§1. Automorphisms

The concepts of automorphism and inner automorphism apply to many classes of linear algebras: inner automorphisms are those which can be built up from multiplication operators. Automorphisms are harder to construct in alternative algebras than they are in associative algebras. We have to take products of conjugations and associations in such a way that the indicator of the product is nuclear. The most important examples are the conjugations T_z and $T_{x,y}$ determined by nuclear elements z and invertible elements x and y; they give rise to the standard inner automorphisms.

In this chapter ALL ALGEBRAS WILL BE UNITAL, because we want to deal with invertible elements. Recall that an **automorphism** of A is an isomorphism of A with itself, i.e. a bijective linear transformation T on A which preserves multiplication

(1.1) $$T(xy) = T(x)T(y).$$

T necessarily preserves anything built up out of multilpications e.g. $T[x,y] = [Tx,Ty]$ and $T[x,y,z] = [Tx,Ty,Tz]$. In particular, T maps the nucleus N(A) into itself: $[TN,A,A] = [TN,TA,TA] = T[N,A,A] = T(0) = 0$. Since any isomorphism sends the unit onto the unit we have $T(1) = 1$, and T sends inverses to inverses, $T(x^{-1}) = T(x)^{-1}$.

The automorphisms form a subgroup, the **automorphism group**

$$\mathrm{Aut}\,(A),$$

of the general linear group $GL(A)$ of all invertible linear transformations on A, since if T_1, T_2, T are automorphisms so are T_1T_2 and T^{-1}: $T_1T_2(xy) = T_1\{T_2(x)T_2(y)\} = T_1T_2(x)T_1T_2(y)$ and $\{T^{-1}(xy)\} = T^{-1}\{(TT^{-1}x)(TT^{-1}y)\} = T^{-1}\{T\{(T^{-1}x)(T^{-1}y)\}\}$ $= (T^{-1}x)(T^{-1}y)$.

The **multiplication group**

$$GM\,(A)$$

is the subgroup of $GL(A)$ generated by all invertible multilpications L_x, R_y (x,y invertible elements). Those automorphisms which can be built out of multiplications are called **inner automorphisms**, and form a subgroup

$$\mathrm{Inaut}(A) = GM(A) \cap \mathrm{Aut}(A).$$

The automorphism condition (1.1) can be written

$$L_{Tx} = TL_xT^{-1}$$
$$R_{Ty} = TR_yT^{-1}$$

which shows the generators and hence all of $GM(A)$ is invariant under conjugation by automorphisms, so

$$\mathrm{Inaut}(A) \triangleleft \mathrm{Aut}(A).$$

Criteria for Inner Automorphisms

Thinking of the associative case, a reasonable candidate for an inner automorphism in an alternative algebra would be the **conjugation**

(1.2) $\quad T_z : x \to z\,x\,z^{-1} \qquad (T_z = L_zR_z^{-1}).$

A calculation shows this is an automorphism iff z^3 lies in the nucleus (see 1.12). Another candidate would be the **association**

(1.3) $L_{z,w}: x \to (zw)^{-1}\{z(wx)\}$ $\qquad (L_{z,w} = L_{zw}^{-1}L_zL_w)$,

but again this is an automorphism iff $zwz^{-1}w^{-1}$ lies in the nucleus (see 1.13 and Exercise 1.1).

1.4 Example In a Cayley algebra $A = \mathbb{C}(B,\mu)$ we have the following formulas for the action of conjugation and association maps (x,y invertible in B):

$$T_x(a + b\ell) = xax^{-1} + \{\overline{b\bar{x}\bar{x}^{-1}}\}\ell$$

$$T_{x\ell}(a + b\ell) = \bar{a} + \{\bar{x}\ b\ x^{-1}\}\ell$$

$$L_{x,y}(a + b\ell) = a + \{b[[yx]]\}\ell$$

$$L_{\bar{x}\ell,y}(a + b\ell) = yay^{-1} + \{x^{-1}y^{-1}x\ b\ y\}\ell$$

$$L_{x\ell,\bar{y}\ell}(a + b\ell) = (yx)^{-1}a(yx) + \{(xy)b(yx)^{-1}\}\ell$$

and therefore

$$L_{\ell,y}(a + b\ell) = yay^{-1} + \{y^{-1}by\}\ell$$

$$L_{x\ell,\bar{y}\ell}L_{\ell,yx}(a + b\ell) = a + \{[[xy]]b\}\ell$$

$$L_{\bar{x}\ell,y}L_{\ell,y^{-1}}(a + b\ell) = a + \{[[x^{-1}y^{-1}]]b\}\ell$$

$$L_{\ell,y}L_{\ell,z}(a + b\ell) = (yz)a(yz)^{-1} + \{(zy)^{-1}b(zy)\}\ell. \blacksquare$$

Having failed to find a general construction of inner automorphisms, and seeing that the expressions involved keep getting more complicated, we will stop trying to guess what form inner automorphisms "ought" to take, and instead make a general investigation of elements of the multiplication group and formulate criteria for when such an element is an automorphism. By examining the criteria we will be able to choose the parameters appropriately in order to guarantee we get inner automorphisms.

It is convenient to "straighten out" the elements of the multiplication group. Our straightening tools are the formulas

(1.5) $$L_z L_w = L_{zw} L_{z,w}$$

(1.6) $$R_z R_w = R_{wz} L_{z^{-1},w^{-1}}$$

(1.7) $$L_z R_w = R_w L_z L_{w,w^{-1}z}$$

(1.8) $$L_{z,w} L_x = L_{L(z,w)x} L_{(zw)^{-1},z(wx)} L_{w,x}$$

for invertible z,w,x. (1.5) is merely the definition (1.4) of the operator $L_{z,w}$. (1.6) follows from $R_{wz} L^{-1}_{z^{-1},w^{-1}} L_{z^{-1}} L_{w^{-1}} = R_{wz} L_{wz} L_z^{-1} L_w^{-1} = U_{wz} L_z^{-1} L_w^{-1} = \{R_z U_w L_z\} L_z^{-1} L_w^{-1}$ (right fundamental) $= R_z U_w L_w^{-1} = R_z R_w$. (1.7) follows from middle Moufang I. 33 op, $L_z R_w = U_w L_{w^{-1}z} = R_w L_w L_{w^{-1}z} = R_w L_z L_{w,w^{-1}z}$ by (1.5). For (1.8), $L^{-1}_{zw} L_z L_w L_x = \{L_{(zw)^{-1}\cdot z(wx)} L^{-1}_{(zw)^{-1}\cdot z(wx)}\} L^{-1}_{zw} \{L_{z(wx)} L^{-1}_{z(wx)}\} \cdot L_z \{L_{wx} L^{-1}_{wx}\} L_w L_x = L_{(zw)^{-1}\cdot z(wx)} \{L^{-1}_{(zw)^{-1}\cdot z(wx)} L_{(zw)^{-1}} L_{z(wx)}\} \cdot \{L^{-1}_{z(wx)} L_z L_{wx}\}\{L^{-1}_{wx} L_w L_x\}$.

1.9 (Normal Form Theorem) Any element of the multiplication group of an alternative algebra can be represented in the normal form

$$T = R_y L_x L_{z_1,w_1} \cdots L_{z_n,w_n}$$

for invertible elements $x, y, z_1, w_1, \ldots, z_n, w_n$.

Proof. The set of multiplications in normal form contains generators L_x, R_y for x,y invertible. If it is invariant under left multiplications by the generators, it will contain all multiplications.

If we multiply a normal T by an L_z, the result is normal since

$$L_z R_y L_x = R_y L_z L_{z,z^{-1}y} L_x$$

by (1.7), and similarly if we multiply it by an R_z because

$$R_zR_yL_x = R_{yz}L_{z^{-1},y^{-1}}L_x$$

by (1.6), where we can move an L_x to the left of a $L_{z,w}$ by (1.8) and collapse two L's by (1.5). ■

Having found a way to represent the elements of the multiplication group, we now want a criterion for when an element is an automorphism. We define the **indicator** of a multiplication operator $T = R_yL_xL_{z_1,w_1}\cdots L_{z_n,w_n}$ in normal form to be zx^3 for $z = R^{-1}_{z_1w_1}R_{z_1}R_{w_1}\cdots R^{-1}_{z_nw_n}R_{z_n}R_{w_n}(1)$

1.10 (Inner Automorphism Criterion) If $x_1,\ldots,x_n,y$ are invertible elements of an alternative algebra, the operator $T = R_yL_{x_1}\cdots L_{x_n}$ is an automorphism iff

$$(\text{A.I}_o)\quad y = L(1)^{-1}$$

$$(\text{A.II}_o)\quad L(1)R(1)L(1) \text{ lies in the nucleus}$$

for $L = L_{x_1}\cdots L_{x_n}$, $R = R_{x_1}\cdots R_{x_n}$.

An operator $T = R_yL_xL_{z_1,w_1}\cdots L_{z_n,w_n}$ in normal form is an automorphism iff

$$(\text{A.I})\quad y = x^{-1}$$

$$(\text{A.II})\quad \text{the indicator } zx^3 \text{ lies in the nucleus}$$

for $z = R^{-1}_{z_1w_1}R_{z_1}R_{w_1}\cdots R^{-1}_{z_nw_n}R_{z_n}R_{w_n}(1)$

Thus all inner automorphisms are built up from conjugations and associations

$$T = T_xL_{z_1,w_1}\cdots L_{z_n,w_n}.$$

such that the product has nuclear indicator.

Proof. Consider any $T = R_y L_{x_1} \ldots L_{x_n}$. Using left and right Moufang repeatedly in the forms

$$L_x(ab) = U_x(a) L_{x^{-1}}(b)$$

$$R_x(ab) = R_{x^{-1}}(a) U_x(b)$$

(from $xax \cdot x^{-1}b = x\{a(x \cdot x^{-1}b)\} = x\{ab\}$ and $ax^{-1} \cdot xbx = \{(ax^{-1} \cdot x)b\}x = \{ab\}x$), we see $T(ab) = R_{y^{-1}} U_{x_1} \ldots U_{x_n}(a) \cdot U_y L_{x_1^{-1}} \ldots L_{x_n^{-1}}(b) = R_y^{-1} U(a) \cdot U_y \tilde{L}(b)$ for $U = U_{x_1} \ldots U_{x_n}$, $\tilde{L} = L_{x_1^{-1}} \ldots L_{x_n^{-1}}$. Then $b = 1$ and $a = 1$ yield $T(a) = R_y^{-1} U(a) \cdot u$ and $T(b) = v\, U_y \tilde{L}(b)$,

(1.11) $\quad R_u^{-1} T = R_y^{-1} U, \; L_v^{-1} T = U_y \tilde{L} \quad (u = U_y \tilde{L}(1),\; v = R_y^{-1} U(1)).$

On the other hand, T is an automorphism iff $T(ab) = T(a)T(b)$. These two expressions for $T(ab)$ agree iff $T(a)T(b) = R_y^{-1} U(a) \cdot U_y \tilde{L}(b) = R_u^{-1} T(a) \cdot L_v^{-1} T(b)$ (by (1.11)), i.e. $xy = (xu^{-1})(v^{-1}y)$ for $x = T(a)$, $y = T(b)$. Since T is surjective, equality holds for all a,b iff it holds for all x,y. With $x = y = 1$ we see $1 = u^{-1}v^{-1}$, so $u^{-1} = v$. Thus T is an automorphism iff $v = u^{-1}$ and $L_x = L_{xv} L_v^{-1}$. But $L_x L_v = L_{xv}$ for all x is precisely the condition that v be nuclear: $[x,v,A] = 0$. Therefore T is an automorphism iff $v = u^{-1}$ is nuclear. Now $v = U(1)y^{-1} = \{L(1)R(1)\}y^{-1}$ for $L = L_{x_1} \ldots L_{x_n}$, $R = R_{x_1} \ldots R_{x_n}$ by repeated use of middle Moufang, and $u = y[x_1^{-1}(\ldots x_n^{-1})]y$, hence $u^{-1} = y^{-1}[(x_n \ldots)x_1]y^{-1} = \{y^{-1}R(1)\}y^{-1}$, so the condition $v = u^{-1}$ reduces to

$$y^{-1} = L(1)$$

and the condition $v = \{L(1)R(1)\}y^{-1} \in N(A)$ reduces to

$$L(1)R(1)L(1) \in N(A).$$

This gives us the criteria $A.I_0$ and $A.II_0$.

All this has been for arbitrary $T = R_y L_{x_1} \ldots L_{x_n}$. But for normal T the $x_1, \ldots, x_n$ have a special form where $L(1) = L_x L_{z_1 w_1}^{-1} L_{z_1} L_{w_1} \ldots L_{z_n w_n}^{-1} L_{z_n} L_{w_n}(1) = x$ since $L_{zw}^{-1} L_z L_w(1) = L_{zw}^{-1}(zw) = 1$, and $R(1) = R_x R_{z_1 w_1}^{-1} R_{z_1} R_{w_1} \ldots\ldots R_{z_n w_n}^{-1} R_{z_n} R_{w_n}(1) = R_x z = zx$. Thus $y = L(1)^{-1}$ becomes $y = x^{-1}$, and $L(1)R(1)L(1) \in N(A)$ becomes $x(zx)x \in N(A)$. Since $xzx^2 \in N(A) \Leftrightarrow x^{-1}[xzx^2]x = zx^3 \in N(A)$, (by II. 1.7) the conditions reduce to

$$y = x^{-1} \text{ and } zx^3 \in N(A)$$

as in A.I and A.II. ∎

1.12 Example. (Brandt's Conjugation Criterion) The conjugation $T_x = L_x R_x^{-1}$ is an automorphism iff x^3 lies in the nucleus.

Proof. T_x has normal form $R_y L_x$ with $y = x^{-1}$ and $z = 1$ (no z_i or w_i), so A.I is met and A.II holds iff the indicator $zx^3 = x^3$ is nuclear. ∎

1.13 Example. (Association Criterion) The association $L_{z,w} = L_{zw}^{-1} L_z L_w$ is an automorphism iff $[[wz]] = wzw^{-1}z^{-1}$ lies in the nucleus.

Proof. $L_{z,w}$ is in normal form with $x = y = 1$ and $\tilde{z} = R_{zw}^{-1} R_z R_w(1) = wz(zw)^{-1} = wz\,w^{-1}z^{-1}$ (using Artin's Theorem with Inverses II. 3.13), so the indicator is $\tilde{z}x^3 = \tilde{z}$. Thus A.I is trivially satisfied and A.II reduces to $\tilde{z} \in N(A)$. ∎

1.14 Example. (Strict inner automorphisms) An inner automorphism $T = R_y L_x L_{z_1, w_1} \ldots L_{z_n, w_n}$ extends naturally to any unital enveloping algebra $\tilde{A} \supset A$ having the same unit as A (since the x, y, z_i, w_i will remain invertible in $\tilde{A}$). However, the extended map $\tilde{T}$ will remain an automorphism iff the indicator zx^3 nucleizes $\tilde{A}$ (not merely A). We say T is a **strictly inner automorphism** if its indicator zx^3 is strictly nuclear; in this case zx^3 remains nuclear in all larger algebras $\tilde A$, so all extensions $\tilde T$ remain inner automorphisms. ∎

Standard Inner Automorphisms

So far our examples have constructed automorphisms only from very special elements or elements related in a very special way. What we need is a way of constructing automorphisms from arbitrary invertible elements.

Referring to the Criterion, we see that in order to make $T = R_{\tilde y} L_{\tilde x} L_{z_1,w_1} \ldots L_{z_n,w_n}$ an automorphism we must choose $\tilde x$ so that the indicator $\tilde z \tilde x^3$ is nuclear and then choose $\tilde y = \tilde x^{-1}$. The only general method of guaranteeing $\tilde z \tilde x^3$ is nuclear is to make it 1, so $\tilde z \tilde x^3 = 1$ and $\tilde z = \tilde x^{-3}$ is a cube. In general $\tilde z = R^{-1}_{z_1 w_1} R_{z_1} R_{w_1} \ldots\ldots R^{-1}_{z_n w_n} R_{z_n} R_{w_n} (1) = \tilde R(1)$ is not a cube, but we can make it one by making R a cube. (Here we use $[[x,y]] = xyx^{-1}y^{-1}$ with double brackets to denote the <u>group</u> commutator, in distinction to the <u>algebra</u> commutator $[x,y] = xy-yx$. Parentheses are unnecessary by Artin's Theorem with Inverses).

1.15 (Conjugation Construction) For any two invertible elements x and y in an alternative algebra the operator

$$T_{x,y} = T_{[[x,y]]} L^3_{x,y} = R^{-1}_{[[x,y]]} L_{[[x,y]]} \{L^{-1}_{xy} L_x L_y\}^3$$

is a strict inner automorphism with indicator 1. On any associative subalgebra containing x and y the action of $T_{x,y}$ is given by conjugation by $[[xy]]$:

$$T_{x,y}(z) = [[xy]]z[[xy]]^{-1}.$$

Proof. If we apply the Criterion to $T_{x,y} = R_{\tilde{y}} L_{\tilde{x}} L^{3}_{x,y}$ for $\tilde{y} = [[x,y]]^{-1}$, $\tilde{x} = [[x,y]]$ we see right away that A.I holds, and since $\tilde{R} = (R^{-1}_{xy} R_x R_y)^3$ is a cube we have $\tilde{z} = \tilde{R}(1)$ $= \{yx(xy)^{-1}\}^3 = \{xyx^{-1}y^{-1}\}^{-3} = [[xy]]^{-3} = \tilde{x}^{-3}$ a cube (using Artin with Inverses), so the indicator $\tilde{z}\,\tilde{x}^3 = 1$ is strictly nuclear. Therefore $T_{x,y}$ is a strictly inner automorphism.

On an associative subalgebra

$$\begin{aligned} T_{x,y}(z) &= [[xy]]\ \{(xy)^{-1}xy\}^3\ z\ [[xy]]^{-1} \\ &= [[xy]]\ z\ [[xy]]^{-1}. \blacksquare \end{aligned}$$

We call $T_{x,y}$ the **conjugation** determined by the invertible elements x and y. Any finite product $T_z T_{x_1,y_1} \ldots T_{x_n,y_n}$ of conjugations (for z nuclear and x_i, y_i invertible) will be called a **standard inner automorphism**; when z is strictly nuclear it is called a **strictly standard inner automorphism**. Strictly standard automorphisms are always strictly inner.

III.1 Exercises

1.1 Prove directly that $T = T_z$ is an automorphism iff $z^3 \in N(A)$ and $T = L_{z,w}$ is an automorphism iff $[[z,w]] \in N(A)$ by computing $T\,L_x\,T^{-1}$.

1.2 Show T_x and $L_{z,w}$ are always at least automorphisms of the Jordan structure.

1.3 Prove directly <u>Jacobson's Criterion</u>: if $c_1(c_2..c_n) = (c_n..c_2)c_1 = 1$ then all c_i are invertible and $T = L_{c_1}..L_{c_n} = R_{c_1}..R_{c_n} = U_{c_1}..U_{c_n}$ is an automorphism. Show the set of such T forms a normal subgroup of Aut(A).

1.4 If x^3, y^3 are nuclear need the same be true of $(xy)^3$? Even if $(xy)^3$ is nuclear, is $T_x T_y = T_{xy}$?

1.5 Show that if $e+f = 1$ for e,f orthogonal idempotents and $x = e+\lambda f$ where $\lambda \neq 1$ but $\lambda^3 = 1$, then x is not nuclear in general yet $x^3 = 1$ is nuclear. Find x^{-1} and T_x.

1.6 If $x_1,\ldots,x_n,y$ lie in a subalgebra $\Phi[a,N(A)]$ generated by a single element (plus nucleus) show $T = R_y L_{x_1}\ldots L_{x_n}$ reduces to T_z for $z = x_1\ldots x_n$ whenever $T(1) = 1$.

1.7 Show $\mathrm{Naut}(A) = \{T_z \mid z \in N(A)\}$ is a normal subgroup of Aut(A). What if you take $\{T_z \mid z^3 \in N(A)\}$?

III. 1.1 Problem Set on Other Normal Forms

1. Show every element of the multiplication group can be written in the following normal forms

(Left normal)	$L_{x_1} \ldots L_{x_n} R_y$
(Right normal)	$R_{x_1} \ldots R_{x_n} L_y$
(Middle normal)	$U_{x_1} \ldots U_{x_n} R_y$
(S-normal)	$U_x S_{x_1,y_1} \ldots S_{x_n,y_n} R_y$
($\tilde{S}$-normal)	$U_x \tilde{S}_{x_1,y_1} \ldots \tilde{S}_{x_n,y_n} L_y$

where

$$S_{x,y} = U_{xy}^{-1} U_x U_y, \ \tilde{S}_{x,y} = U_{xy}^{-1} U_y U_x$$

measure how far U_{xy} is from $U_x U_y$ or $U_y U_x$;

$$S_{x,y} = R_{xy}^{-1} R_x R_y, \ \tilde{S}_{x,y} = L_{xy}^{-1} L_y L_x.$$

2. Find criteria for when an element in one of the above normal forms is an automorphism.

3. Apply the middle - and S- criteria to $T_x = L_x R_x^{-1}$ and $U_{c_1} \ldots U_{c_n}$ (c_i as in Jacobson's Criterion Ex. 1.3). Apply the S-Criterion to $\lambda S_{x,y}$ and the $\tilde{S}$-Criterion to $\lambda \tilde{S}_{x,y}$ where x and y commute up to a scalar, $xy = \lambda yx$. (Think of all those elements $ij = -ji$, $kl = -lk$ etc. in a Cayley algebra).

4. Find necessary and sufficient conditions that $U_{x_1} \ldots U_{x_n}$ be an automorphism; that $L_{x_1} \ldots L_{x_n}$ be an automorphism.

5. Apply S-Criterion to $T = U_{[[xy]]} \{U_{xy}^{-1} U_x U_y\}^3 R_{[[xy]]}$. What does T amount to in an associative algebra? Does this suggest an alternate expression for T?

6. Apply the criterion of your choice to decide whether $\{U_x U_y U_{xy}^{-1}\}\{U_y U_x U_{yx}^{-1}\}$ is an automorphism.

7. Show $S_{x^i, x^j} = 1$, $S_{x,y}^{-1} = S_{y,x}$, $S_{x,y} = S_{y^{-1}, yx} = S_{xyx, x^{-1}}$ $= S_{xy,y}$ $U_x U_y U_z = S_{y,z}$ if $x(yz) = 1$. Develope analogous results for $\tilde{S}_{x,y}$.

8. If $x_1, \ldots, x_n, y$ lie in a subalgebra $\Phi[a, N(A)]$ generated by a single element together with the nucleus, show $T = U_{x_1} \ldots U_{x_n} R_y$ reduces to T_z for $z = x_1 \ldots x_n$ whenever $T(1) = 1$.

III. 1.2 Problem Set on the Structure Group

Just as an automorphism is an isomorphism of an algebra with itself, so an <u>autotopism</u> is an isotopism of an algebra with itself: a triple (T,T',T") of bijections on A satisfying

(*) $T(xy) = T'(x)T''(y) \qquad (x,y \in A)$.

1. Show T is an automorphism iff (T,T,T) is an autotopism. Just as the set of automorphism form a subgroup Aut(A) of the general linear group GL(A), show the set of autotopisms forms a subgroup

Autop(A)

of GL(A) x GL(A) x GL(A).

2. Besides the automorphisms, show we have autotopisms

$(L_x, U_x, L_x{-1}) \qquad (R_x, R_x{-1}, U_x) \qquad (U_x, L_x, R_x)$

coming from multiplications by an invertible element x,

3. As in Schafer's Isotopy Theorem, show any autotopism has the form $(T,T',T") = (T,R_{t"}^{-1}T,L_{t'}^{-1}T)$ $(t' = T'1,\ t" = T"1)$

4. This allows us to permute the entires in an autotopy: if $(T,T',T")$ is an autotopy so are $(T',T,U_{t"}^{-1}T")$ and $(T",U_{t'}^{-1}T',T)$ for $t' = T'1$, $t" = T"1$.

5. The <u>structure group</u> Str(A) of A consists of all bijective linear transformations T on A for which there exist bijective T',T" satisfying the autotopism condition (*), i.e. iff T is the first component of an autotopy (T,T',T"). Show this is equivalent to being the second or third component of an autotopy, and find the defining conditions for these components (T',T" in (*)) in operator form.

6. Show the group Str(A) not only contains all automorphisms, but also all multiplications L_x, R_x, U_x for invertible x, hence the multiplication group GM(A) generated by the invertible L_x, R_y:

$$Str(A) \supset GM(A) \cdot Aut(A).$$

7. The structure group is not too far removed from the multiplication and automorphism groups. In "characteristic $\neq 3$" situations show we have the following polar decomposition of an element into a multiplication and an automorphism: If $T \in Str(A)$ is such that $t't"t'$ has a cube root $(t' = T'1, t" = T"1)$ then there are $x,y,z \in A$ and an automorphism A with

$$T = L_x L_y R_x A.$$

If all invertible elements have cube roots then

$$Str(A) = GM(A)\,Aut(A).$$

8. Show the first component of autotopy determines the other two only up to translations by elements of the nucleus: if (T,T'T")

is an autotopy, another $(T,\tilde{T}',\tilde{T}'')$ is too iff $\tilde{T}' = R_n T'$, $\tilde{T}'' = L_n^{-1} T''$ for $n \in N(A)$.

9. The <u>inner structure group</u> is that part of the structure group which can be built up from multiplications,

$$\mathrm{Instr}(A) = \mathrm{Str}(A) \cap GM(A).$$

Deduce that

$$\mathrm{Instr}(A) = GM(A).$$

10. Establish the following geometric description of the structure group: The structure group consists of all isomorphisms of A with an isotope, or equivalently of all isomorphism from one isotope of A to another. Conclude the structure group of A and its isotopes $A^{(u,v)}$ coincide,

$$\mathrm{Str}(A) = \mathrm{Str}(A^{(u,v)}).$$

11. If $T \in \mathrm{Str}(A)$, show directly that T takes invertible elements to invertible elements.

12. Show T in (T,T',T'') is an automorphism iff $t' = n$, $t'' = n^{-1}$ for some nuclear n.

13. Assuming all c_i are invertible, show via autotopies that if $c_1(c_2(\ldots c_n)) = 1 = ((c_n \ldots)c_2)c_1$ then $U_{c_1} \ldots U_{c_n} = L_{c_1} \ldots L_{c_n} = R_{c_1} \ldots R_{c_n}$ is an automorphism.

14. Show via autotopies that $T_x = L_x R_x^{-1}$ is an automorphism iff $x^3 \in N(A)$.

15. If $1/3 \in \Phi$, show Str(A) contains an anti-automorphism iff A is commutative and associative.

16. Show $GM(A) \triangleleft \mathrm{Str}(A)$.

17. Show $T = L_{x_1} \ldots L_{x_n}$ is an automorphism iff $x_1(x_2(\ldots x_n)) = 1$ and $((x_n \ldots)x_2)x_1$ is nuclear.

18. Show $T \in \mathrm{Str}(A)$ is an automorphism of Jordan structure iff $T(1) =$