Geometrical Modeling of the Heart

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The Project

- Goal: Creation of a precise 3D model of the heart

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- Methodology: Extract heart characteristics from medical images
  - Boundaries of muscle and cavities
  - Fine anatomic details

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- Techniques: Variational Methods
  - Mumford-Shah functional
  - Level Set

Segmentation Problem

Identify significant regions and boundaries

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Identify significant regions and boundaries

Data: CT scan

Courtesy of the
PET Center

University of

Ottawa

Heart Institute

Data: MRI

Courtesy of the Sunnybrook Health Sciences Center

University of

Toronto

Data: Visible Human Project

Courtesy of the
United States
National Library
of Medecin

National Institute

of Health

Overview

1. Variational Methods
2. Numerics

1. Variational methods

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Image:

$$g : \Omega \subseteq \mathbb{R}^N \longrightarrow [0, 1]$$

0: white, 1: black

Mumford-Shah functional

Find u such that

$$F(u) = \int_\Omega |\nabla u|^2 + \int_\Omega (g-u)^2 + \mathcal{H}^{N-1}(J_u)$$

is minimized

$u \in SBV(\Omega)$
J_u is the discontinuity set of u
$\mathcal{H}^{N-1}$ is the Hausdorff measure

Mumford-Shah functional

Find u such that

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Image is smooth away from edges J_u

$u \in SBV(\Omega)$
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Mumford-Shah functional

Find u such that

$$F(u) = \int_\Omega |\nabla u|^2 + \int_\Omega (g-u)^2 + \mathcal{H}^{N-1}(J_u)$$

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Image is close to g

$u \in SBV(\Omega)$

J_u is the discontinuity set of u

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Mumford-Shah functional

Find u such that

$$F(u) = \int_\Omega |\nabla u|^2 + \int_\Omega (g-u)^2 + \mathcal{H}^{N-1}(J_u)$$

is minimized

Edge set is short

$u \in SBV(\Omega)$

J_u is the discontinuity set of u

$\mathcal{H}^{N-1}$ is the Hausdorff measure

The space $BV(\Omega)$

- $u \in W^{1,1}(\Omega) \Leftrightarrow \exists g \in \left[L^1(\Omega)\right]^N$ such that

$$\forall \phi \in [C_c^\infty(\Omega)]^N, \quad \int_\Omega u \mathrm{div} \phi \mathrm{d}x = (-1)^N \int_\Omega \phi \cdot g \mathrm{d}x$$

$$\nabla u := g$$

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- $u \in BV(\Omega) \Leftrightarrow \exists \mu \in \mathcal{M}(\Omega)$ such that

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$$Du := \mu$$

$\mathcal{M}(\Omega)$: vector valued finite measures on Ω

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$$L^1(\Omega) \subseteq BV(\Omega) \subseteq W^{1,1}(\Omega) \subseteq C^1(\Omega)$$

Example

$$f(x) = \begin{cases} 0 & x < 0 \\ x^2/2 & x \geq 0 \end{cases}$$

$$f'(x) = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases}$$

$$f''(x) = H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$$f'''(x) = \delta$$

Example

$$f(x) = \begin{cases} 0 & x < 0 \\ x^2/2 & x \geq 0 \end{cases} \qquad f'(x) = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases}$$

$$f''(x) = H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \qquad f'''(x) = \delta$$

$$f \in C^1(-1,1)$$
$$f' \in W^{1,1}(-1,1)$$
$$f'' \in BV(-1,1)$$

The space $SBV(\Omega)$

Theorem Let $u \in BV(\Omega)$. Then

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u + D^c u.$$

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$$u \in SBV(\Omega) \iff D^c u = 0$$

The space $SBV(\Omega)$

Theorem Let $u \in BV(\Omega)$. Then

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u + D^c u.$$

$$u \in SBV(\Omega) \quad \Longleftrightarrow \quad D^c u = 0$$

Theorem Let $u_n \in SBV(\Omega)$ such that

$$\sup_n \|u_n\|_\infty < \infty, \quad \sup_n \left\{ \int_\Omega |\nabla u_n|^2 + \mathcal{H}^{N-1}(J_u) \right\} < \infty.$$

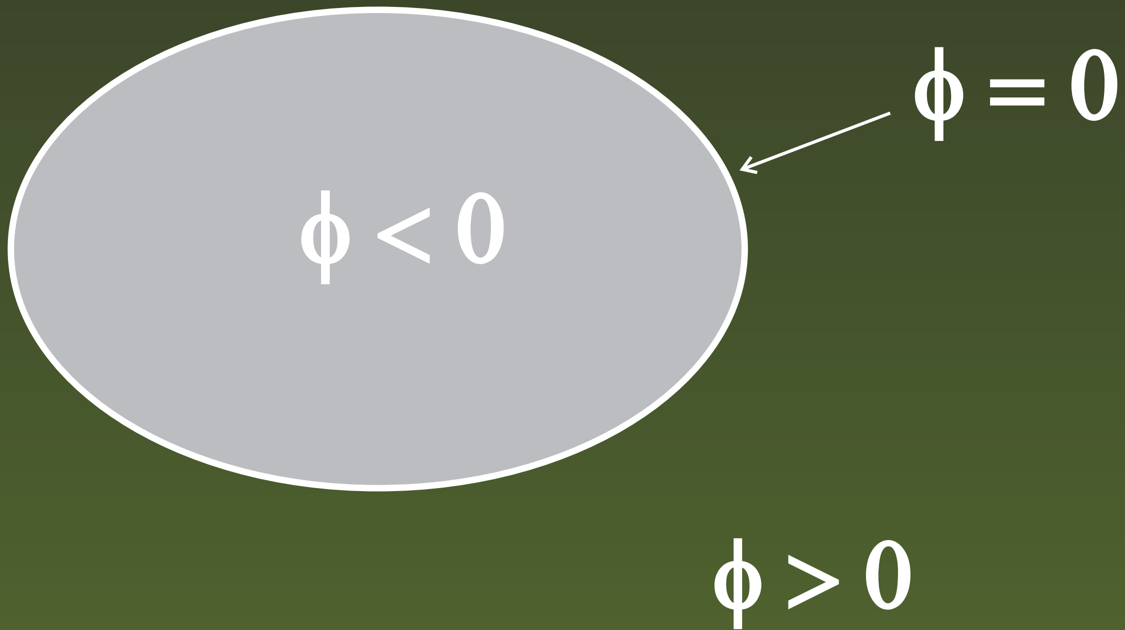
Then $\exists u_{n_k} \stackrel{*}{\rightharpoonup} u \in SBV(\Omega)$.

2. Numerics

Piecewise Constant Mumford-Shah

Restrict to binary segmented images (with values c_1, c_2).
Interface: $C = \{\phi = 0\}$,

$$u = c_1 H(\phi) + c_2(1 - H(\phi))$$



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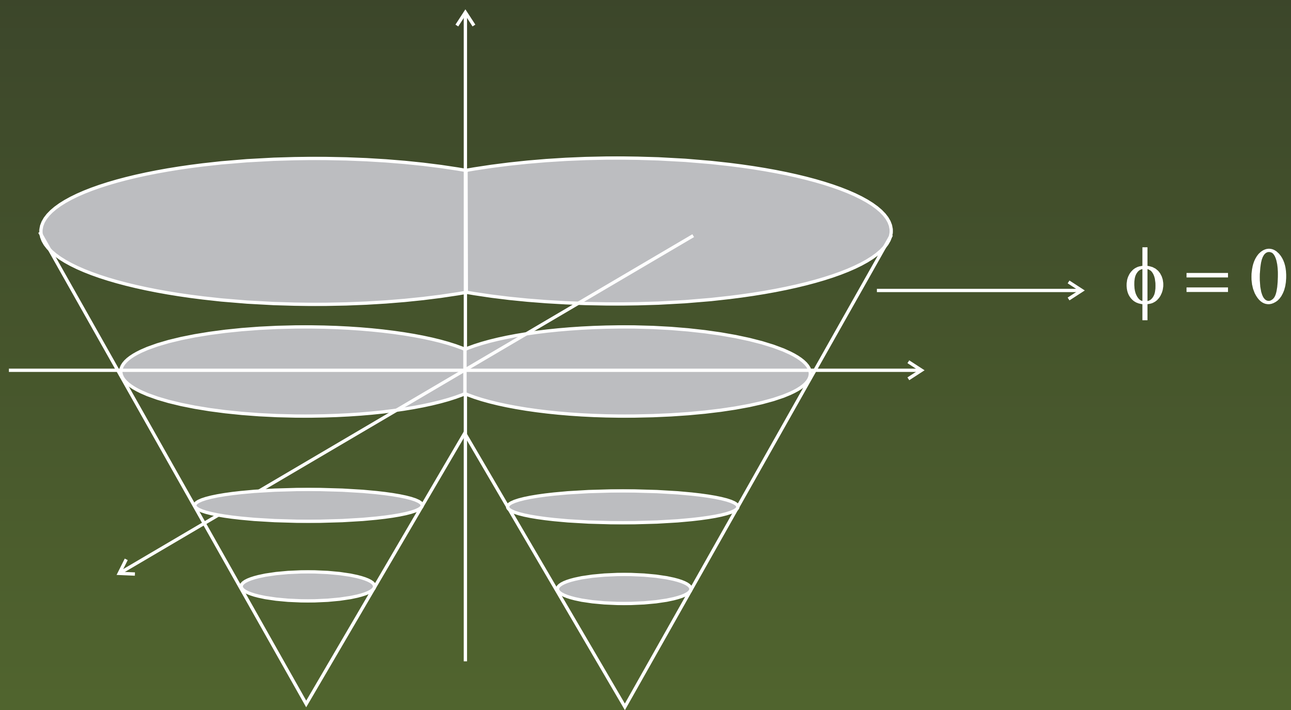


Euler-Lagrange equation

$$0 = H'(\phi)\left[\operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) - (g - c_1)^2 + (g - c_2)^2\right]$$

Euler-Lagrange equation

$$\frac{d\phi}{dt} = H'(\phi)\left[\operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) - (g - c_1)^2 + (g - c_2)^2\right]$$

Euler-Lagrange equation

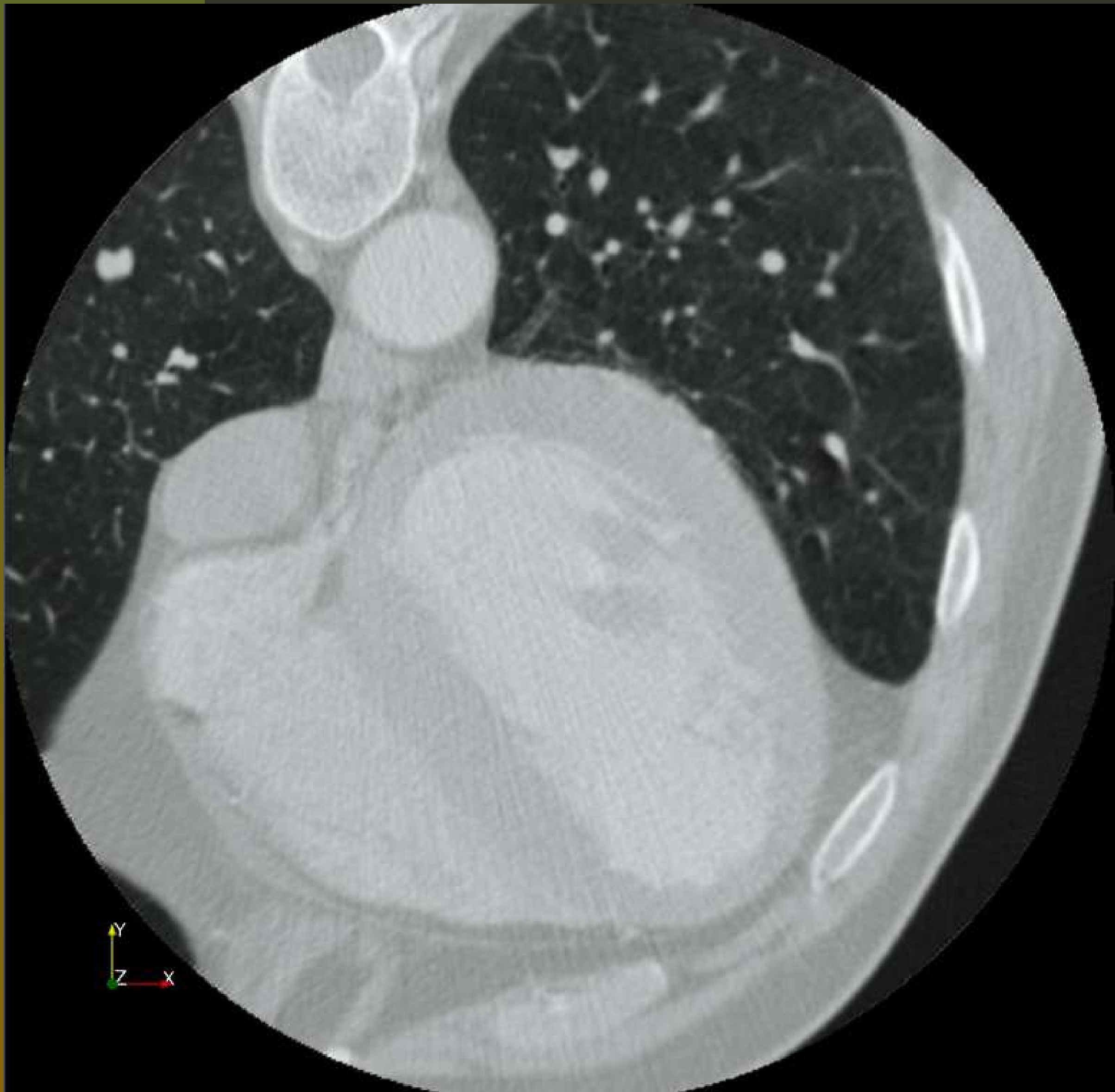
$$\frac{d\phi}{dt} = H'(\phi)\left[\operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) - (g - c_1)^2 + (g - c_2)^2\right]$$

Algorithm

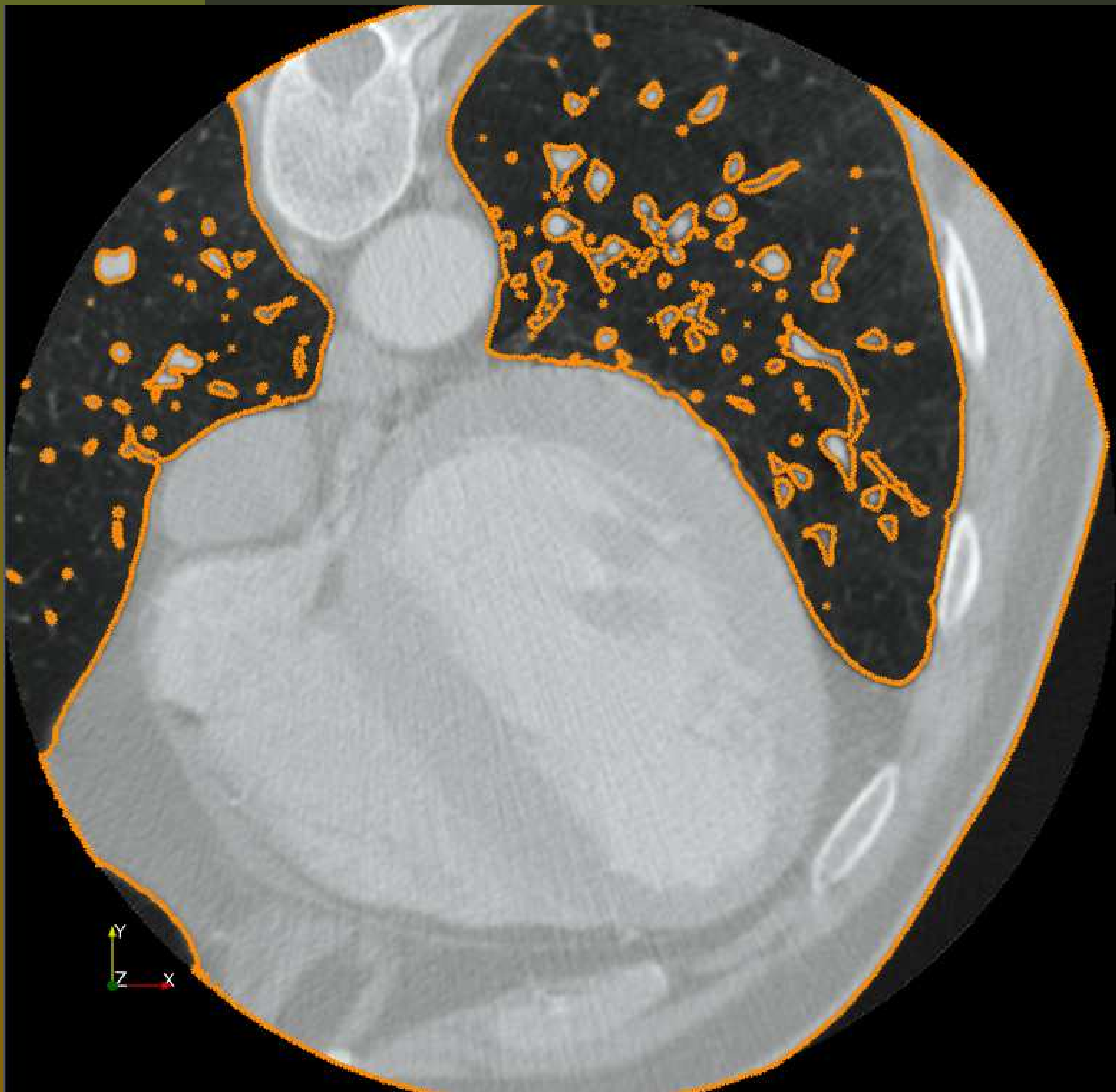
At each time step

- Compute c_1 and c_2 from ϕ^n
- Compute ϕ^{n+1} by finite differences

Results: 2D



Results: 2D



Dimensions:
500×500

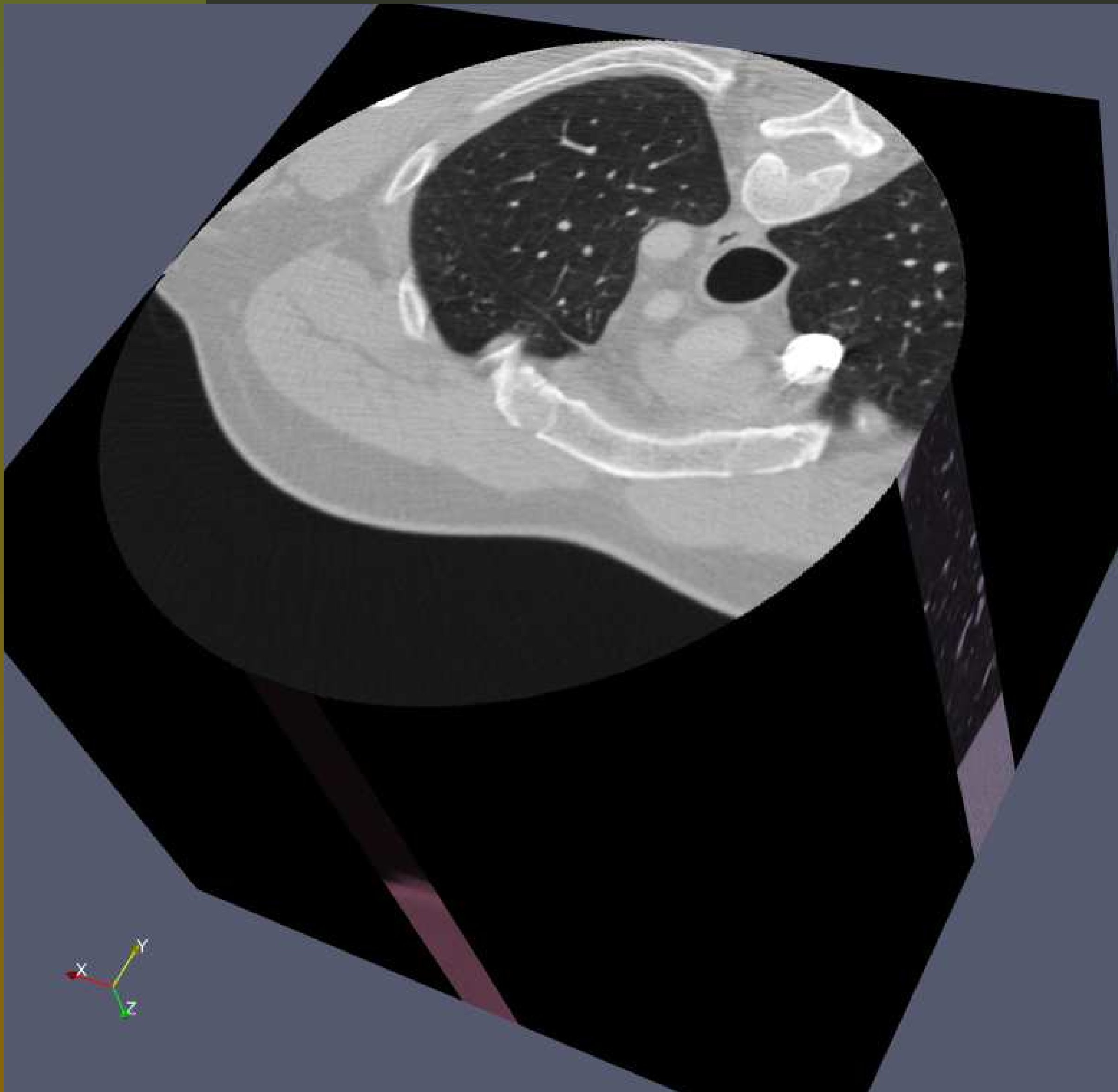
Pixels: 250 000

Time step: 0.1

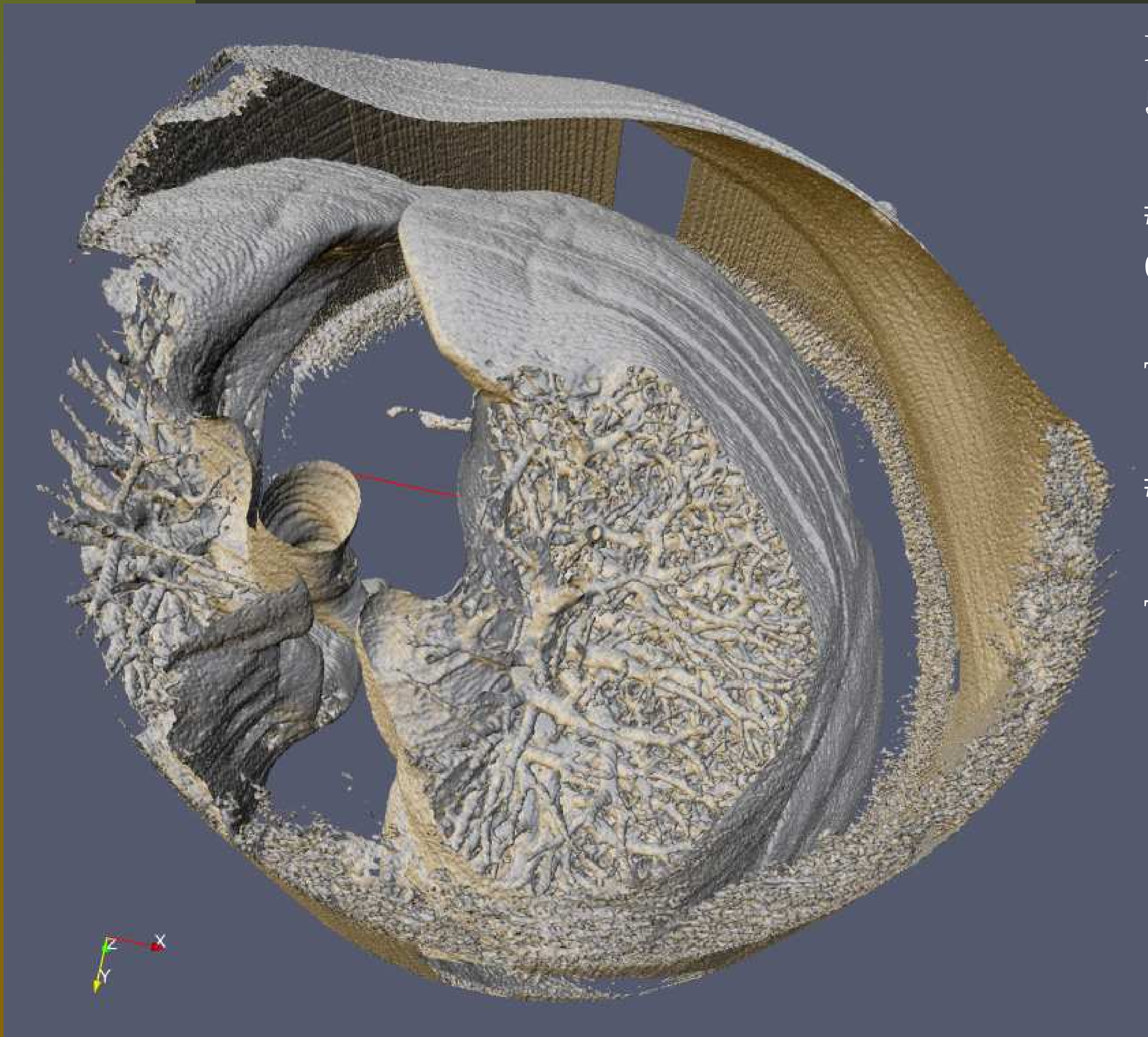
Iterations: 50

Time: 20 sec.

Results: 3D



Results: 3D



Dimensions:
$500 \times 500 \times 250$

Pixels:
62 500 000

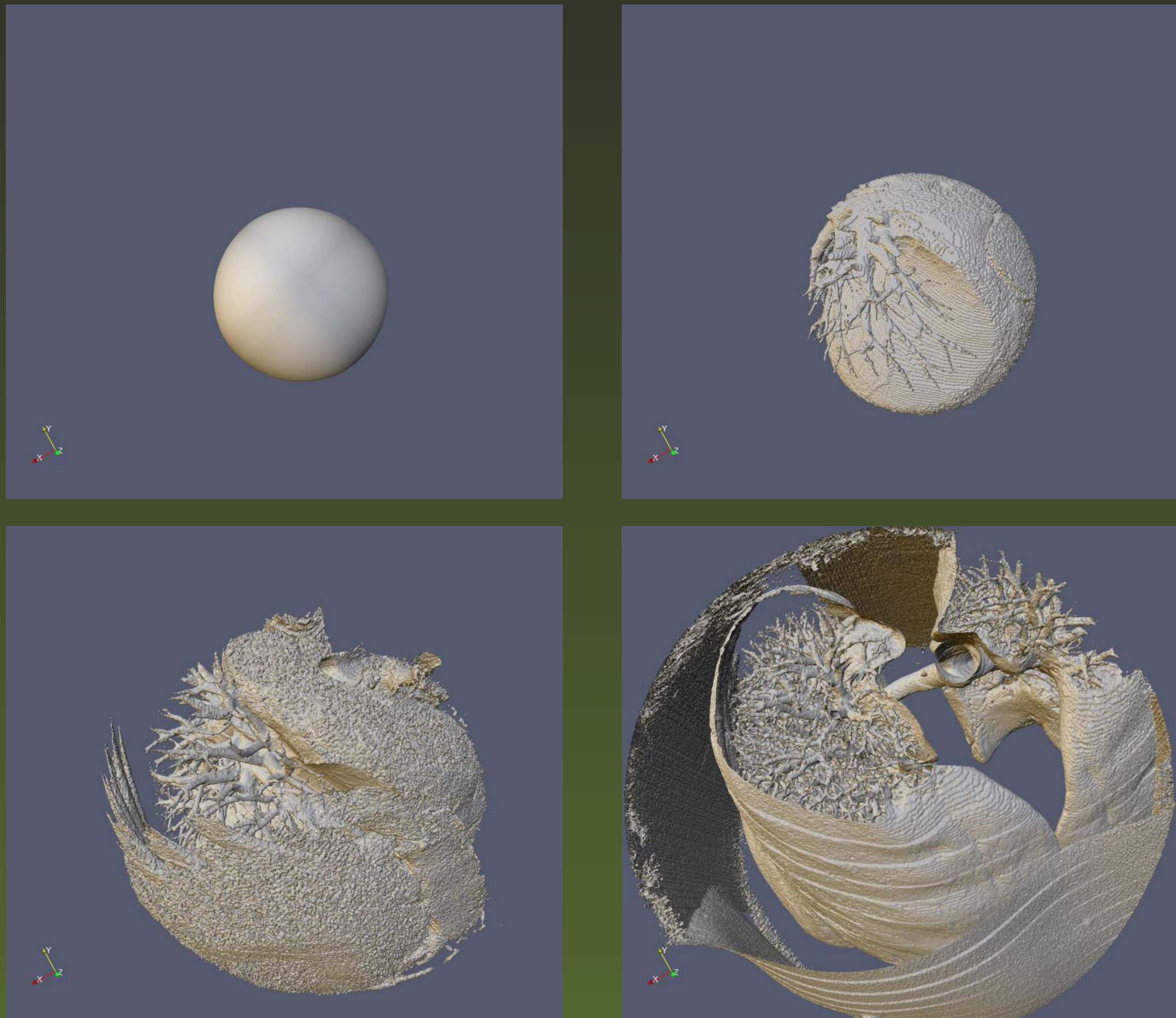
Time step: 0.001

Iterations: 5600

Time: 186 hours

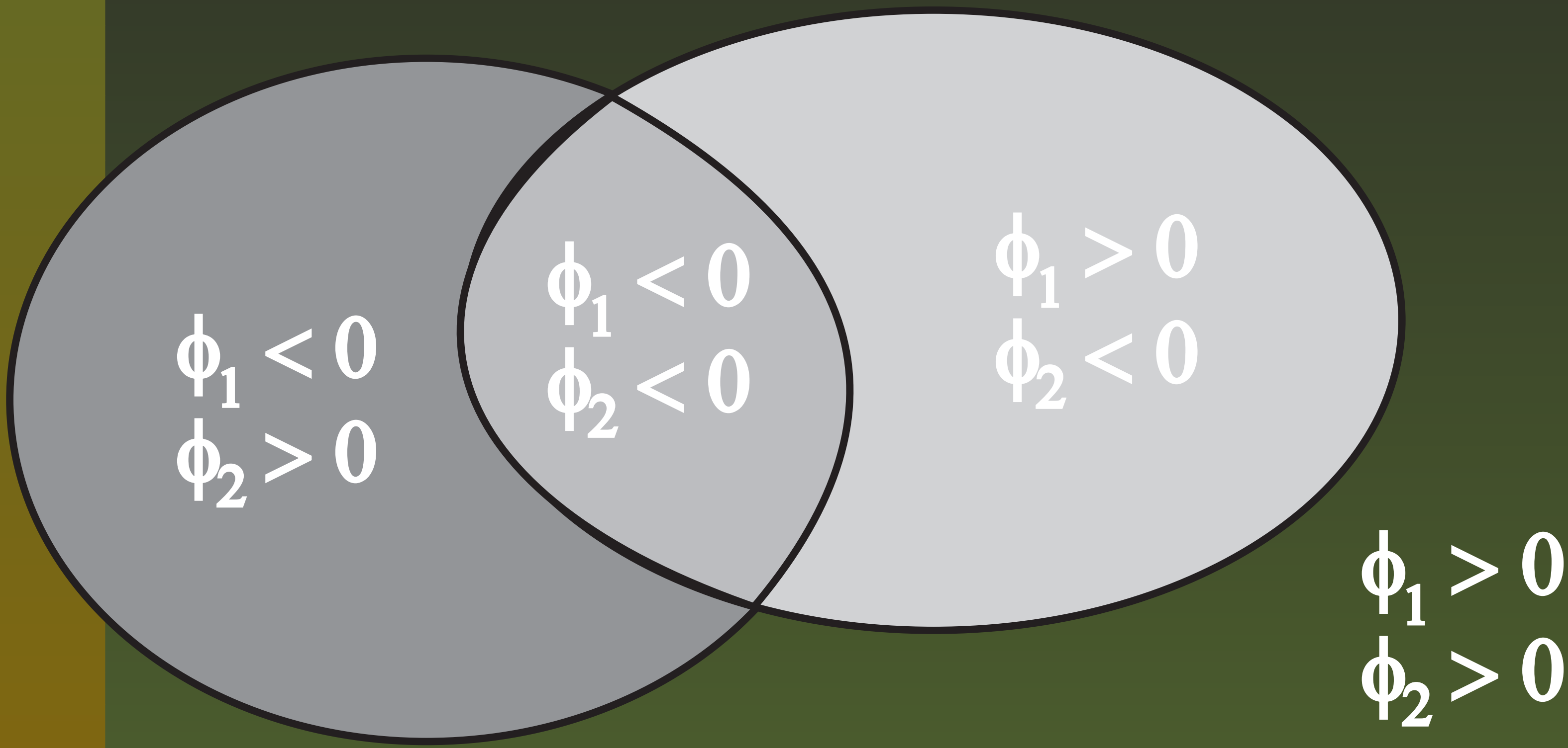
Results: 3D

Results: 3D



Several colours

Evolve n-curves to split the image in 2^n regions:



Further work

- Segment the heart!