A Fictitious Domain Method for Particulate Flows

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QUAECUMQUE VERA

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- Eulerian vs. Lagranian methods
- Fictitious Domain Methods
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Formulation

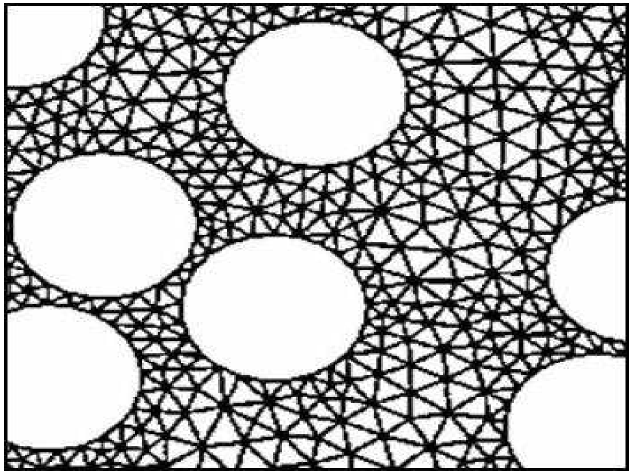
$$\rho_1 \frac{D\hat{\mathbf{u}}_1}{Dt} = \nabla \cdot \hat{\sigma}_1, \quad \nabla \cdot \hat{\mathbf{u}}_1 = 0 \text{ in } \Omega_1$$

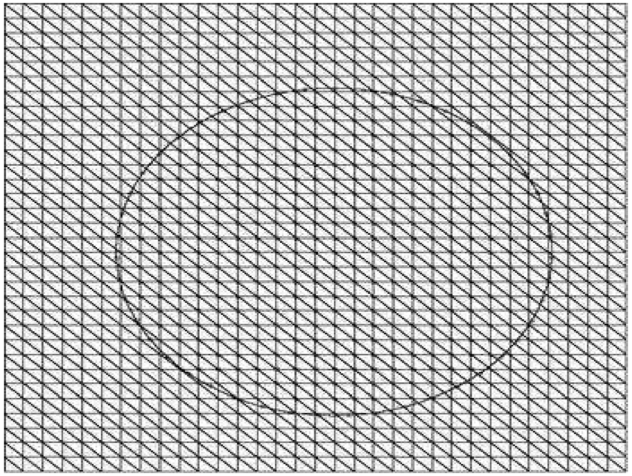
$$\hat{\sigma}_1 = \hat{p}_1 \delta + 2\mu_1 D[\hat{\mathbf{u}}_1]$$

$$M_i \frac{d\mathbf{U}_i}{dt} = M_i \mathbf{g} + \mathbf{F}_i$$

$$\mathbf{I}_i \frac{d\boldsymbol{\omega}_i}{dt} + \boldsymbol{\omega}_i \times \mathbf{I}_i \boldsymbol{\omega}_i = \mathbf{T}_i$$

Fictitious domain formulation





Fictitious domain formulation

$$\int_\Omega \rho_1\left(\frac{d\mathbf{u}}{dt} - \mathbf{g}\right)\cdot \mathbf{v}\,d\mathbf{x} - M\left(\frac{d\mathbf{U}}{dt} - \mathbf{g}\right)\cdot \mathbf{V} - \left(\mathbf{I}\frac{d\varpi}{dt} + \varpi \times \mathbf{I}\varpi\right)\cdot \xi + \int_\Omega \sigma : \mathbf{D}[\mathbf{v}]\,d\mathbf{x} = \langle \lambda, \mathbf{v} - (\mathbf{V} + \xi \times \mathbf{r})\rangle_{P(t)} \quad (1)$$

$$\int_\Omega q \cdot \nabla \mathbf{u}\,d\mathbf{x} = \mathbf{0} \quad (2)$$

$$\langle \eta, \mathbf{u} - (\mathbf{U} + \varpi \times \mathbf{r})\rangle_{P(t)} = 0 \quad (3)$$

Fictitious domain formulation

- $\underline{Step1\ :}$**Advection**
- $\underline{Step2\ :}$**Generalized Stokes**
- $\underline{Step3\ :}$ **Rigid body constraints**

Fictitious domain formulation

$$\mathbf{u}_2(\mathbf{x},t)|_{\Omega_i} = \mathbf{U}_i(t) + \boldsymbol{\omega}_i(t) \times (\mathbf{x} - \mathbf{X}_i(t))$$

$$\frac{d}{dt}\int_{\Omega_{2,i}} \rho_{2,i}\mathbf{u}_2 d\Omega = \int_{\Omega_{2,i}} (\rho_{2,i} - \rho_1)\mathbf{g} d\Omega + \int_{\partial\Omega_{2,i}} \hat{\boldsymbol{\sigma}}_1 \mathbf{n}_i ds$$

$$\int_{\Omega_{2,i}} \frac{D}{Dt}(\rho_{2,i}\mathbf{u}_2)\, d\Omega = \int_{\Omega_{2,i}} (\rho_{2,i} - \rho_1)\mathbf{g} d\Omega + \int_{\Omega_{2,i}} \nabla\cdot\boldsymbol{\sigma}_1 d\Omega$$

$$\int_{\Omega_{2,i}} \frac{D}{Dt}(\rho_2\mathbf{u}_{2,i})\, d\Omega = \int_{\Omega_{2,i}} (\rho_{2,i} - \rho_1)\mathbf{g} d\Omega + \int_{\Omega_{2,i}} \left(-\nabla p_1 + \mu_1 \nabla^2 \mathbf{u}_1\right) d\Omega$$

Fictitious domain formulation

$$\mathbf{F} = \begin{cases} -\rho_1 \frac{D\mathbf{u}_1}{Dt} + \mu_1 \nabla^2 \mathbf{u}_1 - \nabla p_1, & \text{in} \quad \Omega_{2,i}, \quad i = 1, \dots, n \\ 0, \quad \text{in} \quad \Omega_1 \end{cases}$$

$$\rho_1 \frac{D\mathbf{u}_1}{Dt} = -\nabla p_1 + \mu_1 \nabla^2 \mathbf{u}_1 - \mathbf{F}, \quad \nabla \cdot \mathbf{u}_1 = 0 \text{ in } \Omega$$

$$\int_{\Omega_{2,i}} \frac{D}{Dt} \left(\rho_{2,i} \mathbf{u}_2 - \rho_1 \mathbf{u}_1 \right) d\Omega = \int_{\Omega_{2,i}} [(\rho_{2,i} - \rho_1)\mathbf{g} + \mathbf{F}] d\Omega$$

$$\int_{\Omega_{2,i}} \frac{D}{Dt} \left[(\rho_{2,i} - \rho_1) \mathbf{u}_2 \right] d\Omega = \int_{\Omega_{2,i}} [(\rho_{2,i} - \rho_1)\mathbf{g} + \mathbf{F}] d\Omega$$

Fictitious domain formulation

$$\rho_1 \frac{D\mathbf{u}_1}{Dt} = -\nabla p_1 + \mu_1 \nabla^2 \mathbf{u}_1 - \mathbf{F}, \quad \nabla \cdot \mathbf{u}_1 = 0 \text{ in } \Omega$$

$$\Delta M_i \frac{d\mathbf{U}_i}{dt} = \Delta M_i \mathbf{g} + \int_{\Omega_{2,i}} \mathbf{F} d\Omega, \quad i = 1, \ldots, n$$

$$\mathbf{U}_i(t) + \boldsymbol{\omega}_i(t) \times (\mathbf{x} - \mathbf{X}_i(t)) = \mathbf{u}_1, \text{ in } \Omega_{2,i}, \quad i = 1, \ldots, n$$

$$\boldsymbol{\omega}_i(t) V_{\Omega_i} = 0.5 \int_{\Omega_{2,i}} \nabla \times (\mathbf{u}_1 - \mathbf{U}_i(t)) d\Omega, \quad i = 1, \ldots, n$$

Fictitious domain formulation

$$-\alpha\boldsymbol{\lambda} + \mu_1\nabla^2\boldsymbol{\lambda} = \mathbf{F}, \text{ in } \Omega$$
$$\boldsymbol{\lambda} = 0, \text{ on } \Gamma,$$

$$\rho_1\frac{D\mathbf{u}_1}{Dt} = -\nabla p_1 + \mu_1\nabla^2\mathbf{u}_1 + \alpha\boldsymbol{\lambda} - \mu_1\nabla^2\boldsymbol{\lambda}, \quad \nabla\cdot\mathbf{u}_1 = 0 \text{ in } \Omega$$
$$\Delta M_i\frac{d\mathbf{U}_i}{dt} = \Delta M_i\mathbf{g} - \int_{\Omega_{2,i}}\alpha\boldsymbol{\lambda}d\Omega - \mu_1\int_{\partial\Omega_{2,i}}\frac{\partial\boldsymbol{\lambda}}{\partial\mathbf{n}}ds, \quad i = 1,\ldots,n$$

Discretization

- Substep 1 (advection-diffusion)

$$\mathbf{X}_i^{p,n+1} = \mathbf{X}_i^{n-1} + 2\delta t \mathbf{U}_i^n$$

$$\rho_1 \tau_0 \mathbf{u}_1^* - \mu_1 \nabla^2 \mathbf{u}_1^* = -\rho_1 (\tau_1 \tilde{\mathbf{u}}_1^n - \tau_2 \tilde{\mathbf{u}}_1^{n-1}) - \nabla p^n, \text{ in } \Omega$$

$$\mathbf{u}_1^* = 0 \text{ on } \Gamma$$

- Substep 2 (incompressibility)

$$\tau_0 (\mathbf{u}_1^{**} - \mathbf{u}_1^*) = -\nabla (p_1^{n+1} - p_1^n) \text{ in } \Omega$$

$$\nabla \cdot \mathbf{u}_1^{**} = 0 \text{ in } \Omega$$

$$\mathbf{u}_1^{**} \cdot \mathbf{n} = 0 \text{ on } \Gamma,$$

Discretization

- Substep 3 (rigid body constraint)

$$\boldsymbol{\lambda}^{0,n+1} = 0$$

$$\mathbf{u}_1^{0,n+1} = \mathbf{u}_1^{**}$$

$$\tau_0 \mathbf{U}_i^{0,n+1} = -\tau_1 \mathbf{U}_i^{n} - \tau_2 \mathbf{U}_i^{n-1} + \mathbf{g}$$

$$V_{\Omega_i} \boldsymbol{\omega}_i^{0,n+1} = 0.5 \int_{\Omega_i} \nabla \times \mathbf{u}_1^{0,n+1} d\Omega$$

$$\mathbf{u}_2^{0,n+1} = \mathbf{U}_i^{0,n+1} + \boldsymbol{\omega}^{0,n+1} \times (\mathbf{x} - \mathbf{X}_i^{p,n+1})$$

Discretization

$$\begin{cases} (1+\frac{\rho_1}{\rho_{2,i}-\rho_1})(\alpha I-\mu_1\nabla^2)\delta\boldsymbol{\lambda}^{k+1,n+1} & \\ =-(\rho_1\tau_0 I-\mu_1\nabla^2)(\mathbf{u}_1^{k,n+1}-\mathbf{u}_2^{k,n+1}) & \text{in } \Omega_{2,i}, \quad i=1,\ldots,n \\ (1+\frac{\rho_1}{\rho_{2,i}-\rho_1})(\alpha I-\mu_1\nabla^2)\delta\boldsymbol{\lambda}^{k+1,n+1}=0 & \text{in } \Omega_1 \\ \delta\boldsymbol{\lambda}^{k+1,n+1}=0 & \text{on } \Gamma, \end{cases}$$

Discretization

$$\begin{cases} (\rho_1 \tau_0 I - \mu_1 \nabla^2)\delta \mathbf{u}_1^{k+1,n+1} = (\alpha I - \mu_1 \nabla^2)\delta \boldsymbol{\lambda}^{k+1,n+1} & \text{in } \Omega \\ \delta \mathbf{u}_1^{k+1,n+1} = 0 & \text{on } \Gamma \end{cases}$$

$$\begin{cases} \Delta M_i \tau_0 \delta \mathbf{U}_i^{k+1,n+1} = -\frac{\rho_{2,i}-\rho_1}{\rho_{2,i}} \int_{\Omega_{2,i}} \alpha(\mathbf{u}_1^{k,n+1} - \mathbf{u}_2^{k,n+1}) d\Omega \\ +\mu_1 \frac{\rho_{2,i}-\rho_1}{\rho_{2,i}} \int_{\partial\Omega_{2,i}} \frac{\partial(\mathbf{u}_1^{k,n+1} - \mathbf{u}_2^{k,n+1})}{\partial \mathbf{n}} ds \\ V_{\Omega_i} \boldsymbol{\omega}_i^{k+1,n+1} = 0.5 \int_{\Omega_i} \nabla \times \mathbf{u}_1^{k+1,n+1} d\Omega \\ \mathbf{u}_2^{k+1,n+1} = \mathbf{U}_i^{k+1,n+1} + \boldsymbol{\omega}_i^{k+1,n+1} \times (\mathbf{x} - \mathbf{X}_i^{p,n+1}) \text{ in } \Omega_{2,i} \\ \mathbf{u}_2^{k+1,n+1} = 0 \text{ in } \Omega_1 \end{cases}$$

None-Lagrange formulation

$$\mathbf{F} = \begin{cases} \frac{1}{Fr}\mathbf{e}_g + \frac{\rho_1}{\rho_2-\rho_1}\hat{\mathbf{F}}, & \text{in} \quad \Omega_{2,i}, \quad i = 1, \ldots, n \\ 0, \quad \text{in} \quad \Omega_1 \end{cases}$$

$$\frac{D\mathbf{u}_1}{Dt} = -\nabla p_1 + \frac{1}{Re}\nabla^2\mathbf{u}_1 + \frac{\rho_2 - \rho_1}{\rho_1}\left(\mathbf{G} - \mathbf{F}\right), \text{ in } \Omega$$

$$\frac{d\mathbf{U}_i}{dt} = \frac{1}{V_i}\int_{\Omega_{2,i}} \mathbf{F} d\Omega$$

None-Lagrange formulation

$$\tau_0(\mathbf{u}_1^{n+1} - \mathbf{u}_1^{**}) = -\frac{\rho_{2,i} - \rho_1}{\rho_1}\mathbf{F} \text{ in } \Omega,$$

$$\tau_0\left(\mathbf{U}_i^{n+1} - \mathbf{U}_i^{*}\right) = \frac{1}{V}\int\limits_{\Omega_{2,i}} \mathbf{F} d\Omega,$$

$$\mathbf{u}_1^{n+1} - \left(\mathbf{U}_i^{n+1} + \boldsymbol{\omega}_i^{n+1} \times (\mathbf{x} - \mathbf{X}_i^{p,n+1})\right) = 0 \text{ in } \Omega_{2,i}.$$

None-Lagrange formulation

$$-\frac{\rho_{2,i}-\rho_1}{\rho_1}\mathbf{F}-\frac{1}{V}\int\limits_{\Omega_{2,i}}\mathbf{F}d\Omega=\tau_0\left(\mathbf{U}_i^*-\mathbf{u}_1^{**}\right)$$

$$+\frac{\tau_0}{2V}\left(\int\limits_{\Omega_{2,i}}\nabla\times\mathbf{u}_1^{n+1}d\Omega\right)\times\left(\mathbf{x}-\mathbf{X}_i^{p,n+1}\right)\text{ in }\Omega_{2,i}.$$

$$\int\limits_{\Omega_{2,i}}\mathbf{F}d\Omega=\frac{\rho_1}{\rho_{2,i}}\tau_0\int\limits_{\Omega_{2,i}}\left(\mathbf{u}_1^{**}-\mathbf{U}_i^*\right)d\Omega.$$

None-Lagrange formulation

$$\mathbf{u}_1^{n+1} = \mathbf{u}_1^{**} + \Bigg[(\mathbf{U}_i^* - \mathbf{u}_1^{**}) + \frac{1}{2V} \left(\int_{\Omega_{2,i}} \nabla \times \mathbf{u}_1^{n+1} d\Omega \right) \times$$

$$- \frac{1}{V} \frac{\rho_1}{\rho_{2,i}} \int_{\Omega_{2,i}} (\mathbf{u}_1^{**} - \mathbf{U}_i^*) \, d\Omega \Bigg] 1_{\Omega_2} \text{ in } \Omega,$$

$$\mathbf{U}_i^{n+1} = \frac{1}{V} \frac{\rho_1}{\rho_{2,i}} \int_{\Omega_{2,i}} \mathbf{u}_1^{**} d\Omega + \left(1 - \frac{\rho_1}{\rho_{2,i}} \right) \mathbf{U}_i^* .$$

Collision

$$F_i^w = \begin{cases} -6\pi r_i U_\perp \mu_1 \left(\frac{r_i}{\hat{h}} - \frac{r_i}{h} \right), & \text{if} \quad \hat{h} < h \\ 0, & \text{otherwise} \end{cases}$$

$$s_{i,j} = |\mathbf{X}_i - \mathbf{X}_j| - (r_i + r_j)$$

$$\Delta \mathbf{r}_i = \frac{M_j(\epsilon - s_{i,j})}{M_i + M_j}$$

Mesh fitting

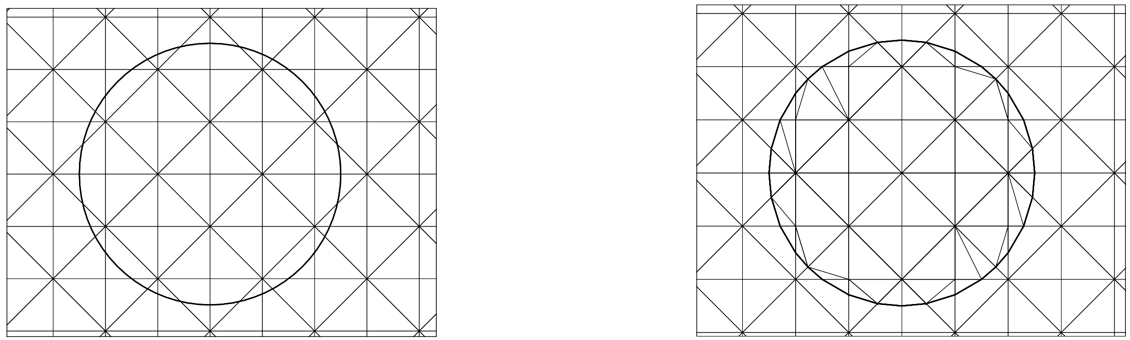
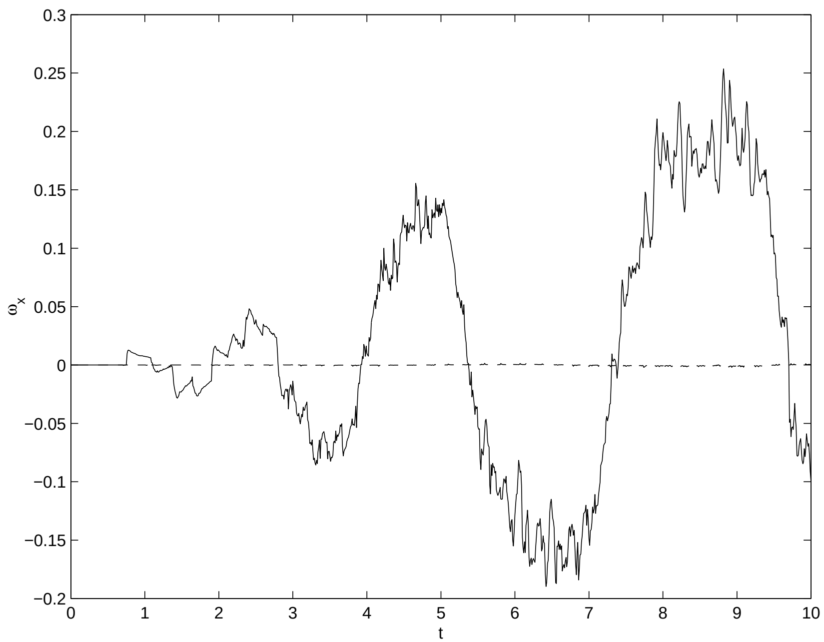
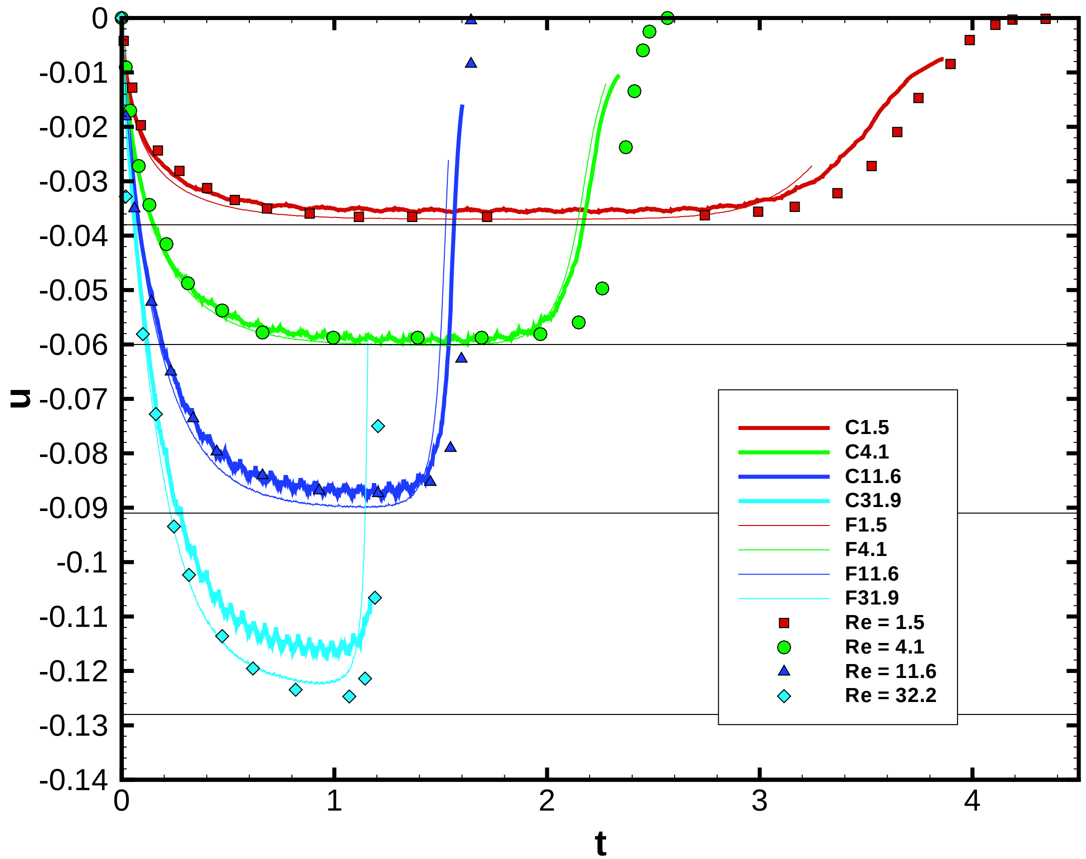


Figure 1: A cross section of the basic tetrahedral grid (left) and the boundary fitted grid (right).

Validation

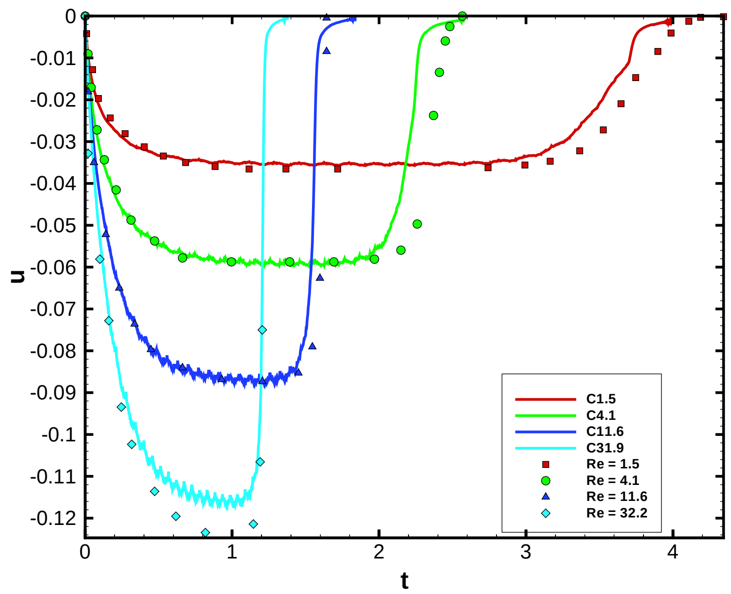


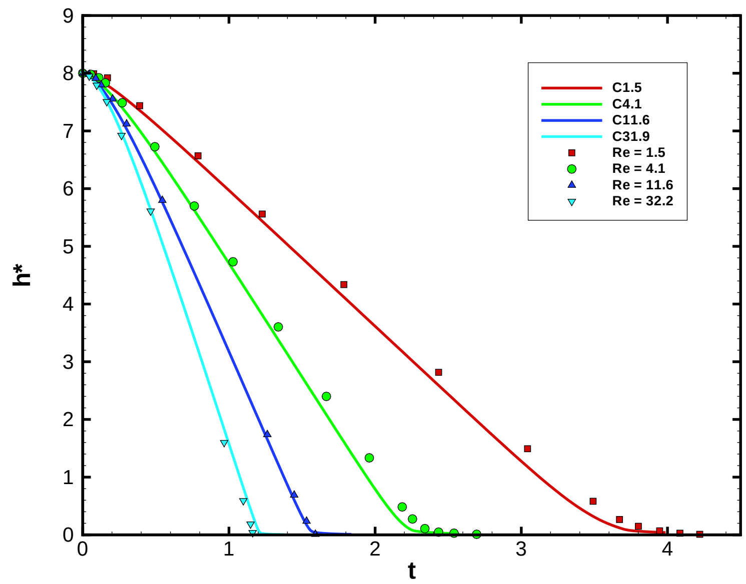
Validation



0
-0.01
-0.02
-0.03
-0.04
-0.05
-0.06
-0.07
-0.08
-0.09
-0.1
-0.11
-0.12
-0.13
-0.14
u
0
1
2
3
4
t
C1.5
C4.1
C11.6
C31.9
F1.5
F4.1
F11.6
F31.9
Re = 1.5
Re = 4.1
Re = 11.6
Re = 32.2

Validation


u
t
C1.5
C4.1
C11.6
C31.9
Re = 1.5
Re = 4.1
Re = 11.6
Re = 32.2


h*
t
C1.5
C4.1
C11.6
C31.9
Re = 1.5
Re = 4.1
Re = 11.6
Re = 32.2

Two spheres

Sedimenting particles

Heavy and neutral particles

Light and neutral particles

Three spheres

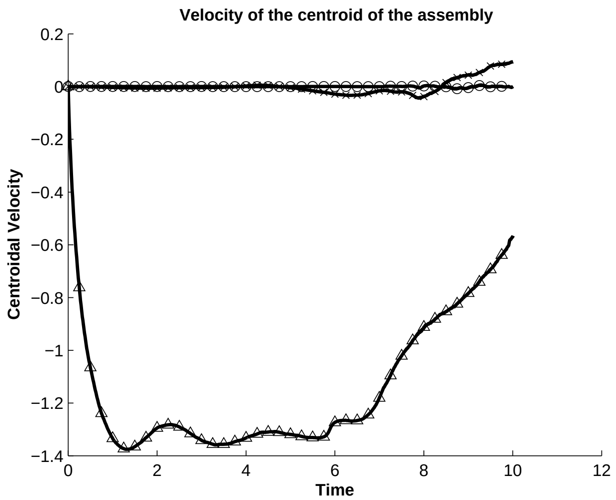
Three spheres interaction

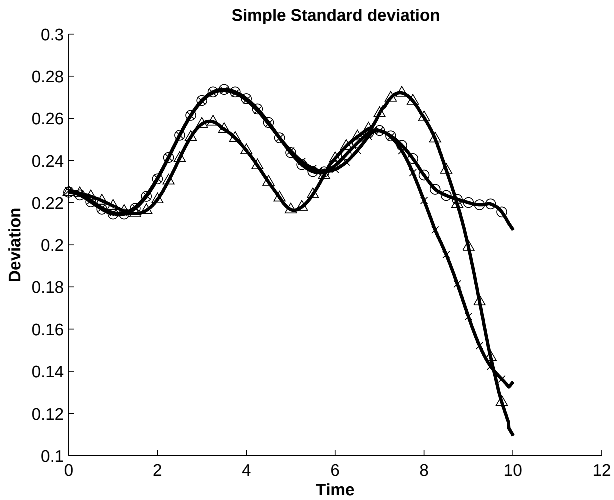
64 spheres

64 particles swirl

64 particles wave

64 spheres

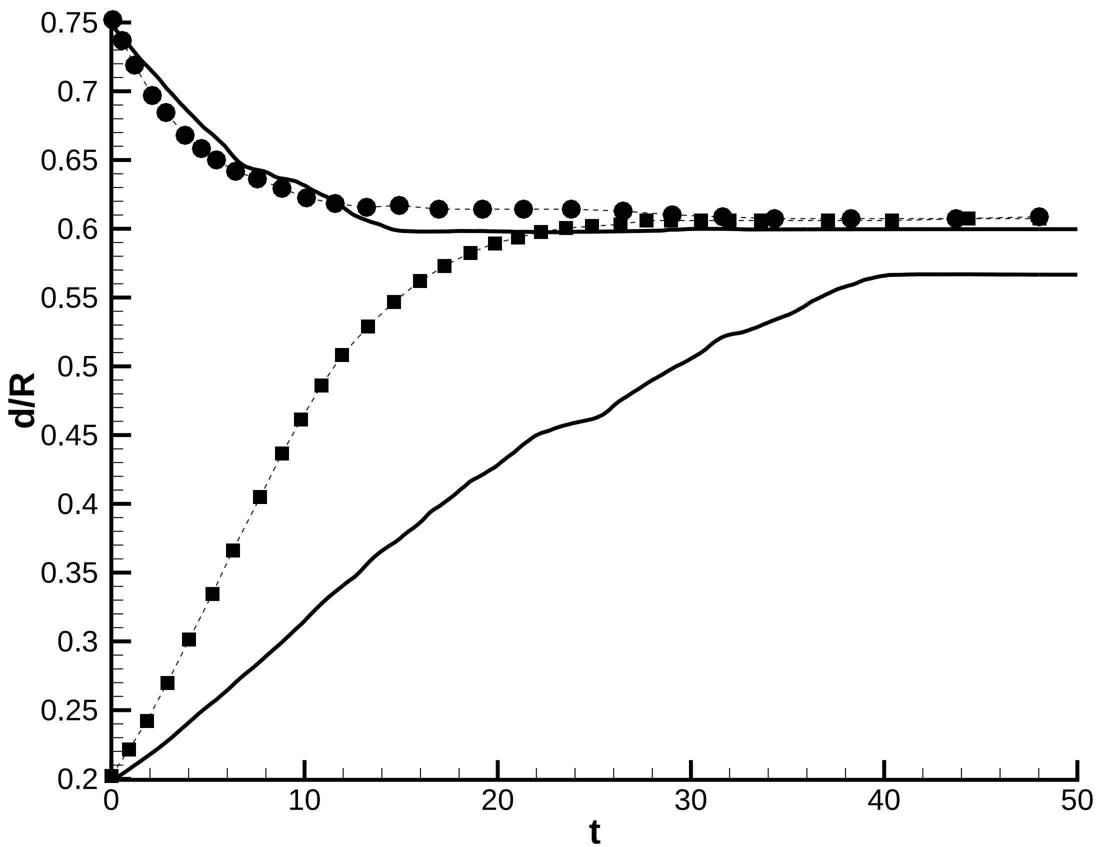

Velocity of the centroid of the assembly
Centroidal Velocity
Time


Simple Standard deviation
Deviation
Time

Poiseuille flow

$U_m = 20.0$	DLM	Present	DLM	Present
d/R	U_y		ω_z	
0.10	19.4965	19.5805	0.7751	0.7812
0.20	18.8841	18.9658	1.5514	1.5734
0.30	17.8656	17.9730	2.3235	2.3537
0.40	16.4442	16.5582	3.0872	3.1523
0.50	14.6210	14.7561	3.8409	3.9060
0.60	12.3957	12.5155	4.5824	4.7483

Poiseuille flow



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- C. Veeramani, P. Minev, and K. Nandakumar, A fictitious domain method for particle sedimentation, to appear in **Lecture Notes in Computer Science**, Springer (2005).

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