4.23 Multiplication Table for Split G_2

(4.24) $[H_i,H_j] = 0,\ [H_i,E_\nu] = \nu(H_i)E_\nu,\ [H_i,F_\nu] = -\nu(H_i)F_\nu$

(4.25) $[E_{\lambda_i},F_{\lambda_i}] = H_i, [E_{\lambda_i},E_{\lambda_{i+1}}] = 2F_{\lambda_{i+2}}, [F_{\lambda_i},F_{\lambda_{i+1}}] = -2E_{\lambda_{i+2}}$

(4.26) $[E_{\mu_i},F_{\mu_i}] = 3(H_i - H_{i+1}),\ [E_{\mu_i},E_{\mu_{i+1}}] = 3F_{\mu_{i+2}},$

$[F_{\mu_i},F_{\mu_{i+1}}] = -3E_{\mu_{i+2}}$

(4.27) $[E_{\mu_i},F_{\mu_j}] = [E_{\mu_i},F_{\lambda_j}] = [E_{\lambda_i},F_{\mu_j}] = 0 \quad (i\neq j)$

(4.28) $[E_{\mu_i},F_{\lambda_i}] = -3F_{\lambda_{i+1}},\ [E_{\lambda_i},F_{\mu_i}] = -3E_{\lambda_{i+1}}$

(4.29) $[E_{\lambda_i},F_{\lambda_{i+1}}] = E_{\mu_i},\ [E_{\lambda_{i+1}},F_{\lambda_i}] = F_{\mu_i}$

(4.30) $[E_{\mu_i},E_{\lambda_j}] = [F_{\mu_i},F_{\lambda_j}] = 0 \qquad (j\neq i+1)$

(4.31) $[E_{\mu_i},E_{\lambda_{i+1}}] = 3E_{\lambda_i},\ [F_{\mu_i},F_{\lambda_{i+1}}] = -3F_{\lambda_i}.$

A Lie algebra L with root decomposition (4.20)-(4.22) and multiplication rules (4.24)-(4.31) is said to be **split of type** G_2.

We now give a plebeian proof, based directly on the multiplication table, that a split G_2 is simple.

4.32 (Simplicity of Split G_2) A split Lie algebra of type G_2 over a field of characteristic $\neq 3$ is simple of dimension 14.

Proof. Suppose M is a nonzero Lie ideal in the (split) L of type G_2. We will show M = L. First consider the case $M\cap H \neq 0$. If M contains nonzero $H = \alpha_1 H_1 + \alpha_2 H_2$ (by (4.21) H_3 is redundant)

To compute the multiplication table for the standard derivations Stander ($\mathbb{C}$) (which equals Inder($\mathbb{C}$) = Der($\mathbb{C}$) in characteristic $\neq 3$) we introduce the abbreviations

$$E_{\lambda_i} = D(e_1, f_i) \qquad F_{\lambda_i} = D(e_2, g_i)$$

(4.19)

$$E_{\mu_i} = D(f_i, g_{i+1}) \qquad F_{\mu_i} = D(f_{i+1}, g_i)$$

$$H_i = D(f_i, g_i)$$

Thus

(4.20) $$\text{Stander}(\mathbb{C}) = H + \Sigma_{i=1}^{3}\{\Phi E_{\lambda_i} + \Phi F_{\lambda_i} + \Phi E_{\mu_i} + \Phi F_{\mu_i}\}$$

(4.21) $$H = \Phi H_1 + \Phi H_2 + \Phi H_3, \quad H_1 + H_2 + H_3 = 0$$

(the last equality results from $\Sigma_{i=1}^{3} D(f_i, f_{i+1}f_{i+2}) = 0$ by (4.14).

We define linear functionals $\nu = \lambda_i, \mu_i$ on H by

$$\lambda_i(H_i) = 2, \qquad \lambda_i(H_j) = -1$$

$$\mu_i(H_i) = 3, \qquad \mu_i(H_{i+1}) = -3, \qquad \mu_i(H_{i+2}) = 0$$

(4.22) $$\lambda_1 + \lambda_2 + \lambda_3 = \mu_1 + \mu_2 + \mu_3 = 0$$

$$\mu_1 = 2\lambda_1 + \lambda_3 = \lambda_1 - \lambda_2, \qquad -3\lambda_1 = \mu_2 + 2\mu_3$$

$$\mu_2 = 2\lambda_2 + \lambda_1, \qquad -3\lambda_2 = \mu_3 + 2\lambda_1$$

$$\mu_3 = 2\lambda_3 + \lambda_2 = -2\lambda_1 - \lambda_2, \qquad -3\lambda_3 = \mu_1 + 2\lambda_2$$

(the relations among the λ and μ follows immediately from their action on the H_i).

We can use our previous tables to compute the multiplication table for the $D_{x,y}$ (using the relation $[D, D_{x,y}] = D_{D(x),y} + D_{x,D(y)}$).

we can assume by symmetry in the indices that $\alpha_1 \neq 0$. Then by (4.24) and (4.22) $[H,E_{\mu_3}] = \{\alpha_1\mu_3(H_1) + \alpha_2\mu_3(H_2)\}E_{\mu_3} = -3\alpha_1 E_{\mu_3}$, so in characteristic $\neq 3$ M contains E_{μ_3}; from $[H,F_{\mu_3}] = +3\alpha_1 F_{\mu_3}$ it also contains F_{μ_3}. From E_{μ_3}, F_{μ_3} we see by (4.26) M contains all $F_{\mu_2}, F_{\mu_1}, E_{\mu_2}, E_{\mu_1}$; once M contains all 6 E_{μ_i}, F_{μ_i} it contains all 6 $E_{\lambda_i}, F_{\lambda_i}$ by (4.31) (characteristic $\neq$ 3 again), hence by (4.25) all H_i, so M contains everything.

From now on assume $M \cap H = 0$. Suppose $M = H + \Sigma\alpha_\nu E_\nu + \Sigma\beta_\nu F_\nu \in M$ has minimum number of nonzero α_ν, β_ν (by assumption all nonzero $M \in M$ have some α_{ν_o} or β_{ν_o} nonzero). Choosing H_o so $\nu_o(H_o) \neq 0$ we see M contains $M' = [H_o, M] = \Sigma\alpha_\nu\ \nu(H_o)E_\nu - \Sigma\beta_\nu\ \nu(H_o)F_\nu$ by (4.24) with $\alpha_{\nu_o}\nu_o(H_o) \neq 0$ or $\beta_{\nu_o}\nu_o(H_o) \neq 0$ but contains no term from H and no new E_ν, F_ν. Replacing M by M' if necessary, we may assume $H = 0$, $M = \Sigma\alpha_\nu E_\nu + \Sigma\beta_\nu F_\nu$. If $\alpha_{\nu_o} \neq 0$ then $\nu_o(H_i)M - [H_i, M] \in M$ has fewer coefficients since no new E_ν, F_ν, H_j appear and the coefficient of E_{ν_o} is now $\alpha'_{\nu_o} = \nu_o(H_i)\alpha_{\nu_o} - \nu_o(H_i)\alpha_{\nu_o} = 0$; by minimality it must have all $\alpha'_\nu, \beta'_\nu = 0$ and therefore be the zero element (recall $M \cap H = 0$): $\nu_o(H_i)M = [H_i, M]$. Equating coefficients of E_ν, F_ν gives, taking into account (4.24),

$$(4.33) \qquad \nu_o(H_i)\alpha_\nu = \nu(H_i)\alpha_\nu, \quad \nu_o(H_i)B_\nu = -\nu(H_i)B_\nu.$$

If $\beta_{\nu_o} \neq 0$ we would use $\nu_o(H_i)M + [H_i, M] \in M$ to get $\nu_o(H_i)M = -[H_i, M]$ and hence the above relation (4.33) with the minus sign on the α_ν rather than the β_ν.

First consider the case when ν_o is a λ; by symmetry in the indices we may assume $\nu_o = \lambda_1$, $\alpha_{\lambda_1} \neq 0$ or $\beta_{\lambda_1} \neq 0$. Now by (4.22) $\lambda_1(H_1) = 2 \neq \mp 1 = \pm \lambda_j(H_1)$ ($j\neq 1$; we need characteristic $\neq 3$ so $2 \neq -1$), so from (4.33) with $\nu_o = \lambda_1$, $\nu = \lambda_j$, $i = 1$ we get $\alpha_{\lambda_2} = \beta_{\lambda_2} = \alpha_{\lambda_3} = \beta_{\lambda_3} = 0$. Similarly $\lambda_1(H_3) = -1 \neq 0 = \pm\mu_1(H_3)$, $\lambda_1(H_2) = -1 \neq 0 = \pm\mu_3(H_2)$ so with $\nu_o = \lambda_1$, $\nu = \mu_j$, $i = 3$ we get $\alpha_{\mu_1} = \beta_{\mu_1} = \alpha_{\mu_3} = \beta_{\mu_3} = 0$, and

$$M = \alpha_{\lambda_1} E_{\lambda_1} + \alpha_{\mu_2} E_{\mu_2} + \beta_{\lambda_1} F_{\lambda_1} + \beta_{\mu_2} F_{\mu_2}.$$

(We can't get rid of E_{μ_2}, F_{μ_2} so easily since in characteristic two $\lambda_i = \mu_{i+1}$ as linear functionals on H). Then M also contains $[M, F_{\lambda_1}] = \alpha_{\lambda_1} H_1$ and $[M, E_{\lambda_1}] = -\beta_{\lambda_1} H_1$ by (4.25) (note $[E_{\mu_2}, F_{\lambda_1}] = [F_{\mu_2}, E_{\lambda_1}] = 0$ by (4.27) and $[F_{\mu_2}, F_{\lambda_1}] = [E_{\mu_2}, E_{\lambda_1}] = 0$ by (4.30)). Since one of $\alpha_{\lambda_1}, \beta_{\lambda_1} \neq 0$ this contradicts $M \cap H = 0$.

Now consider the case when ν_o is a μ: suppose all α_{λ_i} and β_{λ_i} are zero but (say) α_{μ_1} or β_{μ_1} is nonzero. Here $\mu_1(H_3) = 0 \neq \pm 3 = \pm\mu_j(H_3)$ (characteristic $\neq 3$) so from (4.33) with $\nu_o = \mu_1$, $\nu = \mu_j$, $i = 3$ we get $\alpha_{\mu_2} = \beta_{\mu_2} = \alpha_{\mu_3} = \beta_{\mu_3} = 0$ and

$$M = \alpha_{\mu_1} E_{\mu_1} + \beta_{\mu_1} F_{\mu_1}.$$

Then by (4.26) M contains $[M, F_{\mu_1}] = 3\alpha_{\mu_1}(H_1 - H_3)$ and $[M, E_{\mu_1}] = -3\beta_{\mu_1}(H_1 - H_3)$, again contrary to $M \cap H = 0$. ∎

A Lie algebra L is of **type G_2** if it becomes a split G_2 under a suitable extension of the base field, ie, some L_Ω is a split G_2. (We say L is a **form** of the split G_2). If L_Ω is simple of dimension 14 over Ω then L had to be simple of dimension 14 over Φ to begin with, so

4.34 (Simplicity of G_2) Any Lie algebra of type G_2 over a field of characteristic $\neq 3$ is simple of dimension 14. ■

In particular, since (1) Der($\mathbb{C}$) = Stander($\mathbb{C}$) in characteristic $\neq 3$ by the Inner Derivation Theorem 4.5, and (2) Stander($\mathbb{C}$) is a form of Stander$(\mathbb{C})_\Omega$ = Stander$(\mathbb{C}_\Omega)$ which is split G_2 since a suitable $\mathbb{C}_\Omega$ is split Cyaley by I.2.4, we have

4.35 (Simplicity of Der($\mathbb{C}$)) If $\mathbb{C}$ is a Cayley algebra over a field of characteristic $\neq 3$ its derivation algebra Der($\mathbb{C}$) is simple of type G_2. ■

4.36 Remark. In characteristic 3 $E_{\mu_i} = F_{\mu_i} = 0$, $H_i = -I$ so Stander ($\mathbb{C}$) = Ad($[\mathbb{C},\mathbb{C}]$) = Ad($\mathbb{C}$) has dimension 7. Stander ($\mathbb{C}$) is always an ideal in Der($\mathbb{C}$), and Der($\mathbb{C}$) = Inder($\mathbb{C}$) = Stander($\mathbb{C}$) + Assocder($\mathbb{C}$) still has dimension 14, so Der($\mathbb{C}$) is definitely not simple in characteristic 3. ■

Exercise

5.1 Deduce the Inner Derivation Theorem in characteristic $\neq 2$ from the Campbell-Casimir operator (see I.7 Problem Set).

5.2 If $D: Q \to M$ is a derivation of a split quaternion algebra into a unital bimodule which kills e_{11} and e_{22}, show $m_{11} - m_{22} = e_{12}D(e_{21}D(e_{12}))$ lies in the center of M. (More generally, show $e_{12} \circ m$ commutes with e_{12} for any $m \in M$, and $e_{21} \circ m$ commutes with e_{21}).