1. PICK IN DIMENSION 2

1.1. **Pick.**

Lemma 1. *Let P be a bounded measurable set of $\mathbb{R}^2$, and consider for each $z \in \mathbb{Z}^2$ the sets $P + z$ which is P translated by z. If no two of the $P + z$ have intersection of positive measure, then P has area ≤ 1.*

Proof. Write d for the diameter of P. Consider the square array of $(2n+1)^2$ sets $P + z$, for all z whose coordinates are between $-n$ and n; these sets are all included in a square of side length $2n + 2d$. Since no two of them have intersection of positive measure, we have

$$(2n+1)^2 \mathrm{area}(P) \leq (2n+2d)^2,$$

$$\mathrm{area}(P) \leq \left(1 + \frac{2d}{2n+1}\right)^2.$$

But as n can be arbitrarily large,

$$\mathrm{area}(P) \leq 1.$$

□

polytope. em here.

Points of $\mathbb{Z}^k$ will be called *integer points.* The convex hull of d integer points in general position is a *integral simplex* of dimension d, this simplex is *fundamental* if it contains no integer points other than its vertices; an *integral polygon* is a union of a finite number of integral simplexes, it is *simple* if its boundary ∂P is homeomorphic to S^k.

We will be interested in counting the number of integer points in a simple integral polytope P. Let $j_P = \mathrm{card}(P \cap \mathbb{Z}^k)$, $b_P = \mathrm{card}(\partial P \cap \mathbb{Z}^2)$ and $i_p = \mathrm{card}(P \backslash \partial P \cap \mathbb{Z}^2)$.

Theorem 1 (Pick [6])**.** *Let P be an simple integral polygon. Then*

$$\mathrm{area}(P) = j_P - \frac{b_P}{2} - 1.$$

Proof. We first prove the result for a fundamental triangle and extend it to an arbitrary polygon by triangulation. Take a fundamental triangle with vertices at $(0,0), (a,b), (c,d)$. Now consider the parallelogram generated by these two and tile the plane with its translates. As no two of the translates have intersection of positive measure, it follows by Lemma 1 that the area of the original parallelogram is at most 1. But since this area is $|ad - bc| \in \mathbb{N}$, it must be 1. Hence the triangle has area 1/2. For a fundamental triangle $j_P = 3$ and $b_P = 3$, so Pick's formula is correct in this case.

Now suppose that Pick's formula works for two polygons P_1 and P_2 whose intersection is of dimension 1. We have:

$$j_{P_1 \cup P_2} = j_{P_1} + j_{P_2} - j_{P_1 \cap P_2},$$

$$b_{P_1 \cup P_2} = b_{P_1} + b_{P_2} - 2 j_{P_1 \cap P_2} + 2,$$

so that

$$j_{P_1 \cup P_2} - \frac{b_{P_1 \cup P_2}}{2} - 1 = \left\{ j_{P_1} - \frac{b_{P_1}}{2} - 1 \right\} + \left\{ j_{P_2} - \frac{bP_2}{2} - 1 \right\}$$

$$= \mathrm{area}(P_1) + \mathrm{area}(P_2) = \mathrm{area}(P_1 \cup P_2).$$

Pick's formula is therefore additive; since any polygon P can be triangulated into fundamental triangles for each of which Pick's formula works, the formula holds for P. □

1.2. Regular polygons.

Corollary 1. *There exists no integral equilateral triangle, nor integral regular hexagon.*

Proof. An integral equilateral triangle as an irrationnal area, but Pick's formula says that the area of an integral polygon must be a multiple of 1/2. A contradiction.

The case of the regular hexagon is similar, for it is composed of 6 equilateral triangles. ☐

It is obvious that we can have integral squares, but are there integral regular polygons other than squares?

Theorem 2. *There exist no integral regular polygons but squares.*

Proof. The non-existence has already been established for triangles and hexagons. Let us look at the pentagon. If there exist integral regular pentagons, there is certainly one which is smaller than all the others. Take three consecutive vertices of this minimal pentagon and complete the parallelogram: the fourth point will be integer (see figure **??**).

Doing this with all three consecutive vertices, we obtain a smaller integral regular pentagon, contradicting minimality.

The argument is valid with any regular polygon other than the triangle, the square, and the hexagon. ☐

2. PICK IN ARBITRARY DIMENSION

2.1. construction of the formula. The following example shows that Pick's formula in higher dimension must not be as simple.

Example 1. *Consider the family of tetrahedra with verices* $(0,0,0),(1,0,0),(0,1,0),(1,1,r)$*, where* $r \in \mathbb{N}$*. Each of these tetrahedra has the same number of integer points but the volume* $r/6$ *depends on* r*..*

From P, we define the k-polytope nP which is obtained by rescaling P by a factor of $n \in \mathbb{N}$. Define $j_P(n) = j_{nP}$, $i_P(n) = i_{nP}$ and $b_P(n) = b_{nP}$.

Computing $j_P, b_P, j_P(n), b_P(n)$ for a given $n \in \mathbb{N}$ is sufficient to obtain Pick's formula in dimension 3. In dimension k, we need to rescale a polytope P by $k-2$ different integer factors $n_i \in \mathbb{N}$ and to count $j_P(n_i)$ and $b_P(n_i)$, for each i, to have an exact formula for the volume.

Suppose for instance that the formula has the following form:

$$\mathrm{vol}(P) = M_k(P) - \frac{1}{2} M_k(\partial P),$$

with

$$M_k(P)m = a_1 j_P + a_2 j_P(n_1) + \ldots + a_{k-1} j_P(n_{k-2}) + a_k \chi(P),$$

where $\chi(P)$ is the Euler characteristic of the polytope P, and the coefficients a_i are undetermined.

We will see that it is possible to find the coefficients a_i in such a way that the formula gives the exact volume of an integral polytope. The basic idea is that the formula must be additive under the union of polytopes; once it is additive, it only has to give the exact volumes for fundamental simplexes since every polytope is a union of such simplexes. Let P_1 and P_2 be two polytopes having an intersection of dimension $k-1$, some computing gives

$$\mathrm{vol}(P) = \mathrm{vol}(P_1) + \mathrm{vol}(P_2) - M_k(\partial(P_1 \cap P_2)),$$

where $\partial(P_1 \cap P_2)$ is of dimension $k-2$. So, the formula in dimension k is additive if and only if $M(P) = 0$ for each $(k-2)$-polytope P. A way of doing that is to make M_k an additive function, and to set the value of M_k to be 0 on fundamental simplexes.

Take two polytopes P_1 and P_2 of dimension $k-2$, with an intersection of dimension $k-3$. We see that

$$M_k(P_1 \cup P_2) = M_k(P_1) + M_k(P_2) - M_k(P_1 \cap P_2),$$

Therefore the formula is additive for $(k-2)$-polytopes if and only if $M_k(P) = 0$ for every $(k-3)$-polytope P. Inductively, we must begin by setting $M_k(P)$ to be 0 for every 0-polytope P, which consists of a finite number of points.

This provides $k-1$ conditions that the coefficients must satisfy. It is not obvious that there is such coefficients nor that the conditions can be explicited. If these conditions are satisfied, the volume formula becomes additive with respect to the union, so it is only necessary that the formula gives the exact volume for a fundamental simplexes. This gives us a last condition on the coefficients.

A result of Ehrhart will ensure us that everything is working fine.

Each face σ of a polytope in $\mathbb{Z}_k$ define a sublattice $\mathcal{L}$, we can compute the volume $\mathrm{vol}(\sigma)$ of this face with respect to $\mathcal{L}$. We then define $\mathrm{vol}(\partial P)$ to be the sum volumes of all the faces of P.

Theorem 3 (Ehrhart [4])**.** *Let P be a convex simple integral polytope of dimension k. Then $j_P(n)$ is a polynomial in n of degree P. Also if*

$$j_P(n) = a_k n^k + a_{k-1} n^{k-1} + \dots + a_0,$$

then $a_k = \mathrm{vol}(P)$, $a_{k-1} = \mathrm{vol}(\partial P)/2$ and $a_0 = 1$. Moreover, $i_P(n) = (-1)^k j_P(-n)$.

We thus have a nice expression for $j_P(n_i)$, and the relations obtained form a system of k linear equations with coefficients in n_i.

2.2. **Pick in dimension 1.** We want the formula to give the exact volume for a fundamental segment. The Euler characteristic of a line is always 1, while the caracteristic of its boundary is 2, so that $\chi P - 2\chi\partial P = 0$ for any P of dimension 1. We thus have

$$\mathrm{length}(P) = a_1\left(j_P - \frac{b_P}{2}\right) = a_1\left(2 - \frac{2}{2}\right) = a_1 = 1.$$

Hence Pick in dimension 1 is

$$\mathrm{length}(P) = j_P - \frac{b_P}{2},$$

2.3. **Pick in dimension 2.** Our method gives an alternative derivation of Pick in dimension 2.

We first want $M_2(P) = 0$ when P is of dimension 0. In this case $j_P = \chi(P)$, thus

$$M_2(P) = a_1 j_P + a_2\chi(P) = a_1 j_P + a_2 j_P = 0,$$

leading to

$$a_1 = -a_2.$$

We also want the formula to give the exact volume for a fundamental triangle, which has area 1/2. We thus have:

$$\mathrm{area}(P) = a_1\left(j_P - \frac{b_P}{2} - \chi(P) + \frac{\chi(\partial P)}{2}\right) = a_1\left(3 - \frac{3}{2} - 1 + \frac{0}{2}\right) = \frac{a_1}{2} = \frac{1}{2},$$

which implies

$$a_1 = 1.$$

Pick in dimension 2 is then

$$\mathrm{area}(P) = j_P - \chi(P) - \frac{1}{2}\{b_P - \chi(\partial P)\}.$$

This formula is more general than the one proved in the last section. It applies to polygons which can have holes and allows polygons to have self-intersections at some integer points.

2.4. **Pick in dimension 3.** We now look at Pick in dimension 3, taking

$$M_3(P) = a_1 j_P + a_2 j_P(n) + a_3\chi(P).$$

We want $M_3(P) = 0$ when P is of dimension 0. So that

$$M_3(P) = a_1 j_P + a_2 j_P(n) + a_3\chi(P) = a_1 j_P + a_2 j_P + a_3 j_P = 0,$$

which implies

$$a_1 + a_2 + a_3 = 0.$$

We also want $M_3(P) = 0$, when P is a fundamental segment. In that case, we have

$$M_3(P) a_1 j_P + a_2 j_P(n) + a_3\chi(P) = 2a_1 + (n+1)a_2 + a_3.$$

Leading to the relation

$$2a_1 + (n+1)a_2 + a_3 = 0.$$

We now want Pick's formula to give the exact volume of a fundamental tetrahedron. We can always find a linear map with determinant 1 that takes this tetrahedron to one of the family of tetrahedra: $(0,0,0); (1,0,0); (0,1,0); (1,1,r)$, with $r \in \mathbb{Z}$. It is then sufficient that the formula works for such tetrahedra. We need to count the integer points that are in the tetrahedron.

Now let us find the Ehrhart polynomial associated to a fundamental tetrahedron, say T. Its volume is $r/6$ and $\mathrm{vol}(\partial P) = 2$, so that

$$j_T(n) = \frac{r}{6}n^3 + n^2 + a_1 n + 1.$$

We also know that $j_T(1) = 4$, which gives

$$a_1 = 2 + \frac{r}{6}.$$

Thus

$$j_T(n) = \frac{r}{6}n^3 + n^2 + (2 - \frac{r}{6})n + 1.$$

Combining with $b_T(n) = 2(n^2+1)$, we find

$$\mathrm{vol}(T) = \frac{r}{6} = \left\{4a_1 + a_2\left(\frac{r}{6}n^3 + n^2 + \left(2 - \frac{r}{6}\right)n + 1\right) + a_3\right\} - \frac{1}{2}\left\{4a_1 + 2a_2(n^2+1) + 2a_3\right\}.$$

In this instance the third relation is

$$\frac{r}{6} = 2a_1 + a_2\left(\frac{r}{6} = n^3 + \left(2 - \frac{r}{6}\right)n\right).$$

From the two first conditions already obtained, we have

$$a_1 = -a_2 n.$$

So that

$$\frac{r}{6} = -2a_2 n + a_2\left(\frac{r}{6}n^3 + \left(2 - \frac{r}{6}\right)n\right),$$

$$1 = a_2(n^3 - n),$$

leading to

$$a_2 = \frac{1}{(n-1)n(n+1)}, \quad a_1 = \frac{-n}{(n-1)n(n+1)}, \quad a_3 = \frac{n-1}{(n-1)n(n+1)},$$

and

$$(n-1)n(n+1)M_3(P) = j_P(n) - nj_P + (n-1)\chi(P).$$

And then the formula

$$(n-1)n(n+1)\mathrm{vol}(P) = \{j_P(n) - nj_P + (n-1)\chi(P)\} - \frac{1}{2}\{b_P(n) - nb_P + (n-1)\chi(\partial P)\}$$

gives the exact volume of the polyhedron P, for any $n \neq 1$.

2.5. **Remark.** All the results above can easily be adapted to an arbitrary coordinate system with an appropriate linear map that takes $\mathbb{Z}^k$ to the new lattice. The volume of a polytope changes only by a factor of the determinant of the linear map.

2.6. **Asymptotics.** An exact formula for the volume of a polytope P in higher dimensions needs the information on several polytopes n_iP, but this, in practice, is hard to compute. Instead, from Ehrhart, we know that

$$\mathrm{vol}(P) = \frac{1}{n^k}\left\{j_P(n) - \frac{\mathrm{vol}(\partial P)}{2}n^{k-1} + O(k-2)\right\}.$$

Similarly,

$$vol(\partial P) = \frac{1}{n^{k-1}}\left\{b_P(n) + O(k-2)\right\},$$

leading to

$$\mathrm{vol}(P) = \frac{1}{n^k}\left\{j_P(n) - \frac{b_P(n)}{2} + O(k-2)\right\}.$$

We thus have the two leading terms in the asymptotic expansion.

3. MAGIC SQUARES

3.1. **Magic row.** A magic row is a row of n integers such that the sum of the entries is some given integer S. First consider a magic row with positive entries and let $L_n(S)$ be the number of magic rows with sum S. Obviously

$$L_1(S) = 1.$$

If the row has length 2, we have $S+1$ choices for the first entry and no choice for the second, so

$$L_2(S) = S + 1.$$

For $L_3(S)$, we have $\boxed{x}\boxed{y}\boxed{S-x-y}$ with the conditions:

$$x, y \geq 0$$

$$x + y \leq S$$

which parametrize the following triangle:

The number of magic rows with sum S is the number of integer points in this triangle:

$$L_3(S) = \sum_{i=0}^{S}(i+1) = \frac{(S+1)(S+2)}{2}.$$

This is Ehrhart's polynomial associated with the triangle, and we remark that it can be written as

$$L_3(S) = \sum_{i=0}^{S} L_2(i).$$

In fact L_k can be viewed as the number of ways of having the first $k-1$ integer of sum lower or equal to S We obtain the following recurrence:

$$\begin{cases} L_1(S) = 1 \\ L_{n+1}(S) = \sum_{i=0}^{S} L_n(i) \end{cases}$$

Since solutions for a row of length k can be viewed as integer points inside a polytope, we can easily obtain the asymptotic behavior of $L_k(S)$ when S grow large. The volume of the tetrahedron is $1/(n-1)!$, and the vol$n/(n-2)!$ (verifier...), Ehrhart's theorem gives:

$$L_n(S) = \frac{S^{n-1}}{(n-1)!} + \frac{nS^{n-2}}{2(n-2)!} + O(S^{k-3}).$$

3.2. **Magic squares.** A magic square is a square of side length n with n^2 integers such that the sum of the entries of each row or column is some given integer S. Consider first magic squares with positive entries and let $Q_n(S)$ denote the number of solutions. Obviously $Q_1(S) = 1$ and $Q_2(S) = S + 1$. Now we will look at $Q_3(S)$; the magic square is of the form

x	y	$S - x - y$
z	t	$S - z - t$
$S - z - x$	$S - y - t$	$x + y + z + t - S$

with the conditions

$$x,\ y,\ z,\ t \geq 0,$$
$$x + y,\ z + t,\ x + z,\ y + t \geq S,$$
$$x + y + z + t \geq S.$$

This gives an integral polytope of dimension 4. If start with the polytope whose sum is 1, increasing S is equivalent to rescaling the polytope by a factor of S; Ehrhart's theorem tell us that there exists a polynomial in S that gives the number of integer points inside the polytope. In fact, this number is $Q_3(S)$. We know that $Q_3(-1) = Q_3(-2) = 0$, because $Q_3(-S)$ represents the number of solutions with nonzero entries. Then

$$Q_3(S) = (S + 1)(S + 2)(aS^2 + bS + c).$$

We also know that $Q_3(0) = 1 = Q_3(-3)$ and $Q_3(1) = 6$. These condition determine uniquely $a,\ b \mathrm{and} c$. Solving gives

$$Q_3(S) = \frac{S^4}{8} + \frac{3S^3}{4} + \frac{3S^2}{2} + \frac{3S}{2} + 1.$$

It is less evident than for magic lines that we can find a recurrence. But here is a lower bound: if we want to have a magic square $n \times n$, we can always begin by constructing a $(n - 1) \times (n - 1)$ magic square of sum at most S and then complete the square in a unique manner. The number of those magic squares is given by the same recurrence as for magic rows, so there are more magic squares than magic rows.

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