices to cartesian BCOs.

On the other extreme, every PCA $A$ gives rise to a BCO, by letting the ordering on $A$ be discrete and by taking the realizable functions to be those of the form $a \bullet -$, for $a \in A$. The internal pairing of the PCA makes this into a cartesian BCO (every element can serve as top element). There is a full embedding of 2-categories **PCA** $\hookrightarrow$ **CartBCO**.

One can do the same with ordered PCAs, and with ordered PCAs with filters. To describe the latter case, let $A$ be an ordered PCA with filter of truth values $TV(A)$. That is, $TV(A) \subseteq A$ is an upwards closed subset which is closed under the application and contains (some choice of) combinators $k, s$. This induces a BCO-structure on the underlying poset of $A$ by taking the realizable functions to be those of the form $a \bullet -$, where $a \in TV(A)$. We refer to this situation by saying that the BCO-structure on $A$ arises in the canonical way from the ordered PCA with the filter.

**Indexed preorders.** Given a BCO $\Sigma$, preorder the set of functions $\mathbf{Set}(X, \Sigma)$ by defining, for $\alpha, \beta : X \to \Sigma$,

$$\alpha \vdash_X \beta \Leftrightarrow \exists f \in \mathcal{F}_\Sigma . f(\alpha x) \preccurlyeq \beta x.$$

This defines a **Set**-indexed category, denoted $\mathbf{Set}(-, \Sigma)$. Properties of $\Sigma$ can now be related to logical properties of $\mathbf{Set}(-, \Sigma)$. For example, $\Sigma$ is cartesian if and only if $\mathbf{Set}(-, \Sigma)$ has indexed finite limits. Also, $\Sigma$ is a pseudo algebra for the monad $\mathcal{D}$ precisely when $\mathbf{Set}(-, \Sigma)$ has indexed coproducts (i.e. reindexing functors have left adjoints subject to the Beck-Chevalley condition).

There are two important characterization theorems, which we repeat here.

**Theorem 2.1** *Let $\Sigma$ be a cartesian BCO. Then $\mathbf{Set}(-, \mathcal{D}\Sigma)$ is a tripos if and only if $\Sigma$ arises in the canonical way from an ordered PCA with a filter.*

**Theorem 2.2** *Let $\Sigma$ be a cartesian BCO. Then $\mathbf{Set}(-, \Sigma)$ is a tripos if and only if $\Sigma$ has the following properties: it is a pseudo algebra for the monad $\mathcal{D}$, it is an ordered PCA with a filter, and it has implication.*

The last condition (that implication is present) can be formulated in terms of an appropriate interaction between the supremum map and the application map. For details, see [Hof05].

**Computational Density.** We have seen that every morphism of BCOs induces, covariantly, a map between the associated indexed preorders. When studying triposes (and toposes), one is usually more interested in geometric morphisms. In our setting, these can be conveniently characterized by a condition, which is called computational density. For a morphism $\phi : \Sigma \to \Theta$, we say that $\phi$ is **computationally dense** if there exists an element $u \in \mathcal{F}_\Theta$ with the property that

$$\forall g \in \mathcal{F}_\Theta \exists f^g \in \mathcal{F}_\Sigma \forall a \in \Sigma . u\phi(f^g a) \preccurlyeq g(\phi a).$$

Intuitively, this says that realizable function $g$ of $\Theta$ acting on elements in the image of $\phi$ can be lifted to a realizable function $f$ of $\Sigma$, up to the realizer $u$. We call this realizable function $u$ a *witness* for the density of $\phi$, and refer to $f^g$ as a *lift* for $g$.

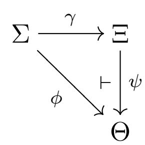
The main characteristic is the following:

**Proposition 2.3** A morphism $\phi : \Sigma \to \Theta$ is computationally dense precisely when the induced map $\mathcal{D}\phi : \mathcal{D}\Sigma \to \mathcal{D}\Theta$ has a right adjoint.

It can be shown that for a map of $\mathcal{D}$-algebras, density simply means having a right adjoint.

# 3 The comma construction

In this section we will work in the category **CartBCO** of cartesian BCOs and cartesian morphisms. We will use the symbol $\wedge$ for the internal pairing operation in a cartesian BCO. Fix a BCO $\Theta$, and consider the *lax slice category* **CartBCO**$//\Theta$. Its objects are morphisms $\phi : \Sigma \to \Theta$. A morphism is a triangle




which need not commute on the nose, but for which there is a 2-cell $\phi \vdash \psi\gamma$.

The 2-cells are inherited from the category **BCO**.

The formation of **CartBCO**$//\Theta$ is a covariant 2-functor: given a map $\beta : \Theta \to \Theta'$, we get, by composition, an induced 2-functor $\beta_! :$ **CartBCO**$//\Theta \to$ **CartBCO**$//\Theta'$.

**Construction.** Let $\phi : \Sigma \to \Theta$ be a cartesian morphism of BCOs. We define a new BCO, denoted $\Sigma \ltimes_\phi \Theta$, by letting the underlying poset be the product $\Sigma \times \Theta$, but where a partial monotone function $h$ on this set is realizable if and only if there exist functions $f \in \mathcal{F}_\Sigma, g \in \mathcal{F}_\Theta$ such that for all $(a, b)$:

$$(f(a), g(\phi(a) \wedge b)) \preccurlyeq h(a, b).$$

We say that the maps $f, g$ *support* the morphism $h$. When no confusion is possible, we simply write $\Sigma \ltimes \Theta = \Sigma \ltimes_\phi \Theta$. The notation is intended to reflect the reminiscence with a semidirect product of, say, groups: we can think of $\phi$ as inducing an action, and then the application on $\Sigma \ltimes_\phi \Theta$ is twisted in the second coordinate by this action.

Let us prove that this is indeed a BCO. To show that the identity is realizable, let $f = Id, g = \pi_2$, the second projection. For composition, assume we are given $h, h'$, supported by $f, g$ and $f', g'$, respectively. On the first coordinate, we simply take $ff'$. On the second, we need $k$ such that

$$k(\phi(a) \wedge b) \preccurlyeq g(\phi f(a) \wedge g'(\phi(a) \wedge b)).$$

Pick $m$ with $m\phi(a) \preccurlyeq \phi f(a)$. Now the function $\phi(a) \wedge b \mapsto m\phi(a) \wedge g(\phi(a) \wedge b)$ is realizable, so composing this with $g'$ does the job.

Observe that the BCO $\Sigma \ltimes_\phi \Theta$ inherits the cartesian structure, given coordinatewise.

**Structure Maps.** The BCO $\Sigma \ltimes_\phi \Theta$ comes equipped with two projections:

$$\Sigma \xleftarrow{\pi_\Sigma} \Sigma \ltimes_\phi \Theta \xrightarrow{\pi_\Theta} \Theta.$$

These are given by $\pi_\Sigma(a, b) = a$ and $\pi_\Theta(a, b) = \phi(a) \wedge b$. Note that the map to $\Theta$ is not a genuine projection, although the notation suggests so.

Let us verify that these are morphisms of BCOs. As for $\pi_\Sigma$, it is evident that it preserves the order on the nose. If $h$ is realizable in $\Sigma \ltimes_\phi \Theta$, say via $f \in \mathcal{F}_\Sigma$ and $g \in \mathcal{F}_\Theta$, then we have

$$f\pi_\Sigma(a,b) = f(a) = \pi_\Sigma(f(a), g(a \wedge \phi(b))) = \pi_\Sigma h(a,b),$$

which proves that realizable functions are preserved by $\pi_\Sigma$.

To prove that $\pi_\Theta$ preserves the order up to a realizer, note that there exists $q \in \mathcal{F}_\Theta$ with $q\phi(a) \preccurlyeq \phi(a')$ for all $a \leq a'$.

This means that (uniformly in $a \leq a'$) $\phi(a) \vdash \phi(a')$, and hence, also uniformly in $b \leq b'$, that $b \wedge \phi(a) \vdash b' \wedge \phi(a')$. This shows that if $(a,b) \leq (a',b')$ then $b \wedge \phi(a) \vdash b' \wedge \phi(a')$, i.e. $\pi_\Theta(a,b) \vdash \pi_\Theta(a',b')$.

To prove that realizable functions are preserved by $\pi_\Theta$, consider a realizable function $h$, supported by $f, g$. To verify

$$\pi_\Theta(a,b) = \phi(a) \wedge b \vdash \pi_\Theta(f(a), g(b \wedge \phi(a))) = \phi(f(a)) \wedge g(b \wedge \phi(a)),$$

note that $\phi(a) \mapsto \phi(f(a))$ is realizable because $\phi$ preserves realizable functions. Clearly the entailment $\phi(a) \wedge b \vdash g(\phi(a) \wedge b)$ also holds.

The square

$$\begin{array}{ccc} \Sigma \ltimes_\phi \Theta & \xrightarrow{\pi_\Sigma} & \Sigma \\ {\scriptstyle \pi_\Theta}\downarrow & \vdash & \downarrow{\scriptstyle \phi} \\ \Theta & \xrightarrow{1} & \Theta. \end{array}$$

commutes up to a 2-cell $1 \circ \pi_\Theta \vdash \phi \circ \pi_\Sigma$, which is realized by second projection. In fact, this exhibits $\Sigma \ltimes_\phi \Theta$ as a *lax comma object*: if we have another BCO $\Lambda$ and maps $\alpha : \Lambda \to \Theta, \beta : \Lambda \to \Sigma$ such that there is a 2-cell $\alpha \vdash \phi \circ \beta$, then there is a map $\langle \alpha, \beta \rangle : \Lambda \to \Sigma \ltimes_\phi \Theta$ (given by $x \mapsto (\alpha(x), \beta(x))$), for which we have and isomorphism $\alpha \dashv\vdash \pi_\Theta \circ \langle \alpha, \beta \rangle$ and an equality $\beta = \pi_\Sigma \circ \langle \alpha, \beta \rangle$. The map $\langle \alpha, \beta \rangle$ is unique up to unique isomorphism.

From this universal property we can deduce a number of other interesting properties. First of all, we get a map $i_\Sigma : \Sigma \to \Sigma \ltimes_\phi \Theta$, given by $i_\Sigma(a) = (a, \phi(a))$. Note that the triangle

$$\begin{array}{ccc} \Sigma & \xrightarrow{i_\Sigma} & \Sigma \ltimes_\phi \Theta \\ & {\scriptstyle \phi}\searrow & \downarrow{\scriptstyle \pi_\Theta} \\ & & \Theta \end{array}$$

does not commute on the nose, but up to isomorphism. We express this by saying that $\pi_\Theta \circ i_\Sigma$ is a pseudo factorization of $\phi$. This factorization is a bicategorical version of the comprehensive factorization; we might say that we have factored $\phi$ as a final map followed by a fibration.

Note also that the composite $\pi_\Sigma \circ i_\Sigma$ is the identity, exhibiting $\Sigma$ as a retract of $\Sigma \ltimes_\phi \Theta$.

**Functoriality.** Another consequence of the universal property of $\Sigma \ltimes_\phi \Theta$ is that the construction is in fact a 2-functor on the lax slice category $\mathbf{CartBCO}//\Theta$.

(It is also a 2-functor on the genuine slice, but later we will find use for the extra generality, see section 5.) Suppose that the following triangle satisfies $\phi \vdash \psi\alpha$:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\alpha} & \Lambda \\ & {}_{\phi}\searrow \;\vdash & \downarrow{\scriptstyle \psi} \\ & & \Theta. \end{array}$$

Now we get an induced map $\alpha \ltimes \Theta : \Sigma \ltimes_\phi \Theta \to \Lambda \ltimes_\psi \Theta$, given by $(a, b) \mapsto (\alpha(a), b)$, such that

$$\begin{array}{ccc} \Sigma \ltimes_\phi \Theta & \xrightarrow{\alpha \ltimes \Theta} & \Lambda \ltimes_\psi \Theta \\ & {}_{\pi_\Theta}\searrow \;\vdash & \downarrow{\scriptstyle \pi_\Theta} \\ & & \Theta \end{array}$$

indeed satisfies $\pi_\Theta \vdash \pi_\psi \circ \alpha \ltimes \Theta$. In fact, this also works on the level of 2-cells. If we have $\alpha \vdash \alpha'$, then there is a function $m \in \mathcal{F}_\Lambda$ with $m\alpha(x) \preccurlyeq \alpha'(x)$. Then for any $(a, b) \in \Sigma \ltimes_\phi \Theta$, we have that the function $(\alpha(a), b) \mapsto (\alpha'(a), b)$ is realizable, in the first coordinate via $m$ and in the second via projection.

**Monadicity.** Just like the ordinary comma construction is a monad on slices of **Cat**, the operation $- \ltimes \Theta$ is a pseudo monad on **CartBCO**$//\Theta$. This can be derived from the universal property, but we will give an explicit description now. We have already seen the unit for the monad, $i_\Sigma : \Sigma \to \Sigma \ltimes_\phi \Theta$. This is a natural map. The multiplication $\mu_\phi : (\Sigma \ltimes_\phi \Theta) \ltimes_{\pi_\Theta} \Theta \to \Sigma \ltimes_\phi \Theta$ at $\phi : \Sigma \to \Theta$ sends a typical element $((a, b), c)$ of $\Sigma \times \Theta \times \Theta$ to $(a, b \wedge c)$. This is, up to isomorphism, a map over $\Theta$. The associativity of $\mu$ follows from that of $\wedge$, and the unit axiom follows from $\mu \circ i(a, b) = \mu((a, b), \phi(a) \wedge b) = (a, \phi(a) \wedge b \wedge b)$, which is isomorphic to $(a, b)$.

**Computational Density.** Since the assignment $(\Sigma, \phi) \mapsto (\Sigma \ltimes_\phi \Theta, \pi_\Theta)$ is functorial, we can ask whether it preserves computational density of morphisms. This is indeed the case.

**Lemma 3.1** *Let a morphism $\alpha$ in* **CartBCO**$//\Theta$ *be given:*

$$\begin{array}{ccc} \Sigma & \xrightarrow{\alpha} & \Lambda \\ & {}_{\phi}\searrow \;\vdash & \downarrow{\scriptstyle \psi} \\ & & \Theta. \end{array}$$

*If $\alpha$ is computationally dense then so is $\alpha \ltimes \Theta$.*

**Proof.** First, since $\phi \vdash \psi\alpha$, there is a $t \in \mathcal{F}_\Theta$ for which $f(\phi(x)) \preccurlyeq \psi\alpha(x)$, all $x \in \Sigma$. Let $u$ be a witness for the density of $\alpha$. We claim that the pair $(u, v)$ is a witness for the density of $\alpha \ltimes \Theta$, where $v$ is a function which projects a pair onto its second coordinate.

Indeed, let $g \in \mathcal{F}_\Lambda, h \in \mathcal{F}_\Theta$ support a realizable map of $\Lambda \ltimes_\psi \Theta$. By density of $\alpha$, there is a lift $f^g \in \mathcal{F}_\Sigma$ of $g$. Let $f^h$ be a morphism which sends $\phi y \wedge z$ to $h(\psi\alpha(y) \wedge z)$. Then $(f^g, f^h)$ supports a realizable map of $\Sigma \ltimes_\phi \Theta$. We claim that this is a lift of $(f, g)$. To verify this, let $(a, b)$ be an element of $\Sigma \ltimes_\phi \Theta$ for which $(g, h)(\alpha \ltimes \Theta(a, b)) = (g, h)(\alpha(a), b) = (g\alpha(a), h(\psi\alpha(a) \wedge b)$ is defined. Now we have the following inequalities:

$$
\begin{array}{rcl}
(u, v)(\alpha \ltimes \Theta(f^g, f^h)(a, b)) & = & (u, v)(\alpha(f^g a), f^h(\phi a \wedge b)) \\
& \preccurlyeq & (u, v)(\alpha(f^g a), h(\psi\alpha a \wedge b)) \\
& = & (u(\alpha(f^g a)), v(\psi(\alpha(f^g a)) \wedge h(\psi\alpha(a) \wedge b))) \\
& \preccurlyeq & (g(\alpha(a)), h(\psi\alpha(a) \wedge b)) \\
& = & (g, h)(\alpha \ltimes \Theta(a, b))
\end{array}
$$

□

**Change of Base.** Let $\phi : \Sigma \to \Theta$ be a BCO over $\Theta$, and let $\beta : \Theta \to \Theta'$ be a morphism. On the one hand we can consider $\Sigma \ltimes_\phi \Theta$ as a BCO over $\Theta'$ (formally we should write $\beta_!(\Sigma \ltimes_\phi \Theta)$ for this object) and on the other hand we have $\Sigma \ltimes_{\beta\phi} \Theta'$, which is also a BCO over $\Theta'$. For $(x, y) \in \Sigma \times \Theta$, the assignment $(x, y) \mapsto (x, \beta y)$ defines a cartesian morphism $\beta^\phi : \Sigma \ltimes_\phi \Theta \to \Sigma \ltimes_{\beta\phi} \Theta'$. The following diagram then commutes up to isomorphism:

$$
\begin{array}{ccc}
\Sigma \ltimes_\phi \Theta & \xrightarrow{\beta^\phi} & \Sigma \ltimes_{\beta\phi} \Theta' \\
\downarrow{\scriptstyle \pi_\Theta} & & \downarrow{\scriptstyle \pi_{\Theta'}} \\
\Theta & \xrightarrow{\beta} & \Theta'
\end{array}
$$

It is in this sense that the comma construction is compatible with change of base.

**Lemma 3.2** *If $\beta : \Theta \to \Theta'$ is computationally dense, then so is $\beta^\phi$.*

**Proof.** Let $v$ be a witness for the density of $\beta$, and let $w$ be a map such that $w(x \wedge y) \preccurlyeq v(y)$. Then we claim that $(Id, w)$ is a witness for the density of $\beta^\phi$. Consider a realizable map $(g, h)$ over $\Sigma \ltimes_{\beta\phi} \Theta'$. First note that we can find a realizable $j$ such that $j(\beta(\phi a \wedge b)) \preccurlyeq h(\beta\phi(a) \wedge \beta b)$. Take a lift $f^j$ for $j$; we claim that $(g, f^j)$ is a lift for $(g, h)$. Consider $(a, b)$ in $\Sigma \ltimes_\phi \Theta$. Then

$$
\begin{array}{rcl}
(Id, w)(\beta^\phi(g, f^j)(a, b)) & = & (Id, w)(ga, \beta(f^j(\phi a \wedge b))) \\
& \preccurlyeq & (ga, w(\beta\phi(ga) \wedge \beta(f^j(\phi a \wedge b)))) \\
& \preccurlyeq & (ga, v(\beta(f^j(\phi a \wedge b)))) \\
& \preccurlyeq & (ga, j(\beta(\phi a \wedge b))) \\
& \preccurlyeq & (ga, h(\beta\phi(a) \wedge \beta b)) \\
& = & (g, h)(\beta^\phi(a, b))
\end{array}
$$

$\square$

# 4 Examples and special cases

Having investigated some of the categorical aspects of the construction, we now turn to some examples. In particular, we apply the construction to some important special kinds of BCOs such as Partial Combinatory Algebras.

**Meet-semi-lattices.** Let $\phi : \Sigma \to \Theta$ be a map of meet-semi-lattices. Then $\Sigma \ltimes_\phi \Theta$ has as underlying meet-semi-lattice the product $\Sigma \times \Theta$, but now there is a realizable function which is not the identity: by definition, the map

$$(a, b) \mapsto (a, \phi(a) \wedge b)$$

is realizable. As a consequence, $\Sigma \ltimes_\phi \Theta$ is not in the image of the embedding **Poset** $\hookrightarrow$ **BCO**. However, it is *equivalent* to a poset: consider the quotient poset obtained from $\Sigma \times \Theta$ by identifying elements $(a, b)$ and $(a, \phi(a) \wedge b)$. Call this poset $Q$. There are morphisms $x : Q \to \Sigma \ltimes_\phi \Theta$, $y : \Sigma \ltimes_\phi \Theta \to Q$, given by

$$x[a, b] = (a, \phi(a) \wedge b), \qquad y(a, b) = [a, b],$$

and it is not hard to verify that these constitute an equivalence of BCOs. The poset $\Sigma \ltimes_\phi \Theta$ has another presentation, namely as the subposet of $\Sigma \times \Theta$ on the pairs $(a, b)$ for which $b \leq \phi(a)$. This is precisely the ordinary comma-poset $\Theta/\phi$. So for posets we see that the lax comma construction (which takes place in the category of BCOs) and the ordinary comma construction (in the category of posets) are equivalent when the result is viewed in the category of BCOs.

**Ordered PCAs.** Let $\phi : \Sigma \to \Theta$ be a map of (ordered) PCAs. We will first show that $\Sigma \ltimes_\phi \Theta$ is again an (ordered) PCA. Remember that $\mathcal{F}_\Sigma$ is the set of partial functions of the form $(a \bullet -)$. Now to say that $\phi$ is a morphism, means that there exist realizers $u, v \in \Theta$ such that $u\phi(a) \leq \phi(b)$ when $a \leq b$ and $v\phi(a)\phi(b) \preccurlyeq \phi(ab)$ when $ab\downarrow$. What are the realizable functions of $\Sigma \ltimes_\phi \Theta$? Well, by definition these are of the form $(x, y) \mapsto (f(x), g(\phi(x) \wedge y))$, for realizable $f, g$. That is, for any two elements $a \in \Sigma, b \in \Theta$, the partial function $(x, y) \mapsto (a \bullet x, b \bullet (\phi(x) \wedge y)$ is realizable. Define application on $\Sigma \times \Theta$ by

$$(a, b) \bullet (x, y) \simeq (a \bullet x, b \bullet (\phi(x) \wedge y)).$$

It is straightforward to show that one can make choices for $k, s$ such that the result is again an ordered PCA. If the ordering on both $\Sigma$ and on $\Theta$ happens to be discrete, then so is the ordering on $\Sigma \ltimes_\phi \Theta$. The structure morphisms are ordered PCA maps.

Now consider the case where $\Sigma$ and $\Theta$ arise in the canonical way from ordered PCAs together with a filter of truth-values. That is, there is an upwards closed subset $TV(\Sigma) \subseteq \Sigma$, which is closed under application and contains (some choice of) $k$ and $s$. The realizable functions are then those of the form $a \bullet -$, for $a \in TV(\Sigma)$, and similarly for $\Theta$. Now a morphism $\phi$ is still a morphism of ordered PCAs as before, but the two witnesses $u, v$ are required to be in $TV(\Theta)$. Note that $\phi$ sends designated truth-values to designated truth-values.

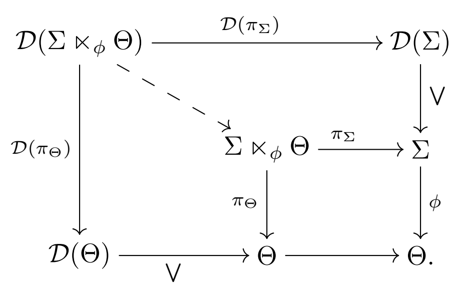
It is easily calculated that the designated truth-values in $\Sigma \ltimes_\phi \Theta$ are exactly the elements $(a, b)$ such that $a \in TV(\Sigma)$ and $b \in TV(\Theta)$, and that the realizable functions on $\Sigma \ltimes_\phi \Theta$ are those of the form $(a, b) \bullet -$, where $(a, b) \in TV(\Sigma \ltimes_\phi \Theta)$.

Taken all together, this shows that the important subcategory of BCOs on the ordered PCAs with filters is closed under the comma construction.

**Complete BCOs.** We now show that the category of pseudo algebras for the downset monad on **CartBCO** is closed under the comma construction. So let $\phi : \Sigma \to \Theta$ be a map of $\mathcal{D}$-algebras. Then the supremum of a family $(a_i, b_i)$ is defined as

$$\bigvee (a_i, b_i) = (\bigvee a_i, \bigvee (\phi(a_i) \wedge b_i)).$$

That this is a sensible definition can be seen by considering the diagram




Taking the route along the left-hand side, we see that a family $(a_i, b_i)$ gets mapped to $\bigvee(\phi(a_i) \wedge b_i)$. Now we have the entailments

$$\bigvee((\phi(a_i) \wedge b_i) \vdash \bigvee \phi(a_i) \vdash \phi \bigvee a_i.$$

The rightmost expression, however, is exactly the result of chasing $(a_i, b_i)$ along the right-hand side of the diagram. Thus we have that $1_\Theta \circ \bigvee \circ \mathcal{D}(\pi_\Theta) \vdash \phi \circ \bigvee \circ \mathcal{D}(\pi_\Sigma)$, and the universal property of the small square now gives the dashed arrow. The diagram also makes clear that the two projections $\pi_\Sigma, \pi_\Theta$ are maps of $\mathcal{D}$-algebras. In fact, the unit and multiplication morphisms are also maps of $\mathcal{D}$-algebras, so that the monad $- \ltimes \Theta$ restricts to a monad on the category of $\mathcal{D}$-algebras.

**Triposes.** We will now prove that when both $\Sigma$ and $\Theta$ have implication, so does the object $\Sigma \ltimes_\phi \Theta$. Define

$$(a, b) \Rightarrow (a', b') = (a \Rightarrow a', (\phi(a) \wedge b) \Rightarrow b').$$

The following computation shows that this defines a right adjoint to the meet map:

$$
\begin{aligned}
(x,y)\wedge(a,b)\vdash(a',b') \quad &\text{iff} \quad (x\wedge a, y\wedge b)\vdash(a',b')\\
&\text{iff} \quad x\wedge a\vdash a' \text{ and } \phi(x\wedge a)\wedge y\wedge b\vdash b'\\
&\text{iff} \quad x\vdash a\Rightarrow a' \text{ and } \phi(x)\wedge\phi(a)\wedge y\wedge b\vdash b'\\
&\text{iff} \quad x\vdash a\Rightarrow a' \text{ and } \phi(x)\wedge y\vdash(\phi(a)\wedge b)\Rightarrow b'\\
&\text{iff} \quad (x,y)\vdash(a,b)\Rightarrow(a',b')
\end{aligned}
$$

We summarize all the special cases in the following theorem.

**Theorem 4.1** *Let $\Sigma, \Theta$ be cartesian BCOs and let $\phi : \Sigma \to \Theta$ be a cartesian morphism. Then:*

1. *If $\Sigma, \Theta$ are meet-semi-lattices then $\Sigma \ltimes_\phi \Theta$ is equivalent to a meet-semi-lattice.*
2. *If $\Sigma, \Theta$ are (ordered) PCAs, then so is $\Sigma \ltimes_\phi \Theta$.*
3. *If $\Sigma, \Theta$ arise in the canonical way from ordered PCAs with a filter of truth-values, then so does $\Sigma \ltimes_\phi \Theta$.*
4. *If $\Sigma, \Theta$ are pseudo algebras for the downset monad and $\phi$ is a map of pseudo algebras, then $\Sigma \ltimes_\phi \Theta$ is a pseudo algebra.*
5. *If the associated indexed preorders of $\Sigma, \Theta$ are triposes and $\phi$ is a map of pseudo algebras, then the associated indexed preorder of $\Sigma \ltimes_\phi \Theta$ is a tripos.*

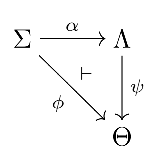
# 5 Iterated realizability

We will now give an interpretation of the comma construction in terms of iterated realizability. This is based on Pitts' iteration theorem for triposes. We focus on the special case of PCAs, but the results remain valid when we replace PCAs by ordered PCAs or ordered PCAs with filters. For a PCA $\Sigma$, we denote the standard realizability topos over $\Sigma$ by $\mathbf{RT}(\Sigma)$.

We start by recalling that a morphism of PCAs $\phi : \Sigma \to \Theta$ exhibits $\Sigma$ as an internal PCA in the topos $\mathbf{RT}(\Theta)$. In fact, the object $(\Sigma, \phi)$ is already an object in the category of partitioned assemblies over $\Theta$. Therefore, as an object of $\mathbf{RT}(\Theta)$, it is a projective PCA.

**Lemma 5.1** *There is a 2-functor from the lax slice category $\mathbf{PCA}//\Theta$ to the 2-category $\mathbf{PCA}(\mathbf{RT}(\Theta))$ of PCAs in $\mathbf{RT}(\Theta)$. This functor is a local equivalence.*

**Proof.** (Sketch.) Consider first a map



such that $\phi \vdash \psi\alpha$. Then the function $\alpha$ is the underlying function of a morphism $(\Sigma, \phi) \to (\Lambda, \psi)$ of partitioned assemblies, for it is tracked by the same function which realizes $\phi \vdash \psi\alpha$. Moreover, the fact that $\phi$ is a morphism of PCAs implies that the same will be the case internally in $\mathbf{RT}(\Theta)$.

The action of the functor on 2-cells is also immediate, as is the fact that it is a local equivalence.

□

This lemma is the main reason for being interested in the lax slice category, as opposed to the genuine slice. The 2-functor just described factors through the 2-category of PCAs internal to the category of partitioned assemblies over $\Theta$. However, it seems that not every PCA in that category is in the image of the functor: the PCAs in the image have the property that the domain $D$ of the application $\Sigma \times \Sigma \rightharpoonup \Sigma$ is a regular subobject of $\Sigma \times \Sigma$, and I see no reason why an arbitrary PCA would have that property.

Given $\phi : \Sigma \to \Theta$, the following presentation of the application on $(\Sigma, \phi)$ as internal PCA in $\mathbf{RT}(\Theta)$ will be useful: as a partial functional relation $App$ : $(\Sigma, \phi) \times (\Sigma, \phi) \rightharpoonup (\Sigma, \phi)$, it can be represented by

$$App(a, b, c) = \{\phi(a) \wedge \phi(b) \wedge \phi(c) | a \bullet b = c\}.$$

For the special case where $\Sigma = \mathbb{N}$, the following theorem was proved by Pitts (see [Pit81]). The general case poses no extra difficulties, but we outline the proof for completeness and later use.

**Theorem 5.2** *Let $\phi : \Sigma \to \Theta$ be a morphism of PCAs. Then the toposes $\mathbf{RT}(\Sigma, \phi)$ and $\mathbf{RT}(\Sigma \ltimes_\phi \Theta)$ are equivalent.*

**Proof.** By the iteration theorem, the topos $\mathbf{RT}(\Sigma, \phi)$ can be obtained from a **Set**-tripos; this tripos is of the form $X \mapsto \mathbf{RT}(\Theta)[\nabla X, \mathcal{P}(\Sigma, \phi)]$. The strategy of the proof is to give a convenient presentation for this tripos, and then to show that it is equivalent to the tripos obtained from the PCA $\Sigma \ltimes_\phi \Theta$. It then follows that the two toposes are equivalent as well.

First, given a set $X$, the realizability tripos over $(\Sigma, \phi)$ sends the constant object $\nabla X$ of $\mathbf{RT}(\Theta)$ to the set $\mathbf{RT}(\Theta)[\nabla X, \mathcal{P}(\Sigma, \phi)]$. This set is in bijective correspondence with the set

$$\mathbb{Q}(X) := \{\gamma : X \times \Sigma \to \mathcal{P}(\Theta) | \alpha(x, a) \vdash_{x,a} \phi(a)\},$$

which is simply the set of strict relations on $X \times \Sigma$.

The bijection $\mathbf{RT}(\Theta)[\nabla X, \mathcal{P}(\Sigma)] \cong \mathbb{Q}(X)$ induces a preorder $\vdash^{\mathbb{Q}}_X$ on the set $\mathbb{Q}(X)$ which can be described as follows: For two strict relations $\alpha, \beta$ : $X \times \Sigma \to \mathcal{P}(\Theta)$ we have $\alpha \vdash^{\mathbb{Q}}_X \beta$ if and only if there exist $a \in \Sigma, b \in \Theta$ for which $b \in (\alpha \to \beta)(x, a)$ for all $x \in X$. Here,

$$(\alpha \to \beta)(x, a) = \{\phi(a)\} \wedge \forall m(\alpha(x, m) \to \exists y.(App(a, m, y) \wedge \beta(x, y))).$$

Next it is demonstrated that the indexed preorder $\mathbb{Q}$ is equivalent to the indexed preorder for the PCA $\Sigma \ltimes_\phi \Theta$. There is a map $l_X : \mathbf{Set}[X, \mathcal{P}(\Sigma \ltimes_\phi \Theta)] \to \mathbb{Q}(X)$, defined for $\alpha : X \to \mathcal{P}(\Sigma \ltimes_\phi \Theta)$ by

$$l_X(\alpha)(x, a) = \{\phi(a) \wedge b | (a, b) \in \alpha(x)\}.$$

Now $l$ is an indexed functor, which not only preserves the preorder, but also reflects it.

There is also an indexed functor in the opposite direction: given $\beta \in \mathbb{Q}(X)$, put

$$r_X(\beta)(x) = \{(a, b) | b \in \beta(x, a)\}.$$

Then $r$ is a pseudo inverse to $l$, making the triposes $\mathbb{Q}$ and $\mathbf{Set}[-, \mathcal{P}(\Sigma \ltimes_\phi \Theta)]$ equivalent.

□

Let us now investigate the naturality of the exhibited equivalence. Consider a morphism $\alpha : (\Sigma, \phi) \to (\Lambda, \psi)$ over $\Theta$; then we claim that the square

$$\begin{array}{ccc} \mathbf{RT}(\Sigma, \phi) & \xrightarrow{\alpha^*} & \mathbf{RT}(\Lambda, \psi) \\ \simeq\downarrow & & \simeq\downarrow \\ \mathbf{RT}(\Sigma \ltimes_\phi \Theta) & \xrightarrow{(\alpha \ltimes \Theta)^*} & \mathbf{RT}(\Lambda \ltimes_\psi \Theta) \end{array}$$

commutes up to isomorphism. Here the horizontal arrows are the induced exact functors (which have right adjoints when $\alpha$ is computationally dense). We verify this on the level of triposes.

First, the map $\alpha : \Sigma \to \Lambda$ is a morphism of internal PCAs $\alpha : (\Sigma, \phi) \to (\Lambda, \psi)$ in the topos $\mathbf{RT}(\Lambda)$. Applying the covariant powerset functor, we get a map $\mathcal{P}(\alpha) : \mathcal{P}(\Sigma) \to \mathcal{P}(\Lambda)$.

We now fix a set $X$, and a strict relation $p : \nabla X \times \Sigma \to \mathcal{P}(\Theta)$, which is an element in the fibre over $X$ for the tripos for $\mathbf{RT}(\Sigma, \phi)$. We can apply $\mathcal{P}(\alpha)$ and obtain a strict relation $p^\alpha : X \times \Lambda \to \mathcal{P}(\Theta)$. Explicitly, we have

$$p^\alpha(x, c) = [\![\exists a \in \Sigma. \alpha(a) = c \wedge p(x, a)]\!].$$

Because $p$ is strict and $\alpha$ is a morphism over $\Theta$, this may be rewritten as:

$$p^\alpha(x, c) = \bigcup_{\alpha(a)=x} p(x, a).$$

Applying the map $r_X$ to $p^\alpha$ gives a function $r_X p^\alpha : X \to \mathcal{P}(\Lambda \ltimes_\psi \Theta)$, which may be given as by

$$r_X p^\alpha(x) = \{(c, b) | b \in p^\alpha(x, c)\},$$

which, by the foregoing, is equivalent to

$$r_X p^\alpha(x) = \{(c, b) | \exists a. \alpha(a) = c, b \in p(x, a)\}.$$

The other way around the diagram, we first apply $r_X$ to $p$ and then apply $(\alpha \ltimes \Theta)^*$, as to obtain a function $(\alpha \ltimes \Theta)^*(r_X p)$, which satisfies

$$(\alpha \ltimes \Theta)^*(r_X p)(x) = \{(\alpha(a), b) | b \in p(x, a)\}.$$

It is obvious that $r_X p^\alpha \dashv\vdash (\alpha \ltimes \Theta)^*(r_X p)$ as elements of the fibre over $X$ in the tripos for $\Lambda \ltimes_\psi \Theta$.

**Change of Base.** When a morphism $\beta : \Theta \to \Theta'$ is given, we have an induced morphism of PCAs $\beta^\phi : \Sigma \ltimes_\phi \Theta \to \Sigma \ltimes_{\beta\phi} \Theta'$. Composing the induced map $(\beta^\phi)^* : \mathbf{RT}(\Sigma \ltimes_\phi \Theta) \to \mathbf{RT}(\Sigma \ltimes_{\beta\phi} \Theta')$ with the the equivalences $\mathbf{RT}(\Sigma \ltimes_\phi \Theta) \simeq \mathbf{RT}(\Sigma, \phi)$ and $\mathbf{RT}(\Sigma \ltimes_{\beta\phi} \Theta') \simeq \mathbf{RT}(\Sigma, \beta\phi)$, we get a diagram of toposes which commutes up to isomorphism:

$$\begin{array}{ccc}
\mathbf{RT}(\Theta') & \longleftarrow & \mathbf{RT}(\Sigma, \beta\phi) \\
{\scriptstyle \beta^*}\downarrow & & \downarrow \\
\mathbf{RT}(\Theta) & \longleftarrow & \mathbf{RT}(\Sigma, \phi)
\end{array}$$

In case $\beta$ is computationally dense, then all morphisms in this square are inverse image functors of geometric morphisms.

# 6 Concluding remarks

There are a few open problems to which we would like to draw the reader's attention. First, there are various questions surrounding the factorization of a morphism in the category **CartBCO**. As remarked, this factorization is analogous to the comprehensive factorization on categories, but it is not clear how close the analogy is, nor how fruitful. Questions that remain to be answered are: can we give elementary characterizations of the two classes of morphisms obtained? Do these classes satisfy some 2-categorical orthogonality properties? Also, while we have proved that every fibration is a computationally dense map, the converse does not hold in general, and it would be nice to know what extra properties are satisfied by fibrations that dense maps lack.

Another issue, which we already hinted at, is the following. Morally, the category of realizability toposes should be closed under iteration. We have shown that it is closed under a certain type of iteration, namely the kind that is the result of viewing $\phi : \Sigma \to \Theta$ as an internal projective PCA in $\mathbf{RT}(\Theta)$. But there may be various non-projective PCAs in $\mathbf{RT}(\Theta)$ over which one can form a realizability topos. Can we describe *all* internal PCAs in $\mathbf{RT}(\Theta)$ as some kind of structure in the category of PCAs over $\Theta$? This should be compared to the classical result that the category of internal locales in $\mathbf{Sh}(X)$ is equivalent to $\mathbf{Loc}/X$.

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