IDEAS IN CRITICAL POINT THEORY

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Abstract. We present some basic tools in critical point theory. We first define the notion of differentiability and then introduce the *Palais-Smale condition*. Using this, we state a very important result called the *deformation lemma* in which the level sets of a function play an fundamental role. We then use this result to prove some theorems about the existence of a critical point for some functions.

1. Introduction

The theory of critical points naturally appears in classical mechanics, with the so called least action principle. If we want to know how a physical system evolves from a state A to a state B, one can define a function that associates to each path from A to B a real quantity called the action. The physical principle says that the path followed by the physical system will be a critical point of this function. This is where the name of the principle is missleading: the physical system may not only follow paths of least action, corresponding to minimums, but also maximums or saddle points. But such a formulation of the problem is quite modern, and physicists found lots of tricks to avoid this kind of space and functions they did not know about.

More recently, critical point theory appeared to be a very useful tool in PDEs. Given a PDE problem, it is sometime possible to find a function whose critical points are solution to the differential problem. For a particular PDE problem, one usualy proceed as follow:

(1) Find the real valued function that should naturally be associated to the problem.
(2) Find or invent a theorem that ensure you that your function has a critical point.
(3) Prove that a critical point of your function is a solution to the problem.

In this paper, we focus only on the second step: how can we prove that a function has a critical point? This step has a flavour of topology, while the others are more analytic. We shall present one of the fundamental result in critical point theory: the deformation lemma and then use it to prove the existence of critical points for some functions.

The deformation lemma tells a lot about topology. In fact, the critical points of a function $f : X \to \mathbb{R}$, highly depends on the topology of X. For example, if X is the surface of a torus and the function f is "non degenerate", it should have at least three critical points, while a function $g : \mathbb{R} \to \mathbb{R}$ could have none. Here, X will be a Banach space; since it is a contractible space, the topology of X cannot tell anything about the critical points of f. What we will be interested on are the level sets of f; that is for $c \in \mathbb{R}$, we have the level set $f^c = \{x \in X : f(x) \leq c\}$. The topology of these sets will be relevant when searching for critical points.

2. Elementary definitions

We shall first define what differentiability of a real function over a Banach space is. We consider the notion of continuity known. From now, let us consider E to be a real Banach space.

Definition 1. A continuous function $f : E \to \mathbb{R}$ is said to be *Fréchet differentiable* (differentiable for short) at $x_0 \in E$ if there exist a continuous linear function L such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - \langle L, x - x_0 \rangle}{\|x - x_0\|} = 0.$$

We write $f'(x_0) = L$.

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We say that the function is of *class* C^1 if it is differentiable for all $x \in E$ and that the function $(x \mapsto f'(x))$ is continous.

Definition 2. Let $f : E \to \mathbb{R}$ be C^1. We say that $x \in E$ is a *critical point* of f if $f'(x) = 0$, as a linear function; then, we say that $f(x)$ is a *critical value* of f. If x is not a critical point, it is said to be a *regular point*; similarly, if $c \in \mathbb{R}$ is not a critical value, it is said to be a *regular value.*

We note $K = \{x \in E : f'(x) = 0\}$. Furthermore, for $c \in \mathbb{R}$, we write $K_c = \{x \in E : f'(x) = 0 \text{ and } f(x) = c\}$ and $f^c = \{x \in E : f(x) \leq c\}$. The last one is called the *level set* of the function f.

In order to obtain the fundamental result in critical point theory, namely the deformation lemma, we need our function to satisfy one more condition.

Definition 3. We say that a function $f : E \to \mathbb{R}$ satisfies the *Palais-Smale condition* (we write (PS)), if every sequence (x_n) in E such that $f(x_n)$ is bounded and $f'(x_n) \to 0$, when $n \to \infty$, admits a convergent subsequence. This subsequence clearly converges to a critical point.

Example 4.
1. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{1/x^2}$ does not satisfy the (PS) hypothesis; consider the sequence of positive integers.
2. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = sinx$ does not satisfy the (PS) hypothesis; consider the sequence of critical points $n\pi$.
3. Any polynomial will satisfy the (PS) condition.

3. Deformation lemma

We are now able to formulate our fundamental result. We will give three versions of the theorem, the last being the more general, but as we will see, the others are sufficient to prove some important critical point theorems. For complete proofs of these theorems, see [1].

Theorem 5 (Deformation lemma I). *Let E be a Banach space and $f \in C^1(E, \mathbb{R})$ satisfying (PS). For $a, b \in \mathbb{R}$, $a < b$, if $[a, b]$ does not contain any critical value of f, then there exists a deformation $\eta \in C([0, 1] \times E, E)$ satisfying*

1. $\eta(0, x) = x \quad \forall x \in E,$
2. $\eta(1, f^b) \subseteq f^a.$

If we were in a Hilbert space, we could prove this result by taking the gradient of the function f, and construct the deformation as follow: for a point $x \in E$, consider the differential problem

$$\begin{cases} \frac{d\eta}{dt} = -\text{grad} f, \\ \eta(0) = x. \end{cases} \tag{3.1}$$

By the existence and unicity theorem of differential equations, this system will have a unique solution. What can be verify, is that the solution will be defined, not only locally, but for all $t \in \mathbb{R}$. It is now sufficient to use the theorem of continuous dependance on initial conditions to obtain the deformation $\eta(t, x)$.

By the fact that there is no critical value in $[a, b]$, and that the function satisfies (PS), it is possible to convince ourselves that the norm of the gradient of f will be bounded below by some positive constant on $f^{-1}([a, b])$. Thus, it is sufficient to follow the solution of the differential equation for a time sufficiently long to get the condition (2) verified. We then rescale time $t \to Ct$ so that condition (2) be statisfied after time 1.

The problem here is that we are in a Banach space, where functions do not necesseraly admit a gradient. We bypass this problem by defining a *pseudo-gradient* which does not have all the nice properties of a gradient, but has at least those we need for the result.

We now state a very useful version of the theorem.

Theorem 6 (Deformation lemma II). *Let E be a Banach space and $f \in C^1(E, \mathbb{R})$ satisfying (PS). If $c \in \mathbb{R}$ is not a critical value of f, then for every $\bar{\epsilon} > 0$, there exists $\epsilon \in (0, \bar{\epsilon}]$ and a deformation $\eta \in C([0, 1] \times E, E)$ satisfying*

1. $\eta(0, x) = x \quad \forall x \in E,$

(2) $\eta(t,x) = x \quad \forall x \notin f^{-1}([c-\bar{\epsilon}, c+\bar{\epsilon}])$,
(3) $\eta(1, f^{c+\epsilon}) \subseteq f^{c-\epsilon}$.

We could obtain (1) and (3) by the last theorem. Since c is not a critical value and f satisfies (PS), it is possible to find $\epsilon \in (0, \bar{\epsilon}]$ such that $[c-\epsilon, c+\epsilon]$ does not contain any critical value.

But to have (2), we have to take the proof from its beginning, we have to consider a modified differential problem, where we multiply the gradient by a continuous function which is 0 on $\{x : x \notin f^{-1}([c-\bar{\epsilon}, c+\bar{\epsilon}]), \}$ and 1 on $f^{-1}([c-\epsilon, c+\epsilon])$.

This theorem tells us that if c is not a critical value, the level sets near the value c all look the same, it is possible to deform one into the other, without changing the topological properties of the sets.

One simple way of finding a critical point would be to look at level sets f^a and f^b: if they have different topology, then we know that there should be a critical value in $[a, b]$. The topological differences could be seen by homotopy or homology theory.

We can ask the question: what happens if c is a critical value?

Theorem 7 (Deformation lemma III). *Let E be a Banach space, $f \in C^1(E, \mathbb{R})$ satisfying (PS) and $c \in \mathbb{R}$. Then for every $\bar{\epsilon} > 0$ and every neighbourhood N of K_c, there exists $\epsilon \in (0, \bar{\epsilon}]$ and a deformation $\eta \in C([0,1] \times E, E)$ satisfying*

(1) $\eta(0,x) = x \quad \forall x \in E$,
(2) $\eta(t,x) = x \quad \forall x \notin f^{-1}([c-\bar{\epsilon}, c+\bar{\epsilon}])$,
(3) $\eta(1, f^{c+\epsilon} \setminus N) \subseteq f^{c-\epsilon}$.

The idea of the proof is the same as earlier, but is quite technical. We need to change the differential problem in order that the points of $f^{c+\epsilon} \setminus N$ do not tend to K_c during the deformation.

The theorem essentially says that if there is a critical value, there should be a change in the topology of the level sets, but this change occurs in an arbitrarily small neighbourhood of the critical points.

4. Critical point theorems

We now apply the deformation lemma to obtain some critical point results.

Theorem 8. *Let E be a Banach space and $f \in C^1(E, \mathbb{R})$ satisfying (PS). If f is bounded below, then*

$$c = \inf_E f$$

is a critical value of f.

Proof. Suppose c is not a critical value. Take $\bar{\epsilon} = 1$ and apply theorem 6. We therefore have $\epsilon \in (0,1]$ and a deformation $\eta \in ([0,1] \times E, E)$ such that

$$\eta(1, f^{c+\epsilon}) \subseteq f^{c-\epsilon}.$$

But, by the definition of c, $f^{c-\epsilon}$ is empty, while $f^{c+\epsilon}$ is not. This is a contradiction. □

The theorem states exactly what one can intuitively think of in a finite dimensional space. The function $f(x) = e^{-1/x^2}$ is bounded below, but does not satisfies (PS); we see that 0 is not a critical value, it is escaping at infinity. That is the kind of behaviour the (PS) hypothesis forbids.

The next one is due to Rabinovitz and Ambrosetti in the 70's (see [1]). Remark that a proof without using the deformation lemma needs really hard work, using Ekeland's variationnal principle and technical tools of analysis, like Radon spaces.

Theorem 9 (Mountain pass theorem). *Let E be a Banach space and $f \in C^1(E, \mathbb{R})$ satisfying (PS). Suppose $f(0) = 0$ and that*

(i) *there exists constants $\rho, \alpha > 0$ such that $f\big|_{\partial B_\rho} \geq \alpha$, and*
(ii) *there exists $e \in E \setminus B_\rho$ such that $f(e) \leq 0$.*

Then f has a critical value $c \geq \alpha$. Morover, c is caracterized by

$$c = \inf_{g \in \Gamma} \max_{t \in [0,1]} f(g(t)),$$

where

$$\Gamma = \{g \in C([0,1]), E) : g(0) = 0, g(1) = e\}.$$

Proof. Conditions (i) and (ii) represent the following situation: imagine that you are at the origin and you want to go to the point e. Condition (i) says that you are circled by a range of mountains that you have to cross to go at e. The theorem states that there exists a path going from you to e that will climb the least and that this path will cross the range at a critical point: a mountain pass.

Remark that c is well defined and that $c \geq \alpha$. We only need to verify that it is a critical value.

Suppose that it is not, theorem 6 says that there exists $\epsilon > 0$ and a deformation η satisfying the conditions (1)-(3) of the theorem.

Now take a path $g \in \Gamma$ such that

$$\max_{t \in [0,1]} f(g(t)) < c + \epsilon,$$

and consider the function $h \in C([0,1], E)$ defined by

$$h(t) = \eta(1, g(t)).$$

Remark that since $g(0) = 0 < \alpha - \bar{\epsilon} < c - \bar{\epsilon}$, then $h(0) = \eta(1, g(0)) = g(0)$ by condition (1) of theorem 6; Similarly $h(1) = e$. Then $h \in \Gamma$.

Since $\eta(1, f_{c+\epsilon}) \subseteq f_{c-\epsilon}$, we have

$$\max_{t \in [0,1]} f(h(t)) = \max_{t \in [0,1]} f(\eta(1, g(t)) \leq c - \epsilon;$$

this contradict the definition of c. □

The mountain pass theorem is a special case of a linking situation. It is possible to prove, by the same arguments, a more general version.

Definition 10. Let A, Q be subsets of E such that $\partial Q \cap A = \emptyset$. We say that ∂Q links A if for every continuous function $\eta \in \Gamma$, where

$$\Gamma = \left\{ \eta \in C(Q, E) \ : \ \eta\big|_{\partial Q} = \mathrm{id} \right\},$$

we have $\eta(Q) \cap A \neq \emptyset$.

Example 11. Here are some simple examples of linking

(1) If $E = E^1 \oplus E^2$, with $\dim E^2 \neq 0$ and $\dim E^1 < \infty$. If $Q = \{x \in E^1 \ : \ \|x\| \leq 1\}$, then $\partial Q = S^1 = \{x \in E^1 \ : \ \|x\| = 1\}$ links $A = E^2$.

(2) Let e be a point in E such that $\|e\| \leq \alpha$. If Q is a line joining e to the origin, then $\partial Q = \{0, e\}$ links $A = \{x \in E \ : \ \|x\| = \alpha\}$. This is the linking used in the mountain pass theorem. The linking means that every path joining the origin to e must cross the circle.

Theorem 12. *Let E be a Banach space, $f \in C^1(E, \mathbb{R})$ satisfying (PS), A, $Q \in E$ such that ∂Q links A and that Q is compact. If*

$$\sup_{\partial Q} f < \inf_{A} f,$$

then f has at least one critical value c caracterized by

$$c = \inf_{\phi \in \Gamma} \sup_{x \in Q} f(\phi(x)),$$

where Γ is given in Definition 10.

References

[1] Paul H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conf. Ser. in Math., **65** Amer. Math. Soc., Providence, 1986.

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