Problem 1. Find the singular values of the matrix \( A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \).

Solution. We compute \( AA^T \). (This is the smaller of the two symmetric matrices associated with \( A \).) We get \( AA^T = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} \). We next find the eigenvalues of this matrix. The characteristic polynomial is \( \lambda^3 - 6\lambda^2 + 6\lambda = \lambda(\lambda^2 - 6\lambda + 6) \). This gives three eigenvalues: \( \lambda = 3 + \sqrt{3}, \lambda = 3 - \sqrt{3} \) and \( \lambda = 0 \). Note that all are positive, and that there are two nonzero eigenvalues, corresponding to the fact that \( A \) has rank 2.

For the singular values of \( A \), we now take the square roots of the eigenvalues of \( AA^T \), so \( \sigma_1 = \sqrt{3 + \sqrt{3}} \) and \( \sigma_2 = \sqrt{3 - \sqrt{3}} \). (We don’t have to mention the singular values which are zero.)

Problem 2. Find the singular values of the matrix \( B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \).

Solution. We use the same approach: \( AA^T = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \). This has characteristic polynomial \( \lambda^2 - 10\lambda + 9 \), so \( \lambda = 9 \) and \( \lambda = 1 \) are the eigenvalues. Hence the singular values are 3 and 1.

Problem 3. Find the singular values of \( A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \) and find the SVD decomposition of \( A \).

Solution. We compute \( AA^T \) and find \( AA^T = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 2 \end{bmatrix} \). The characteristic polynomial is

\[
-\lambda^3 + 10\lambda^2 - 16\lambda = -\lambda(\lambda^2 - 10\lambda + 16) \\
= -\lambda(\lambda - 8)(\lambda - 2)
\]
So the eigenvalues of $AA^T$ are $\lambda = 8, \lambda = 2, \lambda = 0$. Thus the singular values are $\sigma_1 = 2\sqrt{2}, \sigma_2 = \sqrt{2}$ (and $\sigma_3 = 0$).

To give the decomposition, we consider the diagonal matrix of singular values $\Sigma = \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Next, we find an orthonormal set of eigenvectors for $AA^T$. For $\lambda = 8$, we find an eigenvector $p_1 = \left( \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)$. For $\lambda = 2$ we find $p_2 = \left( -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right)$, and finally for $\lambda = 0$ we get $p_3 = \left( \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right)$.

This gives the matrix $P = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

Finally, we have to find an orthogonal set of eigenvectors for $A^TA = \begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix}$.

This can be done in two ways. We show both ways, starting with orthogonal diagonalization. We already know that the eigenvalues will be $\lambda = 8, \lambda = 2, \lambda = 0$. This gives eigenvectors $q_1 = \left( \frac{1}{\sqrt{6}}, \frac{3}{\sqrt{12}}, \frac{1}{\sqrt{12}} \right), q_2 = \left( \frac{1}{\sqrt{3}}, 0, -\frac{2}{\sqrt{6}} \right)$ and $q_3 = \left( \frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2} \right)$. Put these together to get

$$Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{12}} & 0 & -\frac{1}{2} \\ \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{6}} & \frac{1}{2} \end{bmatrix}$$

For a quicker method, we calculate the columns of $Q$ using those of $P$ using the formula

$$p_i = \frac{1}{\sigma_i} A^T p_i.$$  

Thus we calculate

$$p_1 = \frac{1}{\sigma_1} A^T p_1 = \frac{1}{\sqrt{8}} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} = q_1$$

and similarly for the other two columns.

Either way we can now verify that we have $A = P\Sigma Q^T$.  

Problem 4. Find the SDV of the matrix \( A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \).

Solution. We first compute
\[
AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.
\]

We see immediately that the eigenvalues of \( AA^T \) are \( \lambda_1 = \lambda_2 = 2 \) (and hence that the eigenvalues of \( A^T A \) are 2 and 0, both with multiplicity 2), and thus the matrix \( A \) has singular value \( \sigma_1 = \sigma_2 = \sqrt{2} \).

Next, an orthonormal basis of eigenvectors of \( AA^T \) is \( p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). (You can choose any orthonormal basis for \( \mathbb{R}^2 \) here, but this one makes computation easiest.) Thus we set
\[
P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

Lastly we have to find \( Q \). We use the formula
\[
q_1 = \frac{1}{\sigma_1} A^T p_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.
\]

We also need \( q_3 \) and \( q_4 \) but we can’t compute those using the same formula, since we just ran out of \( p_i \)'s. However, we know that the \( q_1, q_2, q_3, q_4 \) should be an orthonormal basis for \( \mathbb{R}^4 \), so we need to choose \( q_3 \) and \( q_4 \) in such a way that this indeed works out. We choose
\[
q_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad q_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.
\]
giving

\[ Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}. \]

It is now easy to check that \( A = P \Sigma Q^T \), where \( \Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \).

Note: we could also have diagonalized \( A^T A \) to obtain \( Q \), but we need to be careful, because if we choose the eigenvectors in the wrong way, we don’t get \( A = P \Sigma Q^T \); however, this can always be fixed by multiplying the eigenvectors by \(-1\) as needed.