Martin-Löf Complexes

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Abstract
In this paper we define Martin-Löf complexes to be algebras for monads on the category of (reflexive) globular sets which freely add cells in accordance with the rules of intensional Martin-Löf type theory. We then study the resulting categories of algebras for several theories. Our principal result is that there exists a cofibrantly generated Quillen model structure on the category of 1-truncated Martin-Löf complexes and that this category is Quillen equivalent to the category of groupoids. In particular, 1-truncated Martin-Löf complexes are a model of homotopy 1-types.

\textit{Dedicated to Per Martin-Löf on the occasion of his retirement.}

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1. Introduction

This paper pursues a surprising connection between Geometry, Algebra, and Logic that has only recently come to light, in the form of an interpretation of the constructive type theory of Martin-Löf into homotopy theory, resulting in new examples of certain algebraic structures which are important in topology.

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This fascinating connection is currently under investigation from several different perspectives ([2, 24, 7, 17, 5, 6]), and these preliminary results confirm the significance of the link. Some of these results will be surveyed in this brief introduction in order to position the present work in its context; for a more detailed and comprehensive introduction, we refer the reader to the survey paper [1].

Martin-Löf type theory [18] is a formal system originally intended to provide a rigorous framework in which to develop constructive mathematics. At heart, it is a calculus for reasoning about dependent types and terms, and equality between those. Under the Curry-Howard correspondence, one may identify types with propositions, and terms with proofs. Viewed in this manner, the system can be shown to be at least as strong as second-order logic, and it is also known to interpret constructive set theory. Indeed, Martin-Löf type theory has been used successfully to formalize parts of constructive mathematics, such as pointless topology (constructive locale theory). Moreover, it has been employed as a framework for the development of programming languages as well, a task for which it is especially well-suited in virtue of its combination of expressive strength and desirable proof-theoretic properties. (See the textbook [19] for a discussion.)

The type theory has two variants: an intensional, and an extensional version. The difference between them lies mainly in the treatment of equality. In the intensional version (with which we are mainly concerned in the present work), one has two different kinds of equality: the first kind is called definitional equality, and behaves much like equality between terms in the simply-typed lambda-calculus, or any other conventional equational theory. The second kind is a more subtle relation, called propositional equality, which, under the Curry-Howard correspondence, represents the equality formulas of first-order logic. Specifically, given two terms \(a, b\) of the same type \(A\), one may form a new type \(\text{Id}_A(a, b)\), which we think of as the proposition that \(a\) and \(b\) are equal; a term of this type thus represents a proof of the proposition that \(a\) equals \(b\) (hence the name “propositional equality”).

In the intensional version with which we shall be concerned here, it can be shown that the identity types \(\text{Id}_A(a, b)\) carry certain structure which was observed by Hofmann and Streicher in [10] to be analogous to that of a groupoid. Specifically, the reflexivity of propositional equality produces identity proofs \(\text{r}(a) : \text{Id}_A(a, a)\) for any term \(a : A\), playing the role of a unit arrow for \(a\); and when \(f : \text{Id}_A(a, b)\) is an identity proof, then (corresponding to the symmetry of identity) there also exists a proof \(f^{-1} : \text{Id}_A(b, a)\), to be thought of as the inverse of \(f\); finally, when \(f : \text{Id}_A(a, b)\) and \(g : \text{Id}_A(b, c)\) are identity proofs, then (corresponding to transitivity) there is a new proof \((g \cdot f) : \text{Id}_A(a, c)\), thought of as the composite of \(f\) and \(g\). Moreover, this structure on each type \(A\) can be shown to satisfy the usual groupoid laws, but only up to propositional equality. We shall return to this point below.

The first non-trivial semantics for intensional type theory was developed by Hofmann and Streicher [10] using groupoids. The category of groupoids is not locally cartesian closed, and the model employs certain fibrations (equivalently, groupoid-valued functors) to model type dependency. A closed type \(A\) will be
interpreted as a groupoid, a term \( a : A \) as an object of this groupoid, and an identity proof \( f : \text{Id}_A(a, b) \) as an arrow \( f : a \to b \) in \( G \). The interpretation no longer validates extensionality, since there can be different elements \( a, b \) related by non-identity arrows \( f : a \to b \). However, the groupoid semantics validates a certain truncation principle, stating that all higher identity types are trivial—a form of extensionality one dimension up. In particular, the groupoid laws for the identity types are strictly satisfied in these models, rather than holding only up to propositional equality. Warren [24] has generalized the groupoid model of [10] to strict \( \omega \)-groupoids, thereby showing that the type theory truly possesses non-trivial higher-dimensional structure. Along similar lines, Garner [7] has used a 2-dimensional notion of fibration to model intensional type theory, and shown that when various truncation axioms are added the theory is sound and complete with respect to this semantics.

**Quillen model categories** capture axiomatically some of the essential features of homotopy of topological spaces, enabling us to “do homotopy” in different mathematical settings, and to express the fact that two categories carry the same homotopical information, even if they are not equivalent in the ordinary sense. The basic result of Awodey and Warren in [2] (see also [24]) is that it is possible to model the type theory in any Quillen model category which is well-behaved in certain ways (essentially using just the basic notion of a weak factorization system). This suggests that intensional type theories are a sort of internal language of (certain kinds of) model categories. Indeed, in [6] it is shown that the type theory itself carries a natural such homotopy structure (i.e., a weak factorization system), so that the theory is not only sound but also complete with respect to such abstract homotopical semantics.

Thus we are justified in thinking of types in the intensional theory as **spaces**. From this point of view, the terms of the type \( A \) are the points of the “space” \( A \), the identity type \( \text{Id}_A(a, b) \) represents the collection of paths from \( a \) to \( b \), and the higher identities are homotopies between paths, homotopies between homotopies of paths, et cetera. The fact that paths and homotopies do not form a groupoid, but only a groupoid up to homotopy, is of course precisely the same observation as the fact that the identity types only satisfy the groupoid laws up to propositional equality. This parallel between type theory and homotopy theory, which was first pointed out by Moerdijk a few years ago, has now been made precise by the recognition that both cases are instances of one and the same abstract axiomatic theory.

Along these lines, it has been shown independently by Lumsdaine [17] and Van den Berg and Garner [5] that the tower of identity types over any fixed base type \( A \) in the intensional theory indeed gives rise to a certain infinite dimensional categorical structure called a weak \( \omega \)-groupoid. The next step in exploring the connection between type theory and topology is to investigate the relationship between type theoretic “truncation” (i.e. higher-dimensional extensionality principles) and topological “truncation” of the higher fundamental groups. Spaces for which the homotopy type is already completely determined by the fundamental groupoid are called **homotopy 1-types**, or simply 1-types. The category of groupoids is Quillen equivalent to the category of 1-types and
therefore the corresponding homotopy categories (obtained by inverting weak equivalences) are equivalent; in this precise sense, groupoids are said to model homotopy 1-types (for more on homotopy types see [4]).

1.1. Contributions of this paper

The current paper aims at further investigation of the relationship between type theory and homotopy theory, but in a way that is somewhat different from the work already mentioned. First of all, our primary objective is not to give a new semantics, although some of the results will depend on a new model which will be presented in a sequel to this paper. Secondly, while earlier work centered around constructing higher-dimensional structures from type theories, we are also interested in understanding the limitations of this process. Finally, we wish to make another connection between model categories and type theory, namely by showing that a category of suitably truncated type theories gives a model of the homotopy 1-types. It is our hope that this picture can then be extended to higher dimensions.

Our first goal is to show how every extension of intensional type theory gives rise to a monad on the category of globular sets. Intuitively, the monad associated to a theory freely adds cells to a globular set in accordance with the structure imposed on the tower of identity types over a base type by the rules of the type theory. For example, the monad will formally add composites and inverses for all cells of dimension strictly greater than 0 in the globular set; however, it adds much more than just these formal composites; it also produces a plethora of new cells which we here call doppelgängers. For every such monad we may consider its category of algebras: these we refer to as Martin-Löf complexes (or ML-complexes), and these are the main objects of study of the paper.

The theories which we shall consider arise from basic intensional Martin-Löf type theory having dependent sums and products as well as a natural numbers object. (The latter plays no conceptual role in this paper but because of its importance in virtually every application of the theory to mathematics and computer science we thought it important to show that our results are not affected by its presence.) We shall then consider extensions of this basic theory obtained by adding truncation axioms, which effectively trivialize the higher identity types above a fixed dimension. Using these theories we get a hierarchy of categories of Martin-Löf complexes, and in this paper we shall investigate the first two dimensions in detail.

The 0-dimensional case is relatively straightforward — we shall prove here that the monad on globular sets is idempotent and is in fact isomorphic to the connected components functor, so that its category of algebras (the 0-dimensional ML-complexes) is equivalent to the category of sets.

Matters become more interesting in dimension 1. Towards an analysis of 1-dimensional ML-complexes we first observe, using the Hofmann-Streicher groupoid semantics, that every ML-complex has an underlying groupoid, and that there is a canonical comparison functor between the underlying groupoid of a free ML-complex and the free groupoid on the same globular set. This functor is
not an isomorphism of groupoids, because the free ML-complex is, intuitively speaking, much larger due to all the doppelgängers produced by the theory. The main technical difficulty then is to prove that there is still an equivalence of groupoids between the two. This result follows from a proof-theoretic analysis of the theories in question, which will be borrowed from [11]. One of the main results in loc. cit. allows us to conclude that every term of the theory represents, up to propositional equality, an object or morphism of the free groupoid. This essentially shows that even though the theory forces the existence of many more objects and arrows than needed to form the free groupoid, it does not force anything which is undesirable from a homotopical point of view.

Once this key result is in place, we turn to an analysis of the category of 1-dimensional ML-complexes as a whole. To start, we set up an adjunction between this category and the category of groupoids. This adjunction is not an equivalence: a ML-complex structure on a globular set carries essentially more information than a groupoid structure. We can, however, make use of the adjunction by transferring along it the standard Quillen model structure on the category of groupoids [13], turning the category of 1-dimensional ML-complexes into a cofibrantly generated model category.

Finally, we prove that the adjunction between groupoids and 1-dimensional ML-complexes is in fact a Quillen equivalence. Because the categories of groupoids and that of homotopy 1-types are Quillen equivalent, this makes precise in which sense the 1-truncated version of the type theory models homotopy 1-types. It also explains why the groupoid semantics is adequate from a homotopical point of view, but is still incomplete because it lacks the possibility (which is present in ML-complexes) of handling different interpretations for doppelgänger terms.

1.2. Plan of the paper

In Section 2 we recall the basics of Martin-Löf type theory as well as several facts about globular sets which will be required later. We also fix notation (some of which is non-standard). The reader who is familiar with this material should feel free to skip ahead.

Section 3 describes the construction of monads on the category of reflexive globular sets coming from type theories. We then define the categories $\text{MLCx}$ and $\text{MLCx}_n$ of Martin-Löf complexes and $n$-truncated Martin-Löf complexes as the Eilenberg-Moore categories of the monads $M_\omega$ and $M_n$, respectively, generated by suitable theories. These monads are shown to be finitary and it therefore follows that the categories $\text{MLCx}$ and $\text{MLCx}_n$ are complete and cocomplete.

In Section 4 we study 0-truncated and 1-truncated Martin-Löf complexes. We first show that the category $\text{MLCx}_0$ is equivalent to the category of sets and, moreover, that if $X$ is a reflexive globular set, then $M_0(X)$ is the set of connected components of $X$. Even the proofs of these eminently plausible results are a bit more complicated than one might at first expect; one of the principal difficulties one faces when proving results about Martin-Löf complexes is that the type theory also adds, in addition to composition and inverses, the doppelgänger terms mentioned earlier. We then turn to 1-truncated complexes,
with a proof that every such complex can be equipped with the structure of a groupoid. Towards a characterization of the free 1-dimensional ML-complexes, it is first shown that the Hofmann-Streicher groupoid semantics induces a comparison functor between the free groupoid $\mathcal{F}(G)$ on a reflexive globular set and the induced groupoid structure on $M_1(G)$, the free 1-dimensional algebra on $G$. The main technical observation, namely that this comparison functor is an equivalence of groupoids, follows from the semantics presented in [11]. Because this technique will also be required later, we shall give, for reasons of self-containment of the present paper, a brief explanation of how it works.

Finally, Section 5 shows that $\text{MLCx}_1$ can be endowed with a Quillen closed model structure. This model structure is obtained from an adjunction with the category of groupoids via Quillen’s path object argument [20]. The main result states that the adjunction between $\text{MLCx}_1$ and the category of groupoids is a Quillen equivalence (Theorem 5.10). Both the existence of the model structure and the verification of the Quillen equivalence depend in a crucial way on the semantics from [11].

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2. Background

The purpose of this section is to provide the reader with a brief introduction to those aspects of Martin-Löf type theory which feature in this paper. We begin by giving a quick exposition of the main features of the most basic version of the theory we shall be concerned with. In particular we explain the different kinds of judgements of the system, dependent products, identity types and the notion of propositional equality. We also use this as an opportunity to fix some notation and terminology, in particular concerning identity types.

We assume that the reader is somewhat familiar with at least simple type theory. For more background on (dependent) type theory we refer to the textbook [12]. The reader who is more familiar with higher-dimensional category theory or homotopy theory might also consult [2] for a “homotopical” view of type theory.

In the last subsection we introduce the basic categorical structures used in the paper, namely globular sets. A more detailed exposition of globular sets may be found in [22], or the textbook [15].

While the machinery developed in this paper only depends on the presence of dependent products and identity types, we stress that the main results apply to versions of ML type theory with dependent sum types and natural numbers
as well. In the appendix the reader can find the precise formulation of these other type constructors.

2.1. Type dependency, contexts and judgements

Dependent type theory, like simple type theory, is concerned with types, and with terms inhabiting those types; but distinguishes itself from simple type theory through the presence of type dependency. Type dependency means that types may depend on variables of other types; for example one can have a type \( T(x) \) depending on a variable \( x \) of type \( S \). Such a type \( T(x) \) is often thought of as being indexed by the type \( S \). To illustrate this, suppose that we let \( S \) denote the type of rings; then the type \( T(x) \) of modules depends on, or varies over, the type of rings.

One may then substitute a term \( a \) of type \( S \) into the type \( T(x) \), as to obtain a new type \( T(a) \). In the above example, \( T(a) \) would be the type of modules over the ring \( a \).

The fact that types may depend on terms has two obvious consequences: first, one can no longer, as in simple type theory, separate the formation of types and that of terms into two inductive definitions; other, types and terms are derived simultaneously. Second, the notion of a variable context also needs to take dependency into account. Explicitly, this means that a variable context \( \Gamma \) is now an ordered sequence of variable declarations \( \Gamma = (x_1 : T_1, \ldots, x_k : T_k) \), where each type \( T_i \) may only depend on the variables declared earlier, i.e. on \( x_1, \ldots, x_{i-1} \). Throughout, we shall always assume that contexts are well-formed in this sense.

The theory is concerned with types and terms in context, and with equalities between such types and terms. Formally, statements about these are called judgements, and these come in four kinds:

\[ \Gamma \vdash T : \text{type} \]

This judgement states that \( T \) is a type, possibly depending on the variables declared in the context \( \Gamma \).

\[ \Gamma \vdash \tau : T \]

This judgement states that \( \tau \) is a term of type \( T \), where both \( \tau \) and the type \( T \) may depend on the variables from \( \Gamma \).

\[ \Gamma \vdash T = S : \text{type} \]

This judgement states that \( T \) and \( S \) are (definitionally) equal types.

\[ \Gamma \vdash \tau = \tau' : T \]

This judgement states that \( \tau \) and \( \tau' \) are (definitionally) equal terms of type \( T \).

In the theory, such judgements are derived from axioms using inference rules. These derivations (which may formally be regarded as finite trees suitably labelled by judgements and inference rules) are the main objects of study. Below
we shall discuss several of the rules which may be used to derive new judg-
ements from old; the axioms typically include judgements stating the existence
of certain basic types and terms.

When the context plays no role in a judgement or rule of the theory, we shall
usually omit it altogether.

2.2. Definitional equality

The notion of equality here is the congruence (in an appropriate sense) gen-
erated by the computation rules of the theory and the qualifier definitional
is used to distinguish it from the different notion of propositional equality, to
be discussed below. The rules governing the behavior of definitional equality
are as expected. Apart from the rules expressing that definitional equality is
an equivalence relation, there are rules which force that it is a congruence with
respect to substitution into types and terms:

\[
\begin{align*}
\vdash a &= b : A \\
\vdash x : A &\vdash B(x) : \text{type} \\
\vdash B(a) &= B(b) : \text{type} \ \\
\vdash a &= b : A \\
\vdash f(x) : B(x) &\vdash f(a) = f(b) : B(a) \\
\vdash a &= b : A \\
\vdash a : A &\vdash a : B \\
\vdash A &= B : \text{type} \\
\end{align*}
\]

The first rule states that substituting equal terms into a type results in equal
types; the second states the same, but now for substitution into terms; the last
rule states that equal types are inhabited by the same terms. A complete set of
rules for definitional equality may be found in the appendix.

2.3. Dependent products

There are several ways to construct new types from old. For each new type
one specifies three things: an introduction rule which generates new terms of
the type; an elimination rule which shows how general terms of the new type
may be used; and a conversion rule which governs the interaction between the
two.

We now discuss the formation of dependent products. Given a type \( B(x) \)
depending on a variable \( x \) of type \( A \), we may form the dependen-
product type \( \prod_{x:A} B(x) \), to be thought of as the type of sections of \( B(x) \) over \( A \). The
rules are as follows:

\[
\begin{align*}
x : A &\vdash B(x) : \text{type} \\
\vdash \prod_{x:A} B(x) : \text{type} &\text{ formation} \\
\vdash x : A &\vdash f(x) : B(x) \\
\vdash \lambda_{x:A}.f(x) : \prod_{x:A} B(x) &\text{ introduction}
\end{align*}
\]

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Thus an introduction term of type $\prod_{x:A} B(x)$ is a lambda expression, thought of as an operation assigning to each $x:A$ a value $f(x):B(x)$. A general term $f$ of type $\prod_{x:A} B(x)$ may be applied to a term $a$ of type $A$, as to return a term $\text{app}(f,a)$ of type $B(a)$. Finally, the conversion rule, commonly known as $\text{beta-conversion}$, allows us to reduce $\text{app}(\lambda x:A.f(x),a)$ to $f(a)$. In the case where the type $B(x)$ does not depend on the variable $x$, we shall often write $B^A$ for the type $\prod_{x:A} B(x)$.

2.4. Identity types

Let $A$ be a type. For every pair of terms $a,b$ of type $A$ we may form a new type $A(a,b)$ called the $\text{identity type}$. This type is thought of as the type of proofs of the fact that $a$ and $b$ are equal. A term $\tau : A(a,b)$ is sometimes referred to as a $\text{propositional identity proof}$. It is important to note that the existence of such a proof term does not necessarily imply that $a = b$ in the definitional sense of equality discussed above. It is perhaps more common to denote the identity type $A(a,b)$ by $\text{Id}_A(a,b)$, but we have chosen to adopt a notation more suggestive of hom-sets.

The formation rule for the identity types is thus as follows (omitting contexts for simplicity)

$$
\vdash a, b : A \\
\vdash A(a,b) : \text{type} \quad \text{Id formation}
$$

where we write $a,b : A$ as an abbreviation for the two judgements $a : A$ and $b : A$. Then, there are the introduction and elimination rules:

$$
\vdash a : A \\
\vdash r(a) : A(a,a) \quad \text{Id introduction}
$$

$$
x : A, y : A, z : A(x,y) \vdash B(x,y,z) : \text{type} \\
x : A \vdash \varphi(x) : B(x,x,r(x)) \\
\vdash f : A(a,b) \\
\vdash J_{[x,y:A,z:A(x,y)]B(x,y,z)}([x:A]\varphi(x), a, b, f) : B(a, b, f) \quad \text{Id elimination}
$$
The introduction term \( r(a) \) is called the **reflexivity term**; it witnesses the fact that \( a \simeq a \). One way to think of the elimination term is as the result of expanding the term \( \phi(x) \) using the propositional equality \( f : A(a,b) \). This viewpoint will be developed in more detail later on.

Note also that the variables \( x, y, z \) in the elimination rule need not necessarily occur in the type \( B(x, y, z) \), and similarly that \( x \) need not occur in \( \varphi(x) \). Also, it may happen that the term \( f \) (and possibly also \( a, b \)) is itself variable, in which case the \( J \)-term depends on those variables. (Indeed, similar remarks apply to all of the rules we have introduced here.)

Finally, there is a conversion rule:

\[
\begin{align*}
\vdash & a : A \\
\vdash & J_{[x, y, A, z : A(x, y, z)]B(x, y, z)} (\pi x : A \varphi(x), a, a, r(a)) = \varphi(a) : B(a, a, r(a)) \\
\end{align*}
\]

Thus, using a trivial identity proof \( r(a) \) to build a \( J \)-term does simply give back \( \varphi(a) \).

### 2.5. Theories and extensions

We shall denote by \( T_\omega \) the system having all of the above constructors and rules, plus possibly extra rules such as those for sum types and natural numbers (for a complete description see the appendix). By a **type theory** we shall mean any extension of the basic system \( T_\omega \) obtained by adding axioms and possibly also inference rules. The axioms are judgements which may assert the existence of basic types or terms, or may assert the equality between certain types or terms. Possible additional inference rules include the so-called **truncation-** and **reflection rules**, which express triviality of certain identity types. See Section 3 for a discussion of these rules.

Given two type theories \( T \) and \( T' \), we say that \( T' \) is an **extension** of \( T \) when every judgement which is derivable in \( T \) is also derivable in \( T' \). Notation: \( T \subseteq T' \). Thus by our definitions, \( T_\omega \) is the smallest type theory.

We introduce a category \( \mathbf{TT} \) of **pointed type theories**; its objects are pairs \((T, T)\) where \( T \) is a type theory extending \( T_\omega \), and where \( T \) is a type of \( T \). A morphism \( (T, T) \to (S, S) \) is a mapping sending types and terms of \( T \) to types and terms of \( S \), in such a way that derivability of judgements is preserved, and that \( T \) is mapped to \( S \).

### 2.6. Expressions

Because the types and terms of such theories are defined simultaneously, in order to formally specify the syntax of the theory it is convenient to first define inductively a class of **expressions** — which need not satisfy any typing conventions — from which the genuine syntactical data of the theory is then extracted via the rules given above (and stated in full in Appendix Appendix A). For example, in order to formally define the theory \( T_\omega \) we first fix a countable set \( V \) of (untyped) variables and then define the class of **expressions of** \( T_\omega \), denoted \( \text{Exp}(T_\omega) \), by
Thus, the expressions are generated by applying all term- and type constructors without regard for well-typedness. The derivation rules of the type theory may then be regarded as carving out from this set of all expressions those which are well-formed and well-typed. The syntax of other the theories extending $\mathbb{T}_\omega$ that we consider later is similarly specified in this way with the evident modifications to the definition of the expressions. Moreover, because the expressions are inductively generated it follows that the sets of the form $\text{Exp}(\cdot)$ possess an obvious universal property.

2.7. Context morphisms

Recall that if $\Gamma$ and $\Delta = (x_1:A_1,\ldots,x_n:A_n(x_1,\ldots,x_{n-1}))$ are contexts, then a context morphism $a : \Gamma \to \Delta$ is a sequence of terms

$$\Gamma \vdash a_1 : A_1, \ldots, \Gamma \vdash a_n : A_n(a_1,\ldots,a_{n-1}).$$

There is a category of contexts with arrows the context morphisms (cf. [9]).

2.8. Globular sets

Globular sets are structures which form the basis for several definitions of higher dimensional category. One way to think of a globular set is as a higher dimensional graph: not only are there vertices and edges between the vertices, but one has edges between edges, and so on. Formally, a globular set $G$ is a tuple $(G_n,s_n,t_n)_{n\in\mathbb{N}}$, where each $G_n$ is a set, and where $s_n,t_n : G_{n+1} \to G_n$ are functions subject to the globular identities

$$d_n d_{n+1} = d_n s_{n+1}; \quad s_n s_{n+1} = s_n d_{n+1}$$

for $d = s,t$. Elements of $G_n$ are referred to as n-cells, and are said to have dimension n. The maps $s_n$ and $t_n$ are called source and target maps, respectively.

If $G$ is a globular set for which $G_n = \emptyset$ for all $n > 1$, then we may simply regard $G$ as a (directed) graph. If there exist elements of higher dimension, then the globular identities ensure that the source $s_n(x)$ and target $t_n(x)$ for such an $n$-dimensional edge are a parallel pair of edges of dimension $n-1$. 

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Because it is often convenient, given a $n$-cell $\alpha$ of a globular set $G$, to be able to refer to the result of iteratively taking the source or target of $\alpha$ we introduce the notation $\alpha^0$, $\alpha^1$ for these corresponding $j$-cells. Explicitly, for $0 \leq j \leq n - 1$,

$$\alpha^j_i := \begin{cases} s_{j} \circ \cdots \circ s_{(n-1)}(\alpha) & \text{if } i = 0 \\ t_{j} \circ \cdots \circ t_{(n-1)}(\alpha) & \text{if } i = 1. \end{cases}$$

By the globular identities, $\alpha^0$ and $\alpha^1$ are the only elements of $G_j$ which are obtainable from $\alpha$ by applying the source and target maps.

A globular set $G$ is said to be reflexive if it comes equipped with a family of maps $i_n : G_n \to G_{n+1}$, such that

$$t_n i_n = 1 = s_n i_n \tag{2}$$

We think of $i_n(x)$ as the identity edge from $x$ to itself. In this paper we shall be working with reflexive globular sets only. For readability we often omit the dimension from the source, target and identity maps of a globular set.

A morphism of globular sets $f : G \to H$ is a family of functions $f_n : G_n \to H_n$ which commute with the source and target maps. Globular sets and their morphisms form a category denoted $\textbf{GSet}$. For reflexive globular sets we also require that the $f_n$ commute with the identity maps; this gives a category $\textbf{rGSet}$.

There is a functor $\Delta : \textbf{Set} \to \textbf{rGSet}$ which takes a set $A$ to the constant globular set with $\Delta(A)_n = A$. A globular set which is isomorphic to one of the form $\Delta(A)$ will be called constant. The functor $\Delta$ has a left adjoint $\pi_0 : \textbf{rGSet} \to \textbf{Set}$; this functor assigns to a globular set $G$ its set of connected components

$$\pi_0(G) = G_0/\sim$$

where the equivalence relation on 0-dimensional elements is generated by

$$x \sim y \iff \exists f \in G_1.s(f) = x, t(f) = y.$$ 

We may express this as a (reflexive) coequalizer diagram:

$$G_1 \xrightarrow{s} G_0 \xrightarrow{t} \pi_0(G).$$

The composite $\Delta\pi_0 : \textbf{rGSet} \to \textbf{rGSet}$ is an idempotent monad, to which we refer as 0-truncation. Often it will be convenient to identify the essential image of this functor (the constant globular sets) with the category of sets.

We may also truncate a globular set at dimension 1: in this case we replace the category of sets by the category $\textbf{rGraph}$ of directed reflexive graphs. There is a functor $\textbf{rGSet} \to \textbf{rGraph}$ which assigns to a globular set $G$ the graph whose vertex set is $G_0$ and whose edge set is $G_1/\sim$, where two edges $h, k$ satisfy $h \sim k$ if there is an $\alpha \in G_2$ with $s(\alpha) = h, t(\alpha) = k$.

In the other direction, any directed reflexive graph $G$ gives a globular set with which is the same as $G$ in dimensions 0 and 1, and is degenerate above
dimension 1. The composite functor \( \text{rGSet} \to \text{rGraph} \to \text{rGSet} \) will be called \textbf{1-truncation}, and a globular set in the essential image of this functor will be said to be \textbf{1-truncated}. We shall often identify the subcategory of 1-truncated globular sets with the category of graphs.

3. The Martin-Löf complex monad

The goal of this section is to state the formal definition of Martin-Löf complexes. Because Martin-Löf complexes are defined to be algebras for a monad on the category of reflexive globular sets the principal matter addressed here is the construction of the appropriate monad. The monad for the theory \( T_\omega \) obtained by the construction below is related to the monads obtained from the operadic constructions due to van den Berg and Garner [5] and Lumsdaine [17]. In particular, the monads (implicitly) constructed by van den Berg and Garner and Lumsdaine are the “operadic coreflections” of the monads constructed here, and they have shown who shown that algebras for their monads can be equipped with the structure of weak \( \omega \)-groupoids. It is worth emphasizing that, because the converse seems not to hold, the problem of determining precisely the higher-dimensional structure of the algebras for these monads remains open. It is to the solution of this problem that the results of the present paper contribute.

Because we will be interested in algebras for the monad generated by theories, such as the theories \( T_n \) described in Section 3.5 below, which extend \( T_\omega \), the description of the monad involved in the definition of Martin-Löf complexes will be described for an arbitrary extension of \( T_\omega \). As such, throughout this section \( T \) is assumed to be an arbitrary theory extending \( T_\omega \). Finally, we mention that although we choose to work with reflexive globular sets, the construction of the monad can be modified to yield a corresponding monad for globular sets.

3.1. Notation for iterated identity types and other conventions

In order to most efficiently (and readably) state some of the additional principles for identity types that we consider it is useful to introduce notation for iterated identity types. Fixing a type \( A \) together with terms \( a, b : A \), we introduce the (at this stage superfluous) notation

\[
A^0 := A, \quad A^1(a, b) := A(a, b).
\]

In general, assuming given terms

\[
\vdash a_{n+1}, b_{n+1} : A^n(a_1, b_1 ; \cdots ; a_n, b_n),
\]

we define

\[
A^{n+1}(a_1, b_1 ; \cdots ; a_n, b_n ; a_{n+1}, b_{n+1}) := A^n(a_1, b_1 ; \cdots ; a_n, b_n)(a_{n+1}, b_{n+1}).
\]
In the sequel we will be dealing extensively with sets of terms from various theories extending \( T_\omega \). We adopt the convention that such terms are always assumed to be identified modulo definitional equality and \( \alpha \)-equivalence.

As a notational convenience we adopt the convention of, given a reflexive globular set \( G = (G_n)_{n \geq 0} \), writing \( G \) for the set \( \sum_{n \geq 0} G_n \).

3.2. The reflexive globular set generated by a type

Fix a type \( A \) in \( T \). It is possible that \( A \) is a type in context, yet we will assume that \( A \) is a type in the empty context. The case where the context is non-empty is obtained in essentially the same way, and so this is a reasonable simplification.

We will now construct a reflexive globular set denoted by \( \Gamma(A) \) and called the reflexive globular set generated by \( A \) (in \( T \)). When the theory \( T \) is fixed we will omit the subscript and write simply \( \Gamma(A) \). This construction will be carried out in such a way that the following conditions are satisfied:

1. Each element of \( \Gamma(A)_n \) is a tuple of \((2n + 1)\) elements of the set of terms of \( T \).
2. If both \((\bar{\alpha}; \beta)\) and \((\bar{\alpha}; \beta')\) are in \( \Gamma(A)_n \), then \( \vdash A^{n+1}(\bar{\alpha}; \beta, \beta') : \text{type} \) is derivable in \( T \).
3. The source and target maps \( s, t : \Gamma(A)_{n+1} \rightarrow \Gamma(A)_n \) must send a tuple \((\alpha_0, \ldots, \alpha_{2n})\) to \((\alpha_0, \ldots, \alpha_{2n-2})\) and \((\alpha_0, \ldots, \alpha_{2n-3}, \alpha_{2n-1})\), respectively.

We begin by defining

\[
\Gamma(A)_0 := \{ a \mid \vdash a : A \},
\]

\[
\Gamma(A)_1 := \{ (a_0, a_1; \alpha) \mid a_0, a_1 \in \Gamma(A)_0 \text{ and } \vdash \alpha : A(a_0, a_1) \},
\]

and the maps \( s, t : \Gamma(A)_1 \rightarrow \Gamma(A)_0 \) are simply the projections \( \pi_0, \pi_1 \) sending \((a_0, a_1; \alpha)\) to \( a_0 \) and \( a_1 \), respectively. Assuming \( \Gamma(A) \) has been constructed up to stage \( n \), we define \( \Gamma(A)_{n+1} \) to be the following set

\[
\{ (\bar{\alpha}; \beta_0, \beta_1; \gamma) \mid (\bar{\alpha}; \beta_i) \in \Gamma(A)_n \text{ for } i = 0, 1, \text{ and } \vdash \gamma : A^{n+1}(\bar{\alpha}; \beta_0, \beta_1) \}.
\]

The source and target maps \( s, t : \Gamma(A)_{n+1} \rightarrow \Gamma(A)_n \) are given by the projections \( (\bar{\alpha}; \beta_0, \beta_1; \gamma) \mapsto (\bar{\alpha}; \beta_i) \), for \( i = 0 \) and \( i = 1 \), respectively.

Lemma 3.1. Given an extension \( T \) of \( T_\omega \) and a (closed) type \( A \) of \( T \), the graded set \( \Gamma(A) \) described above is a reflexive globular set.

Proof. The maps \( i : \Gamma(A)_n \rightarrow \Gamma(A)_{n+1} \) are obtained using reflexivity terms. The equations for reflexive globular sets are then readily verified. \( \square \)

In fact, the above construction is easily seen to give a functor

\[ \Gamma(-) : \mathcal{T}_\ast \rightarrow \mathbf{rGSet} \]

from the category of pointed type theories to the category of reflexive globular sets.
3.3. The type theory associated to a reflexive globular set

Not only does every type \( A \) give rise to a reflexive globular set, but also every reflexive globular set \( G \) gives rise to a type theory \( T_\omega[G] \).

**Definition 3.2.** Given a reflexive globular set \( G \), the type theory \( T_\omega[G] \) generated by \( G \) (or \( T_\omega \) with \( G \) adjoined) is obtained by augmenting the basic type theory \( T_\omega \) with the following additional symbols and rules:

- A type \( \vdash \Gamma G \);
- Terms \( \vdash \Gamma g : \Gamma G \) for each vertex \( g \in G_0 \);
- Terms \( \vdash \Gamma f : \Gamma G \Gamma(\Gamma g, \Gamma h) \) for each element \( f \in G_1 \) with \( s(f) = g \) and \( t(f) = h \);
- Terms \( \vdash \Gamma \alpha : \Gamma G \Gamma^n(\Gamma \alpha_0^0, \Gamma \alpha_1^0; \Gamma \alpha_0^1, \Gamma \alpha_1^1; \cdots; \Gamma \alpha_0^{n-1}, \Gamma \alpha_1^{n-1}) \)

where \( \alpha_i^j \) for \( i = 0, 1 \) and \( 0 \leq j \leq n - 1 \) are as defined in Section 2.8, for each \( \alpha \in G_n \);
- New conversion rules:

\[
\Gamma i(\alpha) = x(\Gamma \alpha) : \Gamma G \Gamma^{n+1}(\cdots; \Gamma \alpha, \Gamma \alpha)
\]

for every \( \alpha \in G_n \).

In the sequel we will refer to types of the form \( \Gamma G \) as **basic types** and to terms of the form \( \Gamma \alpha \) as **basic terms**.

**Remark.** As a matter of notation, we write \( \Gamma \vdash_G J \) to indicate that the judgement \( \Gamma \vdash J \) is derivable in \( T_\omega[G] \). We also write \( \text{Exp}_G \) instead of the more cumbersome \( \text{Exp}(T_\omega[G]) \). Finally, when no confusion will result, we identify the symbol \( \Gamma \tau \) with \( \tau \) itself. E.g., we write \( f : G(g, h) \) instead of the more cumbersome \( \Gamma f : \Gamma G \Gamma(\Gamma g, \Gamma h) \).

In subsequent sections it will be convenient to have at our disposal techniques for constructing maps between the sets of expressions of one type theory \( T_\omega[G] \) and another \( T_\omega[H] \), for \( G \) and \( H \) globular sets. Along these lines, we make the following observation.

**Lemma 3.3.** Given globular sets \( G \) and \( H \), any function

\[
G \xrightarrow{\varphi} \text{Exp}_H
\]

has a unique extension \( \hat{\varphi} : \text{Exp}_G \to \text{Exp}_H \), commuting with the operations from which the expressions are formed, such that the following diagram of sets
commutes:

\[
\begin{array}{c}
\text{Exp}_G \\
\downarrow i_G \\
\downarrow \varphi \\
G \\
\end{array}
\begin{array}{c}
\text{Exp}_H \\
\downarrow i_H \\
\downarrow \varphi \\
H \\
\end{array}
\]

where \( i_G \) is the map sending \( g \in G_n \) to \( \Gamma g \).

Note that the basic type \( \Gamma G \) is sent by the extension \( \hat{\varphi} \) to \( \Gamma H \). Of course, depending on the nature of \( \varphi \) the extension \( \hat{\varphi} \) may or may not preserve derivable judgements. Such a \( \hat{\varphi} \) will, however, commute with substitution. I.e., if \( e(x) \) is an expression of \( T_\omega[G] \) with \( x \) free, then, for any other expression \( f \),

\[
\hat{\varphi}(e)[\hat{\varphi}(f)/x] = \hat{\varphi}(e[f/x]). \tag{4}
\]

We can now easily show that the assignment \( G \mapsto T_\omega[G] \) is a functor

\[ T_\omega[-] : \text{rGSet} \to \text{TT}_\ast. \]

The action on a morphism of globular sets \( \varphi : G \to H \) is given through the canonical extension \( \varphi_* = i_H \circ \varphi \), as in the diagram

\[
\begin{array}{c}
\text{Exp}_G \\
\downarrow i_G \\
\downarrow \varphi \\
G \\
\end{array}
\begin{array}{c}
\text{Exp}_H \\
\downarrow i_H \\
\downarrow \varphi \\
H \\
\end{array}
\]

**Lemma 3.4.** Suppose \( J \) is a judgement derivable in \( T_\omega[G] \), then \( \varphi_*(J) \) is derivable in \( T_\omega[H] \).

**Proof.** The proof is a straightforward induction on the structure of derivations \( \vdash_G J \). For example, suppose \( J \) is the conclusion \( \Gamma \vdash_G \lambda x : A.b(x) : \prod x : A.B(x) \) of the introduction rule for dependent products. Then we have by the induction hypothesis that

\[
\varphi_*(\Gamma), x : \varphi_*(A) \vdash_H \varphi_*(b(x)) : \varphi_*(B(x)).
\]

Applying the introduction rule in \( T_\omega[H] \) yields the appropriate judgement since

\[
\varphi_*(\prod x : A.B(x)) = \prod x : \varphi_*(A).\varphi_*(B)(x),
\]

by definition of \( \varphi_* \). The only case which merits special attention are those judgements of the form (3) which occur as axioms of \( T_\omega[G] \). Such judgements are preserved by the fact that \( \varphi \) is a map of globular sets. \(\square\)
3.4. The induced monad on globular sets

We will now see that the foregoing processes

\[ G \mapsto T_\omega[G], \text{ and} \]
\[ A : \text{type} \mapsto \Gamma(A), \]

actually constitute a pair of adjoint functors

\[ \begin{array}{c}
\text{TT} \leftrightarrow \Gamma(-) \\
rGSet \\
\end{array} \]

As a consequence, this yields a monad \( T \) on the category \( rGSet \) of reflexive globular sets with

\[ T(G) := \Gamma(\Gamma^G). \quad (5) \]

First, given a globular set \( G \), the unit \( \eta_G : G \to \Gamma(T_\omega[G]) \) is the “insertion of generators” defined by setting

\[ \eta_G(g) := (\Gamma^g)^0, (\Gamma^g)^1, \ldots, (\Gamma^g)^n \]

for \( g \in G_n \). This is a globular map which is natural in \( G \) by definition.

Next, we exhibit the counit \( \epsilon(\mathbb{S}, A) : T_\omega[\Gamma(A)] \to (\mathbb{S}, A) \).

This is done by constructing a map of expressions

\[ \hat{s} : \text{Exp}_{\Gamma(A)} \to \text{Exp}(\mathbb{S}) \]

and then showing that this map preserves derivable judgements. We use the universal property of \( t_{\Gamma(A)} : \Gamma(A) \to \text{Exp}_{\Gamma(A)} \), applied to \( s : \Gamma(A) \to \text{Exp}(\mathbb{S}) \), defined by sending each definitional equality class of a term \( t \) to a choice of representative. Clearly \( \hat{s} \) then depends on the choices of representatives, but any two \( s, s' \) will have the property that given a derivable judgement \( J \) of \( T[\Gamma(A)] \), the judgements \( \hat{s}(J) \) and \( s'(J) \) are definitionally equal, simply because \( \hat{s} \) and \( s' \) agree on generators up to definitional equality in \( \mathbb{S} \). The proof that \( \hat{s} \) preserves derivable judgements and hence induces a morphism \( \epsilon(\mathbb{S}, A) : T[\Gamma(A)] \to (\mathbb{S}, A) \) of pointed type theories is again a straightforward induction on derivations and is left to the reader. The triangle laws for these unit and counit maps are also established without complication.

This proves:

**Proposition 3.5.** The composite \( G \mapsto \Gamma(T_\omega[G]) \) underlies a monad on the category of reflexive globular sets.
To illustrate the nature of the multiplication of this monad, suppose \( g \) is a vertex of \( G \); then \( (\overline{g^\gamma})^\gamma \) is likewise a vertex of \( T^2G \). The multiplication \( \mu_G \) acts on such a vertex by removing the outermost \( \overline{\cdot} \). I.e.,

\[
\mu_G((\overline{g^\gamma})^\gamma) = \overline{g^\gamma}.
\]

The action of \( \mu_G \) on composite terms (constructed out of the basic terms of \( T_\omega[TG] \) using the rules of \( T_\omega \)) is then to go through the term recursively removing occurrences of \( \overline{\cdot} \). Thus, the unit acts by adding \( \overline{\cdot} \) and the multiplication acts by removing it.

3.5. Martin-Löf complexes and other categories of algebras

It is possible to extend Proposition 3.5 by allowing the type theory \( T_\omega \) to which we adjoint a globular set to vary. We denote by \( \text{Ext}(T_\omega) \) the category of all extensions of \( T_\omega \). I.e., the objects of \( \text{Ext}(T_\omega) \) are dependent type theories extending \( T_\omega \) (where we only allow those extensions obtained by the addition of set-many new symbols and rules). A morphism \( T \to T' \) in \( \text{Ext}(T_\omega) \) is an inclusion of theories (i.e., such a morphism exists whenever \( T' \) extends \( T \)). We also denote by \( \text{Mon}(rGSet) \) the category of monads on \( rGSet \) (regarded as monoids in \( [rGSet, rGSet] \)).

**Lemma 3.6.** The construction of a monad on \( rGSet \) from an extension of \( T_\omega \) from Section 3.4 gives the action on objects of a functor

\[
\mathcal{T} : \text{Ext}(T_\omega) \rightarrow \text{Mon}(rGSet).
\]

**Proof.** Assume given theories \( T \) and \( T' \) in \( \text{Ext}(T_\omega) \) such that \( T' \) is an extension of \( T \). We will now describe the induced natural transformation \( \xi : T \to T' \), where we write \( T \) and \( T' \) as abbreviations for \( T(T) \) and \( T(T') \), respectively. Given a reflexive globular set \( G \) and an element \( \vec{\alpha} = (\alpha_0, \ldots, \alpha_n^{-1}, \alpha) \) of \( T(G)_n \), we note that since \( T' \) extends \( T \) it follows that each component of the list \( \vec{\alpha} \) is also a term of \( T'(G)_n \). Moreover, all of these terms necessarily possess the appropriate boundaries so that \( \vec{\alpha} \) is also an element of \( T'(G)_n \). As such, we may simply define \( (\xi_G)_n : T(G)_n \to T'(G)_n \) to be the map which sends any \( \vec{\alpha} \) as above to itself (now regarded as a list of terms from \( T'(G) \)). This clearly describes a map of reflexive globular sets; moreover, \( \xi \) is natural and commutes with the multiplication and unit maps for \( T \) and \( T' \). Finally, it is trivial to see that, with this definition \( \mathcal{T} \) is functorial.

\[\square\]

The specific extensions of \( T_\omega \) to which we would like to apply Lemma 3.6 are obtained by augmenting \( T_\omega \) by axioms that force the identity types to be trivial once they have been iterated sufficiently many times. To begin with, recall that the reflection rule for identity types is the principle which states that all identity types are trivial in the sense that

\[
\hvdash a, b : A \quad \hvdash p : A(a, b) \quad \hvdash a = b : A \quad \text{Reflection}
\]
Higher-dimensional generalizations of the reflection rule are then given by “truncating” the identity types only after they have been iterated a certain number of times. Explicitly, the \( n \)-truncation rule is stated as follows:

\[
\vdash a_{n+1}, b_{n+1} : A^n(a_1, b_1; \cdots ; a_n, b_n) \quad \vdash p : A^{n+1}(a_1, b_1; \cdots ; a_{n+1}, b_{n+1})
\]

With these rules at our disposal we are able to describe the type theories extending \( T_\omega \) with which we will be concerned. Explicitly, for \( n \geq 0 \), the theory \( T_n \) is defined to be the result of adding to \( T_\omega \) the (instances of the) principle \( TR_n \). These theories then arrange themselves according to the following hierarchy of theories:

\[
T_\omega \subseteq \cdots \subseteq T_{n+1} \subseteq T_n \subseteq \cdots \subseteq T_1 \subseteq T_0,
\]

since \( TR_m \) clearly implies \( TR_n \), when \( m < n \). The theory \( T_0 \) is also known as extensional type theory as contrasted with the intensional type theory \( T_\omega \).

**Definition 3.7.** Denote by \( M_\omega \) the monad \( T(T_\omega) \). A reflexive globular set \( G \) is a Martin-Löf complex (or ML-complex) if it is an algebra for \( M_\omega \). We write \( MLCx \) for the Eilenberg-Moore category consisting of \( M_\omega \)-algebras and homomorphisms thereof. Similarly, we denote by \( MLCx_n \) the category of \( M_n \)-algebras for \( n = 0, 1, 2, \ldots \), where \( M_n \) denotes the monad \( T(T_n) \).

Corresponding to the hierarchy of theories (6) we obtain, by Lemma 3.6, the following sequence of inclusions of categories:

\[
MLCx_0 \longrightarrow \cdots \longrightarrow MLCx_n \longrightarrow MLCx_{n+1} \longrightarrow \cdots \longrightarrow MLCx_\omega
\]

and it is our goal to understand how these categories relate to the hierarchy of categories of homotopy types discussed in Section 1.

**Remark.** The truncation principles \( TR_n \) are related to several other type theoretic principles. One such principle is (definitional) uniqueness of identity proofs:

\[
\vdash a_{n+1}, b_{n+1} : A^n(a_1, b_1; \cdots ; a_n, b_n) \quad \vdash p : A^{n+1}(a_1, b_1; \cdots ; a_{n+1}, b_{n+1}) \quad \text{UIP}_n
\]

Another related principle is the \( n \)-dimensional ordinary unit principle

\[
\vdash a_{n+1} : A^n(a_1, b_1; \cdots ; a_n, b_n) \quad \vdash p : A^{n+1}(a_1, b_1; \cdots ; a_{n+1}, a_{n+1}) \quad \text{OUP}_n
\]

The truncation and ordinary unit principles have been considered previously by Garner in [7] and by Warren in [24]. The relation between the truncation, uniqueness of identity proofs and ordinary unit principles are clarified in the following lemma (the proof of which we omit, since the idea comes essentially from results from [23]).
Lemma 3.8. Assuming the rules of $\mathbb{T}_\omega$ and the usual rules for identity types, the following implications hold for $n \geq 0$:

1. $\text{TR}_n$ implies $\text{OUP}_n$.
2. $\text{TR}_n$ implies $\text{UIP}_{n+1}$.
3. $\text{UIP}_n$ implies $\text{TR}_n$.

3.6. Skeletal terms

In this section we introduce a convenient technical tool which will later facilitate reasoning about the type theories at hand and their models.

Let us denote by $T^+\kappa$ the theory $T_\kappa[0]$ for $\kappa = 0, 1, \ldots, \omega$. This theory is the same as $T_\kappa$ except that it also has a new distinguished type symbol $\Box 0$. (Hence we could view it as the result of adjoining the initial globular set to $T_\kappa$.) All of the theories of the form $T^\kappa[G]$ are extensions of $T^+\kappa$.

Definition 3.9. A judgement $J$ of $T_\kappa[G]$ is skeletal if no basic term symbols (coming from $G$) occur in $J$.

If $J$ is a skeletal judgement in $T_\kappa[G]$, then there is a corresponding judgement $J^+$ in $T^+\kappa$ obtained by replacing each occurrence of the basic type $\Box G$ by $\Box 0$. Similarly, any judgement $J$ in $T^+\kappa$ has a corresponding translation $J^G$ into any $T_\kappa[G]$ by replacing each occurrence of $\Box 0$ by $\Box G$. We have the following basic observation about the derivability of skeletal judgements:

Lemma 3.10. A skeletal judgement $J$ is derivable in $T_\kappa[G]$ if and only if $J^+$ is derivable in $T^+\kappa$.

Obviously the analogous statement which says that $J$ is derivable in $T^+\kappa$ if and only if $J^G$ is derivable in $T_\kappa[G]$ also holds. Accordingly, we will henceforth not distinguish between the judgements $J$, $J^+$ and $J^G$.

Assume that $H$ is a finite reflexive globular set and define a context $\Delta_H$ in $T^+\kappa$ as follows. $\Delta_H$ consists of:

- For each 0-cell $a$ of $H$, there is a distinct variable declaration $v_a : \Box 0$.
- For each $(n + 1)$-cell $f$ of $H$, there is a distinct variable declaration $v_f : \Box 0^m(v_{s(f)}, v_{t(f)})$.

Because $H$ is finite this determines a well-defined context.

Fix, in addition to the finite reflexive globular set $H$, some $\kappa = 0, 1, \ldots, \omega$. We now define a new category $C^\kappa_H$ as follows.

Definition 3.11. A context relative to $H$ is a (necessarily skeletal) context $\Gamma$ extending $\Delta_H$. Given contexts $\Gamma$ and $\Theta$ relative to $H$, a context morphism $\sigma : \Gamma \to \Theta$ relative to $H$ is a (necessarily skeletal) context morphism such that

$$
\begin{array}{c}
\Gamma \\
\downarrow \sigma \\
\Theta
\end{array}
$$

commutes.
The category $\mathcal{C}_H^\kappa$ has as objects contexts in $T^\kappa_+$ relative to $H$ and as arrows context morphisms in $T^\kappa_+\kappa$ relative to $H$. In fact, we have a comprehension category (cf. [12]) with base $\mathcal{C}_H^\kappa$ and with fibration $P_H^\kappa : T_H^\kappa \to \mathcal{C}_H^\kappa$ determined by letting the fiber $T^\kappa_H(\Gamma)$ consists of the types in context $\Gamma$. This determines a split Grothendieck fibration since types are stable under substitutions and there is an obvious comprehension map $\chi^\kappa_H : T^\kappa_H \to (\mathcal{C}_H^\kappa)^\to$ which sends a type $\Gamma \vdash A$ to the dependent projection $(\Gamma, x : A) \to \Gamma$.

**Lemma 3.12.** The comprehension category $\mathcal{C}_H^\kappa$ is a model of $T^\kappa[H]$.

*Proof.* There is an obvious forgetful functor $\mathcal{C}_H^\kappa \to \mathcal{C}_T^\kappa[H]$, the usual syntactic model of $T^\kappa[H]$, and this functor preserves the comprehension category structure. All type and term formation operations respect skeletalness w.r.t. $\Delta_H$. Therefore, the category $\mathcal{C}_H^\kappa$ also supports dependent products, sums, natural numbers and identity types, and the forgetful functor creates these.

Now $\mathcal{C}_H^\kappa$ becomes a model of $T^\kappa[H]$ by interpreting a the basic type $H$ as $\llbracket 0 \rrbracket$ and a basic term $a$ as the variable $v_a$.

Let us write $sk_H(J)$ for the interpretation of a judgement $J$ of $T^\kappa[H]$. It follows, by induction on derivations, that the substitution $\sigma_H$ given by $v_a \mapsto \llbracket a \rrbracket$ satisfies

$$sk_H(J)[\sigma_H] \equiv J.$$ (7)

Indeed, $\sigma_H$ induces a morphism of models $\mathcal{C}_H^\kappa \to \mathcal{C}_T^\kappa[H]$ where again $\mathcal{C}_T^\kappa[H]$ is the syntactic model. This morphism is in fact an isomorphism of models, with inverse induced by $sk_H$ (which sends contexts and context morphisms to their “skeletalized” counterparts, i.e. it replaces each basic term by the appropriate variable).

In the case where $H$ is no longer finite matters become less straightforward.

**Lemma 3.13.** Given any reflexive globular set $G$ there exists, for any judgement $J$ of $T^\kappa[G]$, a skeletal judgement $sk(J)$ together with a substitution $\sigma_J$ consisting entirely of basic terms such that

$$sk(J)[\sigma_J] \equiv J.$$ (8)

*Proof.* Fix a derivation of $J$. There is a corresponding finite reflexive globular set $H$ obtained as the sub-reflexive globular set of $G$ generated by those cells of $G$ occurring in the fixed derivation of $J$. By the observations above we have $sk_H(J)$ satisfying (7). But $sk_H(J)$ is also a derivable term in $T^\kappa[G]$ and $\sigma_H$ is also a substitution in $T^\kappa[G]$. As such, we may take $sk(J)$ to be $sk_H(J)$ and $\sigma_J$ to be $\sigma_H$. 

Note that there may exist more than one skeletal judgement $sk(J)$ and more than one $\sigma_J$ satisfying the equation from Lemma 3.13 since one may have multiple derivations of the same judgement $J$ which employ different basic terms. We might hope to choose a “minimal” derivation in some way and
define a canonical skeleton of $J$ in that way, but this is also not possible in the presence of truncation rules. For example, if $p, q : A^n(a, b)$ and $t : B(a)$ where we are in the situation that truncation applies at level $n$, then we have two distinct derivations of $t : B(b)$ which employ distinct contexts of basic terms. Nonetheless, any two derivations which employ the same basic terms will give rise to the same skeletal judgement and substitution.

Remark. Instead of the above model theoretic proof that for every derivable judgement there exists a derivable skeletal judgement and a suitable substitution to recover the original judgement one can also work purely syntactically by defining a translation on the level of expressions which replaces every basic term $⌜a⌝$ by a variable $v_a$ and which commutes with all other formation rules. Then one can show by a straightforward induction on derivations that if all basic terms in a derivation of a judgement come from the finite globular set $H$, then the translated judgement is derivable when we work in the context $\Delta H$.

One of the advantages of having Lemma 3.13 at our disposal is that it allows us to give an alternative characterization of the monad $M_\kappa$. Let $\tilde{M}_\kappa(G)$ be the reflexive globular set which has as cells (of the appropriate level) equivalence classes of tuples $(\Gamma, \varphi, \sigma)$ such that $\Gamma \vdash \varphi$ is a skeletal judgement and $\sigma$ is a substitution $(\cdot) \rightarrow \Gamma$ such that $\varphi$ is required to have the appropriate type (i.e., for 0-cells $\Gamma \vdash \varphi : "G"$, et cetera). Here

$$(\Gamma, \varphi, \sigma) \approx (\Delta, \psi, \tau) \text{ if and only if } \Gamma \vdash \varphi[\sigma] = \psi[\tau].$$

Equivalently, by Lemma 3.13 (taking a skeleton of the judgement $\Gamma \vdash \varphi[\sigma] = \psi[\tau]$), $(\Gamma, \varphi, \sigma) \approx (\Delta, \psi, \tau)$ if and only if there exists a skeletal context $\Theta$ extending both $\Gamma$ and $\Delta$ and a substitution $\vartheta : (\cdot) \rightarrow \Theta$ extending both $\sigma$ and $\tau$ such that $\Theta \vdash \varphi = \psi$. $\tilde{M}_\kappa$ is readily seen to constitute a monad on the category of reflexive globular sets using the same approach as in the definition of the monad structure on $M_\kappa$. Furthermore, there is an isomorphism $\lambda : M_\kappa \cong \tilde{M}_\kappa$ of monads described as follows. Given $t$ in $M_\kappa(G)$ choose, by Lemma 3.13, a skeleton $(\Delta, \varphi, \sigma)$ of $t$ and let $\lambda_G(t) := [\Delta, \varphi, \sigma]$. This is independent of choice of representative from the definitional equality class of $t$ and of the choice of skeleton by definition of $\approx$. Going the other way, given $[\Delta, \varphi, \sigma]$ let $\lambda_G^{-1}$ send this data to the definitional equality class of $\varphi[\sigma]$. Again, this is trivially independent of the choice of representative. It is clear that $\lambda$ and $\lambda^{-1}$ constitute a natural isomorphism and that they are compatible with the respective monad structures. Henceforth we will freely employ this isomorphism without explicit mention where convenient.

3.7. Limits and colimits of algebras

The aim of this section is to show that the monads $M_\kappa$ are finitary. One consequence of this is that $\text{MLCx}_\kappa$ is cocomplete as well as being complete.

Assume given a filtered category $I$ and a functor $A : I \rightarrow \text{rGSet}$. Denote by $A^\infty$ the colimit of this functor. By definition, an $n$-cell of $A^\infty$ is an equivalence class $[a]$ of $n$-cells of the coproduct $\coprod_i A(i)$, where $a \in A(i)$ is equivalent to
\(a' \in A(j)\) if and only if there exist arrows \(\varphi : i \to k\) and \(\varphi' : j \to k\) in \(I\) such that
\[
A(\varphi)(a) = A(\varphi')(a').
\]
We would like to prove that
\[
M_\kappa(A^\infty) \cong \lim_{\leftarrow i} M_\kappa(A(i)).
\] (8)

This will require an analysis of the valid derivations of the theory \(T_\kappa[A^\infty]\). To begin with, note that \(T_\kappa[A^\infty]\) is obtained by augmenting \(T_\kappa\) with the new basic type \(\Gamma A^\infty\) as well as with basic terms \(\Gamma [a]\) of the appropriate types as described in Section 3.

Assume we are given a derivable judgement \(J\) of the theory \(T_\kappa[A^\infty]\). Then it follows from Lemma 3.13 that we have a skeleton \(sk(J)\) and the corresponding substitution \(\sigma_J\). Suppose the basic terms occurring in \(\sigma_J\) are \(\Gamma [a_1], \ldots, \Gamma [a_n]\), where it is possible that \(n = 0\), then it follows from the fact that \(I\) is filtered that we may find representatives \(a'_1, \ldots, a'_n\) of the equivalence classes \([a_1], \ldots, [a_n]\) such that \(a'_1, \ldots, a'_n\) are all in the same \(A_i\) for some \(i \in I\). Therefore, we obtain a new substitution \(\sigma'_J\) by substituting \(\Gamma a'_1, \ldots, \Gamma a'_n\) instead of the corresponding terms in \(\sigma_J\). So the judgement \(sk(J)[\sigma'_J]\) is derivable in \(T_\kappa[A_i]\). By considering the case where \(J\) is an appropriate term judgement we are able to use this line of reasoning to show that (8) holds.

**Lemma 3.14.** Given a filtered category \(I\) together with a functor \(A : I \to \text{rGSet}\), there is an isomorphism (8) of reflexive globular sets.

**Proof.** Assume given a reflexive globular set \(X\) together with a cocone \(x_i : M_\kappa A(i) \to X\). We now describe the induced map \(\xi : M_\kappa A^\infty \to X\). Given an element \(\tau\) of \(M_\kappa A^\infty\) we have, by the reasoning above, the term \(\tau' := sk(\tau)[\sigma'_J]\) in \(M_\kappa A^\infty\). Therefore, we define:
\[
\xi(\tau) := x_i(\tau').
\]
This is immediately seen to be independent of the choice of \(a'_1, \ldots, a'_n\). To see that the definition does not depend on the choice of skeleton we use the fact that two skeletons \(\tau\) can be obtained as restrictions of another skeleton with a larger ambient context. Moreover, it follows from this definition that each \(x_i\) can be recovered by precomposing \(\xi\) with the map \(M_\kappa A(i) \to M_\kappa A^\infty\). Finally, for uniqueness of \(\xi\), observe that implicit the construction of \(\xi\) above we have proved that for each cell \(\tau\) of \(M_\kappa A^\infty\) there exists some \(i\) such that \(\tau\) is in the image of the map \(M_\kappa A(i) \to M_\kappa A^\infty\).

Since the monads \(M_\kappa\) are finitary monads on a locally finitely presentable category, it now follows that their categories of algebras are also locally finitely presentable. We may then conclude (see e.g. [3]):

**Proposition 3.15.** For each \(\kappa = 0, 1, \ldots, \omega\), the category \(\text{MLCx}_\kappa\) is complete and cocomplete.
4. Doppelgängers, $M_0$-algebras and $M_1$-algebras

Our purpose in this section is to characterize the category $\text{MLC}x_0$ of algebras for the monad $M_0$ by proving that it is equivalent to the category of sets and to introduce the basic machinery which will allow us, in Section 5 below, to characterize the category $\text{MLC}x_1$. The 0-dimensional case is already instructive and provides us with an opportunity to introduce some ideas and concepts which will be put to work in a more complicated setting in the 1-dimensional case.

We begin by discussing the reason why the results are nontrivial by explaining the various ways in which the type theory $\mathbb{T}_0[G]$ proves the existence of infinitely many duplicates of all of the vertices, edges, and higher edges of the globular set $G$. These duplicates (here called doppelgängers) must all be shown to be propositionally equal to elements of the original globular set $G$.

Next, we establish the characterization of the $M_0$-algebras in a number of steps, making use of the set-theoretic interpretation of extensional type theories and the semantics from [11]. We will concentrate on stating the main concepts and theorems and omit some of the detailed proofs, allowing the reader to follow the line of argument.

4.1. Doppelgängers

Fix a globular set $G$ and consider the type theory $\mathbb{T}_\omega[G]$ (or any extension of it). It is clear from the definition of the theory $\mathbb{T}_\omega[G]$ that every vertex $a \in G_0$ is represented as a term in $\mathbb{T}_\omega[G]$, namely $a : G$. (We shall, as before, not distinguish between an actual element in $G$ and its “name” in the type theory.) Similarly, every 1-dimensional edge $f \in G_1$ is represented by $f : G(a,b)$, where $s(f) = a, t(f) = b$, and so on in higher dimensions. As is well-known, the rules of the type theory allow us to construct formal composites and inverses of these edges (see Section 4.4 for details). One might, at first sight, conjecture that these are the only judgements of this form, i.e., that whenever $\mathbb{T}_\omega[G]$ derives $\tau : G$ for a closed term $\tau$, then $\tau$ must be an element of $G$ already, and whenever $\mathbb{T}_\omega[G]$ derives $\sigma : G(a,b)$ then $\sigma$ is a formal composite of edges in $G_1$ already. However, things are more complicated than that, due to the elimination rule for identity types.

Suppose, for example, that we have $a, b, c \in G_0$ and a non-reflexivity term $f : a \to b$ in $G_1$. Now we can consider the following derivation:

\[
\begin{align*}
x : G, y : G, z : G(x, y) & \vdash G : \text{type} \\
x : G & \vdash c : G \\
& \vdash f : G(a, b) \\
& \vdash J([x : G|c, a, b, f]) : G
\end{align*}
\]

This creates a new term of type $G$ which we denote by $c(f)$; we call it the doppelgänger of $c$ (at $f$). This term is not definitionally equal to any of $a, b, c$. However, it is propositionally equal to $c$: this we can see from the derivation.
\[
x : G, y : G, z : G(x, y) \vdash G(c, J([v : G]c, x, y, z)) : \text{type}
\]
\[
x : G \vdash r(c) : G(c, J([v : G]c, x, x, r(x))
\]
\[
\vdash f : G(a, b) \quad \text{Id elimination}
\]
showing that there is a term witnessing \(c \simeq c(f)\) (note that by the conversion rule the second premise reduces to \(x : G \vdash r(c) : G(c, c)\) so that the trivial term is well-defined).

Of course, this idea works in general: given any term \(\tau : T\) and any (non-reflexivity) identity proof \(f : A(a, b)\) we may form
\[
\tau(f) := J([x : A]\tau, a, b, f) : T
\]
and then show that \(\tau \simeq \tau(f)\).

There are other ways to create doppelgängers: consider again \(f : a \to b \in G_1\) and form
\[
f^\# := J([x, y, G, z : G(x, y)]G([x : G]x, a, b, f) : G.
\]
This term is a new vertex which is homotopic to both \(a\) and \(b\) (again this is proved by defining a suitable witness using the J-rule).

Yet another possibility is to construct
\[
f^\flat := J([x, y, G, z : G(x, y)]G([x : G]r(x), a, b, f) : G(a, b),
\]
which turns out to be homotopic to \(f\).

While in the above examples of doppelgängers it is easy to show that each of the newly created terms is, up to homotopy, equal to a basic term coming from the original globular set, it is not clear why this would always be the case, i.e. why for every term derivable in \(\mathbb{T}_0[G]\) there is a suitable homotopy. Moreover, it will be seen in the next section that the elimination rule for identity types does in certain instances give genuinely new terms which are not homotopic to any basic term (namely, the formal composites which are used to give the Martin-Löf complexes their categorical structure).

4.2. \(M_0\)-algebras

We now study the category of algebras \(\text{MLCx}_0\) for the monad \(M_0\). We fix a reflexive globular set \(G\), and consider \(M_0(G)\), the free algebra on \(G\).

**Lemma 4.1.** The reflexive globular set \(M_0(G)\) is constant.

**Proof.** Since the theory \(\mathbb{T}_0[G]\) satisfies the reflection rule, it follows that any term \(\tau : G^n(a, b)\) is definitionally equal to a reflexivity term (see Remark 3.5 above). Hence for \(n > 0\), the elements of \(M_0(G)_n\) are all degenerate (i.e., occurring in the image of the identity map \(i : M_0(G)_{n-1} \to M_0(G)\)), and the globular set \(M_0(G)\) is completely determined by its vertices. \(\square\)
Thus in order to characterize the globular set $M_0(G)$, it suffices to understand the set $M_0(G)_0$ of its vertices. Recall from the construction of the monad $M_0$ that the elements of $M_0(G)_0$ are equivalence classes of closed terms $\tau : G$, where two of these are identified if the theory proves that they are definitionally equal. We begin by noting that there is a canonical map from $\pi_0(G)$ to $M_0(G)_0$, induced by the coequalizer

\[
\begin{array}{ccc}
G_1 & \xrightarrow{s} & G_0 \\
\downarrow_t & & \downarrow \pi_0(G) \\
M_0(G)_0 & \xrightarrow{\eta_0} & \end{array}
\]

(9)

Here, the map $\eta_0$ is the component of the unit $\eta : G \to M_0(G)$ at dimension 0. For every $f \in G_1$ with $s(f) = a, t(f) = b$ there is an axiom $f : G(a, b)$ in $T_0[G]$; by truncation this forces $a = b$ in the theory, and hence $a$ and $b$ are identified as well in $M_0(G)$; hence $\eta_0s = \eta_0t$.

We would like to show that $p$ is a bijection; this would prove that $M_0$ is isomorphic to the (idempotent) monad $\Delta \pi_0$ on $\mathbf{rGSet}$, and in particular it would follow that the category of $M_0$-algebras is just the category of sets.

The first step in proving this is to exploit the fact that extensional ML type theories may be modelled in locally cartesian closed categories (see the original work of Seely [21], or the expository texts [12, 9]). In particular, the theory $T_0$ may be soundly interpreted in the category of sets and this interpretation can be extended to give a model of $T_0[G]$ by interpreting the basic type $G$ by the set $\pi_0(G)$, and interpreting the basic terms $a : G$ by the elements $[a]$, where $[a]$ is the connected component of $a$ in $G$.

**Lemma 4.2.** The above interpretation extends to a model of $T_0[G]$ in the category of sets.

*Proof.* We need only verify the new axioms and the new conversion rule of the theory; if these are valid under the interpretation then the result follows by soundness. By construction, the judgements $a : G$ for $a \in G$ are valid. Identity types $T(t, s)$ are interpreted as equalizers in $\mathbf{Set}$ and therefore the interpretation of such a type is either the emptyset (when the interpretations of $t$ and $s$ are not equal) or the one element set (when the interpretations of $t$ and $s$ are equal). Therefore, the interpretation of a basic term $f : G(a, b)$ is simply the canonical element of the one element set (since the existence of such a edge in $G$ ensures that $a$ and $b$ are in the same connected component). Similarly, the basic terms corresponding to higher-dimensional cells from $G$ are interpreted as the canonical element of the one element set. Finally, the new conversion rule $i(a) = \pi(a)$ holds under the interpretation since both sides of the equation will be interpreted as $[a]$ and is trivial for higher-dimensional cells $a$. \hfill \Box

The soundness of this interpretation guarantees that the map $p$ is injective: indeed, given two connected components $[a]$ and $[b]$ of $G$, suppose that $p[a] =$
p[b]. Then $T_0[G]$ proves that $a = b$. But then this equation should hold in the model $\pi_0(G)$, i.e. $[a] = [b]$ as elements of $\pi_0(G)$. In particular, the interpretation yields a map $q : M_0G \to \pi_0(G)$ of reflexive globular sets which is a retract of $p$.

In [11] it is shown how to construct models of type theories such as $T_0[G]$ in such a way that the interpretations of terms will provide additional data regarding the syntax of these theories. These models are called **combinatorial realizability models** and can be seen as a generalized form of realizability model in the usual sense where the realizers of terms can be, intuitively, some kind of combinatorial data (in the cases we care about they will usually be edges constructed in the syntax of the theory). We will explain the conditions required in order for such a model to exist, but the proof of this fact is somewhat involved and can be found in [11].

The 0-dimensional variant of the model construction allows us to construct sound and complete models for theories of the form $\lambda \rightarrow \beta \log \forall \exists$. The starting data for the model is a Set-valued functor on $M_0(G)$, where the latter is regarded as a discrete groupoid. We call such a functor a notion of 0-realizability, since it assigns to each object of $M_0(G)$ a set of object which act as realizers for that object. We suggestively write $\alpha \vdash t : \Gamma G^\alpha$ when $\alpha$ is an element of the set of realizers of the term $t$.

**Theorem 4.3.** Given a notion of 0-realizability for $G$ satisfying the condition that for each basic term $a \in G$ there is a realizer $\alpha \vdash a : \Gamma G^\alpha$, there exists a sound and complete model of $T_0[G]$ in which a closed term $t$ of type $\Gamma G^\alpha$ is interpreted as a realizer $\alpha_t \vdash t : \Gamma G^\alpha$.

It now follows from Theorem 4.3 that the map $q : M_0G \to \pi_0G$ is the inverse of $p : \pi_0G \to M_0G$ (both of these are described in Section 4.2 above). Indeed, enote by $\bar{\tau} : M_0G \to M_0G$ the function $p \circ q$. Let $\tau \vdash t : \Gamma G^\alpha$ if and only if $\tau : \Gamma G^\alpha(t, \bar{t})$. This is a notion of 0-realizability. Moreover, the hypothesis of Theorem 4.3 is satisfied since $r(\Gamma a^\alpha) \vdash a^\alpha : \Gamma G^\alpha$. Therefore, this determines a combinatorial realizability model. Let us denote by $\alpha_t$ the interpretation of $t : \Gamma G^\alpha$. Then we have, for any vertex $t$ in $M_0G$, $\alpha_t : \Gamma G^\alpha(t, \bar{t})$ and so, by 0-truncation, $t = \bar{t}$, as required.

This proves the main result of this section:

**Proposition 4.4.** There is an isomorphism of categories $\text{MLCx}_0 \cong \text{Set}$.

### 4.3. 1-Dimensional realizability models

We now briefly discuss the 1-dimensional variant of the semantics from [11], and the results we use in the remainder of the paper.

**Definition 4.5.** Given a reflexive globular set $G$, a notion of 1-realizability for $G$ is a functor real : $\Pi_1(M_1(G)) \to \text{Set}$. (Here, $\Pi_1(X, \alpha)$ is the underlying “fundamental” groupoid of the complex $(X, \alpha)$, see next section for details.)

We will often write $\tau \vdash t : \Gamma G^\alpha$ to indicate that $t$ is an element of $\text{real}(t)$ and, given $f : t \to s$ in $\Pi_1(M_1(G))$ we write $\tau \cdot f$ for $\text{real}(f)(\tau)$. The theorem regarding combinatorial realizability models from [11] can then be stated precisely as follows:
Theorem 4.6 ([11]). Given a notion of 1-realizability for $G$ satisfying the following conditions:

- For each vertex $a$ of $G$, there exists a realizer $\alpha_a \vdash \ulcorner a \urcorner : \ulcorner G \urcorner$.
- For each edge $f : a \to b$ in $G$, we have $\alpha_a \cdot f = \alpha_b$.

there exists a sound and complete model of $T_1[G]$ in which a closed term $t$ of type $\ulcorner G \urcorner$ is interpreted as a realizer $\alpha_t \vdash t : \ulcorner G \urcorner$ and in which a closed term $f : \ulcorner G \urcorner(t,s)$ is interpreted as a proof that $\alpha_t \cdot f = \alpha_s$.

The following application of these models will be used twice in the remainder of the paper:

Theorem 4.7. ([11]) Let $G$ be a graph, $H$ a groupoid and let $P, Q : M_1G \to H$ be two functors. Suppose furthermore that we are given a morphism $\alpha_a : P(a) \to Q(a)$ for each basic term $a$ such that, for each basic term $f : a \to b$, the following diagram commutes:

\[
\begin{array}{ccc}
P(a) & \xrightarrow{\alpha_a} & Q(a) \\
P(f) \downarrow & & \downarrow Q(f) \\
P(b) & \xrightarrow{\alpha_b} & Q(b).
\end{array}
\]

Then there exists a natural transformation $\alpha : P \Rightarrow Q$ whose component at a basic term $a$ is $\alpha_a$.

Proof. Define a notion of 1-realizability for $G$ by letting the set of realizers for a closed term $t : G$ be $H(P(t), (Q(t))$. The functorial action is given by conjugation: given $f : t \to s$ and $\tau \vdash t : G$, set $\tau \cdot f =_{\text{def}} Q(f)\tau P(f)^{-1}$. A basic term $a$ is then realized by the given $\alpha_a$. This gives a model, the soundness of which tells us in particular that each $t : G$ has a realizer, and that these form a natural transformation as desired. \hfill $\square$

4.4. Fundamental groupoids of $M_1$-algebras

The aim of this section is to generalize the basic setup from Section 4.2 to the case of $M_1$-algebras and the category $\text{MLC}_{x_1}$. Contrary to what one might expect, this category is not equivalent to the category of groupoids. However, there is an adjunction

\[
\begin{array}{ccc}
\text{MLC}_{x_1} & \xrightarrow{\perp} & \text{Gpd}
\end{array}
\]  

(10)

analogous to the adjunction between topological spaces or, better yet, homotopy 1-types, and groupoids.

We first describe the right adjoint $\Pi_1 : \text{MLC}_{x_1} \to \text{Gpd}$ which allows us to regard $M_1$-algebras as groupoids (where groupoids are themselves as reflexive globular sets in which all $n$-cells are degenerate for $n \geq 2$ provided with the
additional structure of composites and inverses). That a $M_1$-algebra can be endowed with the structure of a groupoid follows immediately from the construction of composition and inverse operations — as well as the corresponding propositional equalities witnessing the associativity, unit and inverse laws — by Hofmann and Streicher [10]. However, we will later require some of the details of the proof of this fact and we therefore describe the construction explicitly.

First, recall that, given any type $A$ together with terms $a, b : A$ and $f : A(a, b)$, the inverse $f^{-1} : A(b, a)$ of $f$ is defined to be the following elimination term:

$$f^{-1} := \mathsf{J}_{\Delta(x,y)}(\Delta(x,y), ([x : A] x, a, b, f)).$$

Moreover, when there exists a further propositional equality $g : A(b, c)$, the composite $(g \cdot f)$ of $g$ with $f$ is defined to be the term $\mathsf{app}(\mathsf{J}(\Delta(x,y), A), f)$, where the $\mathsf{J}$-term here is written in full as

$$\mathsf{J}_{\Delta(x,y), A}([x : A] \lambda_v. v, b, c, g, f) : A(a, c).$$

We will use these operations on terms of identity type to define the composition and inverses for $M_1$-algebras. To this end, let an object $G$ of $\mathbf{MLC}_{X_1}$ be given with action $\gamma : M_1(G) \to G$. Of course, we will regard $G$ as a groupoid with objects the vertices of $G$ and arrows the edges of $G$. Identities are given by the edges of the form $i(a)$ for $a$ a vertex. In order to define composition in $G$ let a composable pair of edges $f, g$ in $G$ be given with $f : G(a, b)$ and $g : G(b, c)$ in the theory $T_1[G]$. As such, the composite $(g \cdot f) : G(a, c)$, as defined above, exists and we define the result of composing $f$ with $g$ in $G$ to be the edge obtained by applying the action of $G$ to $(g \cdot f)$. I.e.,

$$(g \circ f) := \gamma(\mu a \gamma, \mu c \gamma; (\mu g \gamma \cdot \mu f \gamma)).$$

This edge possesses the appropriate source and target since $\gamma$ is an arrow in $\mathbf{rGSet}$. Likewise, the inverse $f^{-1}$ of $f$ is defined by setting

$$f^{-1} := \gamma(\mu b \gamma, \mu a \gamma; \mu f \gamma^{-1}),$$

where $f^{-1}$ on the right-hand side is the inverse of the term $\mu f \gamma$, as defined above.

With these definitions, the groupoid laws are a consequence of their up-to propositional equality counterparts (for which see [10]) together with the 1-truncation rule. That is, we have described a groupoid $\Pi_1(G, \gamma)$ constructed from a $M_1$-algebra $(G, \gamma)$.

In slightly more abstract terms this construction can be described as follows. For $G$ an arbitrary reflexive globular set, let $\mathcal{F}(G)$ denote the free groupoid
(regarding the free groupoid monad as a monad on reflexive globular sets) on $G$. Recall that $\mathcal{F}(G)$ has the same vertices as $G$, and arrows $a \to b$ in $\mathcal{F}(G)$ are zig-zag paths

\[
\begin{array}{cccccc}
a & \xleftarrow{a_1} & a_2 & \cdots & a_n & \xrightarrow{a_{n-1}} & b \\
\end{array}
\]

of edges in $G$ modulo the evident relations forcing the groupoid laws to hold. There is then, for each $G$, a map $\Phi_G : \mathcal{F}(G) \to M_1(G)$ of globular sets which sends an equivalence class of such "formal composites" from $\mathcal{F}(G)$ to the term representing the result of taking inverses and composites of its edges using the type theoretic inverses and composites described above. These maps constitute a morphism of monads $\mathcal{F} \to M_1$ and therefore induce a functor $\Pi_1 : \text{MLCx}_1 \to \text{Gpd}$. Explicitly, $\Pi_1(G, \gamma)$ is given by the underlying globular set $G$ together with the action $\gamma \circ \Phi_G : \mathcal{F}(G) \to G$. Moreover, $\Phi_G$ is actually the canonical functor $\Phi_G : \mathcal{F}(G) \to \Pi_1(M_1(G))$ extending the unit $G \to M_1(G)$ along the unit $\eta_G$ for $\mathcal{F}$:

\[
\begin{array}{ccc}
\mathcal{F}(G) & \xrightarrow{\Phi_G} & \Pi_1(M_1(G)) \\
& \searrow_{\eta_G} & \downarrow_{\eta_G} \\
& G & \\
\end{array}
\]

We sometimes call $\Pi_1(G, \gamma)$ the \textit{fundamental groupoid} of $(G, \gamma)$. It follows from a general result of Linton [16] that $\Pi_1$ possesses a left-adjoint $K : \text{Gpd} \to \text{MLCx}_1$. We will return to a discussion of this adjunction later. First we will turn to a proof that the maps $\Phi_G$ constitute an equivalence of categories.

4.5. Interpretation of $\mathbb{T}_1[G]$ using the free groupoid on $G$

The theory $\mathbb{T}_1[G]$ is soundly modelled using groupoids by extending the interpretation from [10] by the following additional clauses:

- The new type $\Gamma G$ is interpreted as the free groupoid on $G$:

\[
[\Gamma G] := \mathcal{F}(G).
\]

- The new terms basic $\Gamma a$ of type $\Gamma G$ are interpreted by the objects of $\mathcal{F}(G)$ which they represent:

\[
\]

- The new basic terms $\Gamma f$ of identity type $\Gamma G(\Gamma a, \Gamma b)$ are likewise interpreted as the equivalence classes of the arrows they represent

\[
[\Gamma f] := [f].
\]
• If \( \gamma_\alpha \) is a new basic term of type \( \gamma G^{-m}(\gamma_0^{\alpha_0}, \ldots, \gamma_n^{\alpha_n}) \), for \( n > 1 \), then \( [\gamma_\alpha] \) is the canonical element of the one element set.

With these definitions, the axioms of \( T_1[G] \) are clearly satisfied. We now remind the reader how the particular kinds of terms we are interested in are interpreted in this model. To begin with recall that the identity type \( x, y : \gamma G \vdash \gamma G(x, y) : \text{type} \) is interpreted as the functor \( I_G : \mathcal{F}(G) \times \mathcal{F}(G) \to \text{Gpd} \) which sends a pair of objects \((a, b)\) of \( \mathcal{F}(G) \) to the discrete groupoid \( \mathcal{F}(G)(a, b) \) and which sends an arrow \((\alpha, \beta) : (a, b) \to (a', b')\) to the functor \( \mathcal{F}(G)(a, b) \to \mathcal{F}(G)(a', b') \) with action \( f \mapsto (\beta \circ f \circ \alpha^{-1}) \). The extended context \((x, y : \gamma G, z : \gamma G(x, y))\) is interpreted as the result of applying the Grothendieck construction \( \int I_G \) to \( I_G \). In this instance, \( \int I_G \) coincides with the arrow category \( \mathcal{F}(G)^\to \). As such, the elimination data \( x : \gamma G \vdash \varphi(x) : B(x, x, \pi(x)) \) is interpreted by a functor \([B] : \mathcal{F}(G)^\to \to \text{Gpd}\) together with a functor \([\varphi] : \mathcal{F}(G) \to \int[B]\) such that

\[
\begin{array}{ccc}
\mathcal{F}(G) & \xrightarrow{\varphi} & \int[B] \\
\downarrow & & \downarrow \pi \\
\mathcal{F}(G)^\to & & \\
\end{array}
\]

commutes. I.e., for an object \( a \) of \( \mathcal{F}(G) \), \( \varphi(a) \) is a tuple composed of \( 1_a : a \to a \) together with an object, which we denote by \( a_\varphi \), of the groupoid \([B](1_a : a \to a)\).

For an arrow \( \alpha : a \to a' \) of \( \mathcal{F}(G) \), \( \varphi(\alpha) \) is then a tuple composed of \( \alpha \) itself together with an arrow

\[
[B]\left(\begin{array}{c}
a \xrightarrow{1_a} a \\
a \xrightarrow{\alpha} a' \\
1_{a'} \xrightarrow{1_a} a' \\
\end{array}\right) \xrightarrow{(a_\varphi) \xrightarrow{\alpha_\varphi}} a'_\varphi
\]

in the groupoid \([B](1_a' : a' \to a')\).

The resulting elimination term \( x, y : \gamma G, z : \gamma G(x, y) \vdash J(\varphi, x, y, z) : B(x, y, z) \) is interpreted as the section \( J \) of the projection \( \int[B] \to \mathcal{F}(G)^\to \) which sends an object \( f : a \to b \) of \( \mathcal{F}(G)^\to \) to the pair consisting of \( f \) and the object

\[
[B]\left(\begin{array}{c}
a \xrightarrow{1_a} a \\
1_a \xrightarrow{f} b \\
a \xrightarrow{f} b \\
\end{array}\right) \xrightarrow{(a_\varphi)}
\]

of \([B](f : a \to b)\). Similarly, the action of \( J \) on an arrow

\[
\begin{array}{ccc}
a & \xrightarrow{f} & b \\
\xrightarrow{\alpha} & & \xrightarrow{\beta} \\
a' & \xrightarrow{g} & b' \\
\end{array}
\]

(11)
from \( f : a \to b \) to \( g : a' \to b' \) in \( \mathcal{F}(G)^{\to} \) is the pair consisting of the arrow itself together with

\[
\left[ B \right] \left( \begin{array}{c}
\alpha' \\
\phantom{1} \\
\alpha' \to g \\
\downarrow g
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\frac{a' \to a'}{1_{a'}} \\
\frac{1_{a'}}{\alpha'} \\
\alpha' \to b' \\
\downarrow g
\end{array} \right)
\]

So, for example, given a term \( h : \gamma G^\gamma(g, g') \) in \( T[G] \), consider \( h^{-1} \). The pattern type \( B(x, y, z) \) in this instance is \( G(y, x) \) and \( \left[ B \right] \) is the functor sending \( f : a \to b \) to the discrete groupoid \( G(b, a) \) and which sends an arrow \((11)\) in \( \mathcal{F}(G)^{\to} \) to the functor \( \lambda_{\alpha, \gamma} : \mathcal{F}(a, b) \to \mathcal{F}(b', a') \). As such, it is straightforward to verify with the description of the interpretation given above that \( \left[ h^{-1} \right] \) is equal to the inverse \( \left[ h \right]^{-1} \) in \( \mathcal{F}(G) \). Similarly, given \( f : \gamma G^\gamma(a, b) \) and \( g : \gamma G^\gamma(b, c) \) in \( T[G] \), it is straightforward to verify that the interpretation commutes with composition in the sense that \( \gamma g^{-1} \gamma f \) is equal to \( g \circ f \) in \( \mathcal{F}(G) \). These observations yield the following:

**Lemma 4.8.** The assignment \( \Psi_G : \Pi_1(M_1(G)) \to \mathcal{F}(G) \) which sends an \( n \)-cell

\[
\vec{\alpha} = (\alpha_0, \ldots, \alpha_n^{-1}, \alpha)
\]

of \( M_1(G) \) to \( \left[ \alpha \right] \) is functorial.

**Proof.** By the results of Section 4.4 it follows that \( M_1(G) \) is a groupoid in which the result of composing 1-cells \((a, b, f)\) and \((b, c, g)\) is \((a, c, g \circ f)\). Thus, because the interpretation function commutes with composition it follows that \( \Psi_G \) is functorial (that \( \Psi_G \) preserves identities is also straightforward).

**Theorem 4.9.** Given a reflexive globular set \( G \), \( \Phi_G : \mathcal{F}(G) \to \Pi_1(M_1(G)) \) is an equivalence of categories.

**Proof.** It is an immediate consequence of the universal property of \( \mathcal{F}(G) \) that \( \Phi_G \) is a section of \( \Psi_G \). Construct a combinatorial realizability model (as described in Section 4.3 above) of \( T[G] \) where realizers of terms of type \( \gamma G^\gamma \) are given by

\[
\vdash \varphi : \gamma G^\gamma(t, \Phi_G(\Psi_G(t)))
\]

which satisfies \( \Psi_G(\varphi) \) \( = \) \( 1_{\mathcal{F}(G)}(t) \). Realizers for basic terms \( \gamma a^\gamma \) are given by \( \text{r}(\gamma a^\gamma) \). It follows from the results of [11] that this determines a model of type theory. Let us denote the interpretation (i.e., realizer) of \( t : \gamma G^\gamma \) by \( \alpha_t \). Then, by virtue of the interpretations of identity types in the realizability model, it follows that these \( \alpha_t \) give a natural isomorphism \( \alpha : \Pi_1(M_1(G)) \to \Phi_G \circ \Psi_G \).

Theorem 5.8 shows that free \( M_1 \)-algebras are, up to equivalence, free groupoids. This might lead one to conjecture that the category of \( M_1 \)-algebras is equivalent to the category of groupoids. However, that is not the case. The following example makes clear that different algebras may have the same fundamental groupoid.
Example 4.10. Consider the following groupoid $G$: it has two objects $a$ and $b$, exactly one arrow $f : a \to b$ and its inverse $g : b \to a$. We may define a $M_1$-algebra structure $\gamma : M_1(G) \to G$ on $G$ as follows: on objects, $\gamma$ is defined by

$$\gamma(v) = \begin{cases} a & \text{if } \vdash v = \uparrow a : G \text{ is derivable} \\ b & \text{otherwise} \end{cases}$$

Thus, all doppelgängers of vertices are sent to $b$. On 1-cells we define:

$$\gamma(w) = \begin{cases} f & \text{if } \gamma(s(w)) = a, \gamma(t(w)) = b \\ g & \text{if } \gamma(s(w)) = b, \gamma(t(w)) = a \\ 1_a & \text{if } \gamma(s(w)) = a = \gamma(t(w)) \\ 1_b & \text{if } \gamma(s(w)) = b = \gamma(t(w)) \end{cases}$$

It is readily seen that this is a map of globular sets. To see that it is a $M_1$-algebra, we remark that the unit law is trivially satisfied because the algebra map sends any generator $\uparrow v$ of $M_1(G)$ to $v$. For the associativity law, consider an element $\tau$ of $M_2^1(G)$; this is a term of the theory $T_1[M_1(G)]$, which is generated by basic terms of the form $\uparrow \sigma \uparrow$, where $\sigma$ is a term of the theory $T_1[G]$. Note that on the one hand

$$(\gamma \circ M_1 \gamma)(\tau) = a \iff M_1 \gamma(\tau) = \uparrow a \iff \tau = \uparrow \uparrow a \uparrow,$$

while on the other hand

$$(\gamma \circ \mu)(\tau) = a \iff \mu(\tau) = \uparrow a \iff \tau = \uparrow \uparrow a \uparrow,$$

showing that both maps agree in dimension 0. To show that they agree in dimension 1 as well, one reasons in a similar fashion.

But clearly by symmetry there is another algebra structure on $G$, call it $\delta$, defined by sending all doppelgängers to $a$ instead of $b$. The identity map $G \to G$ is, however, not a map of $M_1$-algebras. Indeed, any map of $M_1$-algebras commutes with the formation of doppelgängers; for example, if $k$ is a map of algebras then $M_1(k)$ must send the doppelgänger $a(\uparrow f)$ to $k(a)(k(f))$, and hence we must have $k\gamma(a(\uparrow f)) = \delta k(a)(k(f))$, which is impossible if $k$ is the identity. For the same reason the only other possible map of groupoids, which interchanges $a$ and $b$, cannot be a map of $M_1$-algebras.

Thus $M_1$-algebras carry more information than their fundamental groupoids, and this information tells us how the formal composites and doppelgängers are interpreted. The fact that non-isomorphic algebras may have the same fundamental groupoid is of course the analogue of the fact that non-homeomorphic topological spaces may have the same fundamental groupoid.

In addition, the above example shows that $\Pi_1$ is not a full functor. (However, it is easily seen to be faithful.) Nonetheless, it will be shown in Section 5 below that $K \dashv \Pi_1$ constitutes a Quillen equivalence and it is to this that we now turn.
5. The Quillen model structure on $\text{MLCx}_1$

In this section we will only consider 1-truncated complexes, and we reduce clutter in the notation by dropping subscripts indicating this one-dimensionality. Given an object $(A, \alpha)$ of $\text{MLCx}_1$, we will sometimes denote the composition in the resulting groupoid $\Pi_1(A, \alpha)$ by $\circ\alpha$.

We begin by defining the three classes of morphisms for the model structure on $\text{MLCx}_1$:

Fibrations a map $f$ of complexes is a fibration when $\Pi_1(f)$ is an isofibration of groupoids. We denote the class of fibrations by $\mathcal{F}$.

Weak Equivalences a map $f$ of complexes is a weak equivalence when $\Pi_1(f)$ is a weak categorical equivalence. We denote by the class of weak equivalences by $\mathcal{W}$.

Cofibrations A map of complexes is a cofibration when it has the left lifting property with respect to maps which are simultaneously fibrations and weak equivalences. The class of cofibrations will be denoted by $\mathcal{C}$.

The remainder of the section is devoted to showing that these indeed constitute a Quillen model structure on the category of 1-dimensional complexes. In what follows, we shall for sake of expositional simplicity work with reflexive graphs instead of 1-truncated globular sets.

5.1. Cotensor of complexes with graphs

Let a complex $(A, \alpha)$ be given together with a graph $X$. We define a new complex $(A^X, \chi)$ as follows. The underlying graph has as 0-cells graph homomorphisms $F : X \to A$ and as 1-cells natural transformations. Here naturality of a transformation $\eta : F \to G$ means that for each vertex $x$ of $X$ we have a 1-cell $\eta_x : Fx \to Gx$ in $A$ such that, for $h : x \to y$ in $X$, we have

$$\eta_y \circ \alpha_h Fh = G h \circ \alpha_x \eta_x.$$ 

Now, fix a vertex $x$ in $X$. There is a graph homomorphism $ev_x : A^X \to A$ which evaluates at $x$. We define an evaluation map $\tilde{\varepsilon}_x : M_1(A^X) \to M_1 A$ as the map of expressions (trivially seen to preserve derivable judgements) which sends $\Gamma F^\alpha$ to $\Gamma Fx^\alpha$ and $\Gamma \alpha^\alpha$ to $\Gamma \alpha_x^\alpha$; that is, $\tilde{\varepsilon}_x$ is the result of applying the free functor $r\text{Graph} \to \text{MLCx}_1$ to $ev_x$. Before we can go any further we must make some observations regarding these evaluation maps. We begin with the following fact which follows immediately from the definition of $\tilde{\varepsilon}_x$:

Lemma 5.1. The evaluation map $\tilde{\varepsilon}_x : M_1(A^X) \to M_1 A$ is functorial.

Of course, what is meant here is that $\tilde{\varepsilon}_x$ induces a functor on the level of the underlying fundamental groupoids. In the sequel, we will often continue this abuse of terminology and notation by speaking of functors between complexes instead of their fundamental groupoids.
Let $\varepsilon_x : M_1(A^X) \to A$ be the composite functor $\alpha \circ \tilde{\varepsilon}_x$. For the following theorem we must construct a combinatorial realizability model (see Section 4.3) of $T_1[A^X]$.

**Theorem 5.2.** Given an edge $f : x \to y$ in $X$, there is an induced natural transformation $\varepsilon_f : \varepsilon_x \to \varepsilon_y$.

**Proof.** This is a consequence of Theorem 4.7 where we take the component of the natural transformation at a basic term $\langle\Phi\rangle : A^X$ to be the maps $Ff : Fx \to Fy$. That realizers of basic terms of type $A^X$ are stable under reindexing along basic edges in $A^X$ follows the definition of 1-cells in $A^X$.

We will write the component $\varepsilon_x(t) \rightarrow \varepsilon_y(t)$ of $\varepsilon_f$ at a vertex $t$ as $\varepsilon_f(t)$ and we will assume that $\varepsilon_1_x$ is the identity. We define the map $\chi : M_1(A^X) \to A^X$ by

$$\chi(t)(x) := \varepsilon_x(t)$$

for $t$ a vertex of $M_1(A^X)$ and $x$ a vertex of $X$, and, for $f : x \to y$ in $X$, we have

$$\chi(t)(f) := \varepsilon_f(t).$$

Next, for $g : t \to s$ in $M_1(A^X)$ we define $\chi(t) \to \chi(s)$ by taking at a vertex $x$ the map

$$\chi(g)_x := \varepsilon_x(g).$$

That this is a natural transformation is by naturality of the $\varepsilon_f : \varepsilon_x \to \varepsilon_y$. I.e., we have proved the following:

**Lemma 5.3.** The map $\chi$ is a graph homomorphism $M_1(A^X) \to A^X$.

It now remains to show that this map gives $A^X$ a $M_1$-algebra structure.

**Lemma 5.4.** $(A^X, \chi)$ is a complex.

**Proof.** The unit law is trivial. For the multiplication law assume given a term of $T_1[M_1(A^X)]$ of the form $\varphi(\langle \xi \rangle)$ where $\xi \in M_1(A^X)$ and where $\varphi(\langle \rangle)$ is skeletal (note that we should really take an arbitrary list of basic terms $\langle \xi \rangle$, but in the more general case the argument is identical to the one given here). Then we must show that

$$\chi(\varphi(\xi)) = \chi(\varphi(\langle \chi(\xi) \rangle)).$$

It suffices to evaluate on $x \in X$. We then have

$$\chi(\varphi(\langle \chi(\xi) \rangle))(x) = \alpha(\tilde{\varepsilon}_x(\varphi(\langle \chi(\xi) \rangle)))$$

$$= \alpha(\varphi(\langle \alpha(\varepsilon_x(\xi)) \rangle))$$

$$= \alpha(\varphi(\varepsilon_x(\xi)))$$

$$= \alpha(\tilde{\varepsilon}_x(\varphi(\xi))),$$

where the third equation is by the fact that $\alpha$ is an algebra and the fourth equation is by the fact that $\varphi(\langle \rangle)$ is skeletal. $\square$
We will denote by $X \triangleleft (A, \alpha)$ the algebra $(A^X, \chi)$ when we do not want to have to mention the action $\chi$ and we call this the **cotensor of** $(A, \alpha)$ **with** $X$. Note that by construction of the map $\chi$, the diagram

\[
\begin{array}{ccc}
M(A^X) & \xrightarrow{M(ev_x) = e_x} & MA \\
\downarrow \chi & & \downarrow \alpha \\
A^X & \xrightarrow{ev_x} & A
\end{array}
\]

commutes, where $ev_x(F) = F(x)$. Thus the evaluation maps are actually algebra morphisms.

### 5.2. The path object argument

Where $\mathcal{F}$ denotes the free groupoid functor, we have the following extremely useful fact:

**Lemma 5.5.** For any graph $X$ and any complex $(A, \alpha)$, there is an isomorphism of groupoids

\[
\Pi_1(X \triangleleft (A, \alpha)) \cong \Pi_1(A, \alpha)^{FX}.
\]

**Proof.** This is routine using the universal property of $FX$ and the definition of the edges in $X \triangleleft (A, \alpha)$ as natural transformations. \hfill $\square$

**Lemma 5.6.** Each object $(A, \alpha)$ of $MLC\times_1$ has a path object factorization.

**Proof.** Let $I$ be the graph with two vertices 0 and 1 and one non-trivial edge $0 \rightarrow 1$. Then $FI$ is the usual “interval” $I$ in the category of groupoids. For any $(A, \alpha)$ we have

\[
\begin{array}{ccc}
(A, \alpha) & \xrightarrow{r} & I \triangleleft (A, \alpha) \\
\downarrow \Delta & & \downarrow p \\
(A, \alpha) \times (A, \alpha)
\end{array}
\]

where

\[r(a)(x) := a\]

and

\[p(H) := (H(0), H(1)).\]

It is routine to verify that these are algebra homomorphisms.
Using Lemma 5.5 and the fact that $\Pi_1$ is a right-adjoint it follows that the result of applying $\Pi_1$ to (12) is

\[
\begin{array}{c}
\Pi_1(A) \\
\downarrow \Delta \downarrow \\
\Pi_1(A) \times \Pi_1(A)
\end{array}
\xrightarrow{\text{morph}}
\begin{array}{c}
\Pi_1(A)^I \\
\end{array}
\]

which, as is well known, constitutes a path object for $\Pi_1(A)$ in $\text{Gpd}$. Therefore, the original diagram (12) is a path object in $\text{MLCx}_1$. \qed

Recall that Quillen’s path object argument provides conditions under which it is possible to transfer a model structure from a category $\mathcal{C}$ to a category $\mathcal{D}$ along an adjunction $F \dashv G$ for $F : \mathcal{C} \to \mathcal{D}$.

**Theorem 5.7** (Quillen). Assume given a cofibrantly generated model category $\mathcal{C}$ together with an adjunction $F \dashv G$ for $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is a cocomplete category with finite limits. Assume furthermore that the following conditions are satisfied:

1. The left-adjoint $F$ preserves small objects.
2. $\mathcal{D}$ has a fibrant replacement functor.
3. $\mathcal{D}$ has a functorial path objects for fibrant objects.

Then there is a model structure on $\mathcal{D}$ in which a map is a fibration (weak equivalence) if and only if its image under $G$ is a fibration (weak equivalence) in $\mathcal{C}$.

From this we are able to obtain the following:

**Theorem 5.8.** The definition of fibration and weak equivalence in $\text{MLCx}_1$ given above determines a model structure on $\text{MLCx}_1$.

**Proof.** By Theorem 5.7, Lemma 5.6 and the fact that every object in $\text{Gpd}$ is fibrant (and hence that every object of $\text{MLCx}_1$ is fibrant by definition), it suffices to prove that the left-adjoint $K : \text{Gpd} \to \text{MLCx}_1$ of $\Pi_1$ preserves small objects. Note that both forgetful functors $\text{MLCx}_1 \to \text{rGSet}$ and $\text{Gpd} \to \text{rGSet}$ preserve and reflect filtered colimits, since both are finitarily monadic. Therefore, the functor $\Pi_1 : \text{MLCx}_1 \to \text{Gpd}$ must preserve filtered colimits, and in particular colimits of chains. This last statement is equivalent to the preservation of small objects by $K$. \qed

5.3. The construction of the left adjoint $K$ of $\Pi_1$

In order to prove that the adjunction $K \dashv \Pi_1$ is a Quillen equivalence it will be necessary to consider the transfinite construction of $K$ from [14] in more detail\(^4\). Henceforth $G$ denotes a fixed groupoid.

\(^4\)The construction can also be seen as combining the transfinite construction for coequalizers in categories of algebras as detailed, for example, in [3] with the fact that the left adjoint to the functor $\Pi_1$ can be rendered as a coequalizer.
Let us set $G(-1) := G$ and $G'(-1) := G$. We construct $G(0)$ as the following coequalizer (taken in the category of graphs):

\[
\begin{array}{c}
M_1 \xrightarrow{\Phi_G} M_1^2 \xrightarrow{\gamma} M_1 G \\
\downarrow \quad \downarrow \quad \downarrow \\
M_1 F G \xrightarrow{\eta G(-1)} G(0)
\end{array}
\]

where $\gamma$ is the action of $G$ (qua groupoid) and $\Phi_G$ is the canonical map induced by the groupoid structure on $M_1 G$. Let $i(-1) : G(-1) \to G(0)$ denote the composite $e(-1) \circ \eta G(-1)$.

In the next stages of the construction of $K$ we obtain $G(n+2)$ from $G(n)$ and $G(n+1)$ as the following coequalizer

\[
\begin{array}{c}
M_1 G(n) \xrightarrow{\eta G(n)} M_1 G(n+1) \\
\downarrow \quad \downarrow \\
M_1^2 G(n) \xrightarrow{\xi G(n)} G(n+2)
\end{array}
\]

and we define $i(n+1) : G(n+1) \to G(n+2)$ to be the composite $e(n+1) \circ \eta G(n+1)$.

$KG$ is then defined as the colimit (taken in the category of reflexive graphs) $\lim^\to_n G(n)$ of the diagram consisting of the maps $i(n)$. The action $\nu : M_1 KG \to KG$ is the canonical map induced by the maps $M_1 G(n) \to G(n+1) \to KG$ together with the fact that $M_1$ is finitary and hence preserves the colimit of the chain $G(i)$. We also point out that the composites $M_1 G(n) \to G(n+1) \to KG$ are in fact functors. (This is merely a general feature of the transfinite construction of the left adjoint.)

### 5.4. The Quillen equivalence

We now begin working towards the proof that the adjunction $K \dashv \Pi_1$ is a Quillen equivalence. This will be done in several steps: first, with the aid of the groupoid semantics we construct a sequence of functors $[\mathord[-]]_n : M_1 G(n) \to G$ with suitable properties. Next, we construct, by induction on $n$, a realizability model of $T_1[G(n)]$ which, using Theorem 4.7, gives a natural transformation fitting in the square

\[
\begin{array}{c}
M_1 G(n) \xrightarrow{e(n)} G(n+1) \\
\downarrow \quad \downarrow \\
[-1]_n \xrightarrow{\eta} G \xrightarrow{\eta} KG
\end{array}
\]

The existence of these natural isomorphisms will then turn out to be sufficient to conclude that the unit $\eta : G \to KG$ is a weak equivalence.

Before going on to the construction we first recall some basic facts about the kinds of models of type theory considered in this paper (the Hofmann-Streicher style groupoid models and the combinatorial realizability models described in
Section 4.3). These models are genuine denotational models in the sense that each term, type and judgement is assigned a canonical interpretation (as opposed to many realizability models where a given judgement may have many different realizers). A consequence of this fact, and of the interpretation of substitution in these models, is that the interpretation is \textit{compositional} in the sense that if we are given an open judgement \( x : A \vdash J \) and a term \( a : A \), then the interpretation of \( J[a/x] \) is completely determined by the interpretations of \( x : A \vdash J \) and \( a : A \). In particular, in order to prove that \( J[a/x] \) and \( J[b/x] \) receive the same interpretation it suffices to show that \( a \) and \( b \) receive the same interpretation. E.g., in the Hofmann-Streicher style groupoid models \( [\langle\langle b(a)\rangle\rangle] \) is canonically determined as the canonical section induced by \( [\langle\langle b(x)\rangle\rangle] \) and \( [\langle\langle a\rangle\rangle] \) (cf. the discussion in Section 4.3 following the proof of Theorem 4.6). explicitly, if \( \Gamma, x : A \vdash b(x) : B(x) \) and \( \Gamma \vdash a : A \), then we have a pullback diagram

\[
\begin{array}{ccc}
[\Gamma, z : B(a)] & \longrightarrow & [\Gamma, x : A, z : B(x)] \\
\downarrow & & \downarrow \\
[\Gamma] & \longrightarrow & [\Gamma, x : A]
\end{array}
\]

and a section \( [\Gamma, x : A \vdash b(x) : B(x)] \) of the projection \( [\Gamma, x : A, z : B(x)] \rightarrow [\Gamma, x : A] \). This induces a canonical section \( [\langle\langle a\rangle\rangle^*([\langle\langle b(x)\rangle\rangle])] \) of the projection \( [\Gamma, z : B(a)] \rightarrow [\Gamma] \) which we define to be the interpretation of \( \Gamma \vdash b(a) : B(a) \).

Similar remarks apply to the realizability models. We will make use of this compositionality of the interpretations at several places below.

We begin by constructing by induction a sequence of functors

\[ [-]_n : M_1 G(n) \rightarrow G \]

as well as a sequence of graph homomorphisms \( \alpha(n) : G(n) \rightarrow G \) such that

\[
\begin{array}{ccc}
M_1 G(n) & \xrightarrow{e(n)} & G(n + 1) \\
\downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \\
[-]_n & \xrightarrow{\alpha(n+1)} & G
\end{array}
\]

is commutative.

For \( n = -1 \), we set \( \alpha(-1) : G(-1) = G \rightarrow G \) to be the identity. Note that in order to specify the functor \( [-]_n \) it suffices to specify a graph homomorphism \( G(n) \rightarrow G \) and then to use the Hofmann-Streicher semantics to extend this to \( M_1 G(n) \). (When the notation overloading.) Thus at each stage, we may let \( [-]_n \) be the the functor induced by \( \alpha(n) \). It remains to be shown then that it coequalizes the relevant maps so that it factors through the coequalizer \( e(n) \) resulting in the desired \( \alpha(n + 1) \) as in the above diagram.
We begin with the base case, where we have to show that \([-\] \_1 : M_1 G \to G \) makes the diagram

\[
\begin{array}{c}
M_1 \mathcal{F} G \xrightarrow{M_1 \gamma} M_1 G \xrightarrow{[-]-1} G \\
\downarrow M_1 \Phi_G \hspace{2cm} \downarrow M_1 \eta_G \\
M_1^2 G \xrightarrow{\mu_G} M_1^2 G
\end{array}
\]

commute. Consider an object of \(M_1 \mathcal{F} G\), regarded as a term \(\varphi(\gamma \circ \cdot f)\), where \(\cdot\) denotes the formal composition in the free groupoid. Here, we assume that \(\varphi(x)\) is skeletal and that \(g, f\) are basic edges in \(G\). (Technically we must consider arbitrary strings of formal composites in the free groupoid and a skeletal term \(\varphi(x_1, \ldots, x_n)\), but the reasoning in that case is identical to the reasoning given here.) On the one hand, \(M_1 \gamma\) sends this term to \(\varphi(\gamma \circ f)\) (where \(g \circ f\) is the composite in \(G\) using the groupoid structure \(\gamma\)), while on the other hand \(\mu_G : M_1 \Phi_G\) sends it to \(\varphi(\gamma \cdot \cdot \cdot f)\) (where \(\cdot\) is the formal composition in \(M_1(G)\)). Thus to prove that the two composites are equal, we must show that

\[
\varphi(\gamma \circ f) \_1 = \varphi(\gamma \cdot \cdot \cdot f) \_1
\]

By compositionality of the interpretation it suffices to show that \(\varphi(\gamma \circ f) \_1 = \varphi(\gamma \cdot \cdot \cdot f) \_1\), which holds by functoriality of the interpretation.

Next, assume that we have defined \([-] \_n\) and \(\alpha(n + 1)\). We must then show that \([-] \_n+1\), obtained by interpreting \(M_1 G(n + 1)\) in \(G\), makes the diagram

\[
\begin{array}{c}
M_1^2 n G(n) \xrightarrow{M_1 \eta_G(n)} M_1 G(n) \xrightarrow{M_1 \alpha(n)} M_1 G(n + 1) \xrightarrow{[-]-n+1} G \\
\downarrow 1_{M_1^2 G(n)} \hspace{2cm} \downarrow \mu_G(n) \\
M_1^2 G(n) \xrightarrow{M_1 \eta_G(n)} M_1 G(n)
\end{array}
\]

commute. To this end, we first establish the following useful lemma.

**Lemma 5.9.** The morphisms \(\alpha(n + 1)\) and the interpretations \([-] \_n+1, [-] \_1\) form a commutative diagram

\[
\begin{array}{c}
M_1 G(n + 1) \xrightarrow{M_1 \alpha(n+1)} M_1 G \\
\downarrow [-][-n+1] \hspace{2cm} \downarrow [-/-]_1 \\
[-][-n+1] \circ \eta = \alpha(n + 1) = [-][-1] \circ \eta \circ \alpha(n + 1) = [-][-1] \circ M_1(\alpha(n + 1)) \circ \eta
\end{array}
\]

**Proof.** By compositionality, it suffices to verify that the diagram commutes when we precompose with the unit \(\eta : G(n + 1) \to M_1 G(n + 1)\). But then by naturality of the unit and the definition of \([-] \_n+1\) we get

\[
[-][-1] \circ \eta = \alpha(n + 1) = [-][-1] \circ \eta \circ \alpha(n + 1) = [-][-1] \circ M_1(\alpha(n + 1)) \circ \eta
\]

as required. \(\square\)
Now consider the diagram

\[
\begin{array}{ccc}
M_1\eta G(n) & \xrightarrow{\eta} & G \\
\downarrow & & \downarrow \eta \\
M_1 G(n) & \xrightarrow{\eta} & M_1 G
\end{array}
\]

\[
\begin{array}{ccc}
M_1 G(n) & \xrightarrow{\alpha(n)} & M_1 G \\
\downarrow & & \downarrow \alpha(n+1) \\
M_1 G(n+1) & \xrightarrow{\alpha(n+1)} & M_1 G
\end{array}
\]

The outer diagram commutes because by IH we have

\[\alpha(n+1) \circ \epsilon(n) \circ \eta = \alpha(n).\]

The two triangles commute by the argument given above. Therefore, to show that (13) commutes, it suffices (again by compositionality) to show that for a term \(\mu s^3\), where \(s\) an element of \(G(n)\), we have \([\mu s^3]_n = [M_1 \epsilon(n)\mu s^3]_{n+1}\). But that is immediate from \(\mu s^3 = s\) and the IH. This completes the proof that we have a well-defined sequence of functors \([\cdot]_n : M_1 G(n) \to G\).

**Theorem 5.10.** The adjunction \(K \dashv \Pi_1\) is a Quillen equivalence.

**Proof.** It suffices to show that \(\eta_G\) is essentially surjective on objects and full. Denote the composite \(M_1 G(n) \to G(n+1) \to KG\) by \(\epsilon(n)\), and recall that this is actually a functor. We may then consider, for each \(n\), the (non-commutative) square

\[
\begin{array}{ccc}
M_1 G(n) & \xrightarrow{\epsilon(n)} & G(n+1) \\
\downarrow \downarrow & & \downarrow \eta \\
G & \xrightarrow{\eta} & KG
\end{array}
\]

According to Theorem 4.7, we may specify a natural transformation in this square by giving the components at the basic terms of \(M_1 G(n)\) and verifying that these are natural. We will do this first for \(n = -1\). Then a basic term of \(M_1 G\) is simply an element of \(G\), and we take the component of the natural transformation to be the identity at that element. This gives the natural transformation \(\tau(-1) : \eta[-1] \Rightarrow \epsilon(-1)\).

In the inductive step, we assume we have constructed \(\tau(n) : \eta[-]_n \Rightarrow \epsilon(n)\), and we wish to define \(\tau(n+1)\). Again by Theorem 4.7, we only have to specify the components at basic terms. Given such basic term \(\mu t\), where \(t\) is an element of \(M_1 G(n)\), we take this component to be \(\tau(n)_t\). Naturality is then inherited from \(\tau(n)\), and it also is immediate that this is independent of the choice of representative of \([t]\) because diagram (14) commutes.

To see that \(\eta_G\) is essentially surjective on objects, let \([t]\) in \(KG\) be given. So, \(t\) is in some \(G(n)\) and by the construction of the natural transformations
above we get a component \( \tau(n) : [t] \to \eta[\cdot]_n(t) \), where the latter is in the image of \( \eta_G \), as required. Similarly, given an arrow \([g] : \eta(a) \to \eta(b) \) in \( KG \) we it follows that \( g \) is in some \( G(n) \). Because the components of \( \eta(a) \) and \( \eta(b) \) of the natural transformation \( \tau(n) \) are identities, the naturality square at \( g \) of \( \tau(n) \) then simply exhibits \( g \) as equal to a map in the image of \( \eta_G \). \( \square \)

Appendix A. Rules of type theory

In this appendix we describe the syntax of the system \( T_\omega \). All rules below are stated in an ambient context which is omitted for ease of presentation.

Appendix A.1. Structural rules

\[
\frac{}{\Delta, \Gamma \vdash J} \quad \text{Weakening}
\]

where \( J \) ranges over judgements and we assume without loss of generality that the variables declared in \( \Delta \) and \( \Gamma \) are disjoint.

\[
\frac{a : A \quad \Delta, x : A, \Delta \vdash B(x) : \text{type}}{\Delta[a/x] \vdash B(a) : \text{type}} \quad \text{Type substitution}
\]

\[
\frac{a : A \quad x : A, \Delta \vdash b(x) : B(x)}{\Delta[a/x] \vdash b(a) : B(a)} \quad \text{Term substitution}
\]

\[
\frac{A : \text{type}}{x : A, \Delta \vdash x : A} \quad \text{Variable declaration}
\]

Appendix A.2. Rules governing definitional equality

\[
\frac{A : \text{type}}{A = A : \text{type}} \quad \frac{A = B : \text{type}}{B = A : \text{type}}
\]

\[
\frac{A = B : \text{type} \quad B = C : \text{type}}{A = C : \text{type}}
\]

\[
\frac{a : A}{a = a : A} \quad \frac{a = b : A}{b = a : A}
\]

\[
\frac{a = b : A \quad b = c : A}{a = c : A}
\]
\[
a = b : A 
\quad x : A \vdash B(x) : \text{type}
\]
\[
B(a) = B(b) : \text{type}
\]
\[
a = b : A 
\quad x : A \vdash f(x) : B(x)
\]
\[
f(a) = f(b) : B(a)
\]
\[
A = B : \text{type} 
\quad a : A
\]
\[
a : B
\]

**Appendix A.3. Formation rules**

\[
\frac{
\text{x : A} \vdash B(x) : \text{type}
}{
\prod_{x : A} B(x) : \text{type}
}\text{ \(\prod\) formation}
\]

\[
\frac{
\text{x : A} \vdash B(x) : \text{type}
}{
\sum_{x : A} B(x) : \text{type}
}\text{ \(\sum\) formation}
\]

\[
\frac{
a, b : A
}{
\quad \vdash A(a, b) : \text{type}
}\text{ \(\text{Id} \) formation}
\]

\[
\quad \vdash \mathbb{N} : \text{type}
\text{ \(\mathbb{N}\) formation}
\]

**Appendix A.4. Introduction and elimination rules for dependent products**

\[
\frac{
\text{x : A} \vdash f(x) : B(x)
}{
\lambda_{x : A} f(x) : \prod_{x : A} B(x)
}\text{ \(\prod\) introduction}
\]

\[
\frac{
f : \prod_{x : A} B(x) 
}{
\quad \text{app}(f, a) : B(a).
}\text{ \(\prod\) elimination}
\]

**Appendix A.5. Introduction and elimination rules for dependent sums**

\[
\frac{
a : A 
\quad b : B(a)
}{
\quad \text{pair}(a, b) : \sum_{x : A} B(x)
}\text{ \(\sum\) introduction}
\]

\[
\frac{
p : \sum_{x : A} B(x) 
\quad \text{x : A, y : B(x) \vdash } \psi(x, y) : C(\text{pair}(x, y))
}{
\quad \text{R}([x : A, y : B(x)]\psi(x, y), p) : C(p)
}\text{ \(\sum\) elimination}
\]
Appendix A.6. Introduction and elimination rules for identity types

\[\frac{a : A}{r(a) : A(a,a)}\] \text{Id introduction}

\[\frac{x : A, y : A, z : A(x,y) \vdash B(x,y,z) : \text{type}}{x : A \vdash \varphi(x) : B(x,x,r(x))}\]

\[f : A(a,b)\]

\[J[x,y : A, z : A(x,y)]B(x,y,z)([x : A]\varphi(x),a,b,f) : B(a,b,f)\] \text{Id elimination}

Appendix A.7. Introduction and elimination rules for natural numbers

\[0 : \mathbb{N}\] \text{N introduction (i)}

\[\frac{n : \mathbb{N}}{S(n) : \mathbb{N}}\] \text{N introduction (ii)}

\[\frac{n : \mathbb{N}}{c : C(0) \quad x : \mathbb{N}, y : C(x) \vdash \gamma(x,y) : C(S(x))}{\text{rec}(n,c,[x : \mathbb{N}, y : C(x)]\gamma(x,y)) : C(n)}\] \text{N elimination}

Appendix A.8. Conversion rules

\[\frac{\lambda x : A f(x) : \Pi_{x : A} B(x)}{\text{app}(\lambda x : A f(x), a) = f(a) : B(a)}\] \text{Π conversion}

\[\frac{a : A \quad b : B(a) \quad x : A, y : B(x) \vdash \psi(x,y) : C(\text{pair}(x,y))}{R([x : A, y : B(x)]\psi(x,y),\text{pair}(a,b)) = \psi(a,b) : C(\text{pair}(a,b))}\] \text{Σ conversion}

\[\frac{a : A}{J[x,y : A, z : A(x,y)]B(x,y,z)([x : A]\varphi(x),a,a,r(a)) = \varphi(a) : B(a,a,r(a))}\] \text{Id conversion}

\[\frac{\text{rec}(0,c,[x : \mathbb{N}, y : C(x)]\gamma(x,y)) = c : C(0)}{\text{N conversion (i)}}\]

\[\frac{n : \mathbb{N}}{\text{rec}(S(n),c,[x : \mathbb{N}, y : C(x)]\gamma(x,y),n) : C(S(n))}\] \text{N conversion (ii)}
References


