§5. Degree 2 algebras

We have said that most well-behaved left Moufang algebras are alternative. Another general class for which this holds, besides the division algebras, are the degree 2 algebras. In fact, if a degree 2 algebra is left alternative it is necessarily left Moufang (even in characteristic 2) and moreover alternative.

According to our general definition, A is of **degree two** over a field ϕ if

(5.1) $$x^2-t(x)x+n(x)1 = 0 \quad (t(1) = 2,\ n(1) = 1)$$

for linear t and quadratic n. A left alternative degree 2 algebra is automatically left Moufang, indeed even alternative.

(5.2) (Degree 2 Theorem). A left alternative algebra of degree 2 over a field is alternative.

Proof. To prove A is alternative we need by (1.8) to establish flexibility. Now $[L_x,R_x]z = [L_x,V_x]z = x(x\circ z)-x\circ xz = x\{t(x)z+t(z)x-n(x,z)1\} - \{t(x)xz+t(xz)x-n(x,xz)1\} = t(z)\{t(x)x-n(x)1\} -n(x,z)x-t(xz)x+n(x,xz)1$ by linearized (5.1), and $x^*(xz) = t(x)xz-x^2z = n(x)z$ for $x^* = t(x)1-x$, so

$$[L_x,R_x]z = \{n(x,xz)-t(x^*(xz))\}1 - \{n(x,z)-t(x^*z)\}x.$$

Thus flexibility will follow if we can show the bilinear form

(5.3) $$f(x,y) = n(x,y)-t(x^*y) = n(x,y)+t(xy)-t(x)t(y)$$

vanishes identically. Since f vanishes when x or y is 1 $(n(x,1) = t(x)$

follows by linearizing $x \to x,1$ in (5.1) and using $t(1) = 2$), and also when $x = y$ (taking traces of (5.1)) we may assume $1,x,y$ are linearly independent.

First suppose xy is linearly dependent on $1,x,y$. Then so is $yx = x\circ y - xy$,

$$(5.4) \qquad \begin{aligned} xy &= \alpha 1+\beta x+\gamma y \\ yx &= \alpha' 1+\beta' x+\gamma' y \end{aligned} \qquad \left\{ \begin{aligned} \alpha+\alpha' &= -n(x,y) \\ \beta+\beta' &= t(y) \\ \gamma+\gamma' &= t(x) \end{aligned} \right.$$

From $t(x)xy-n(x)y = x^2y = x(xy) = \alpha x+\beta x^2+\gamma xy$ we can (by independence of $1,x,y$) identify coefficients of x to see $t(x)\beta = \alpha+\beta t(x)+\gamma\beta$. Thus $\alpha+\beta\gamma = 0$, and dually with x and y interchanged

$$(5.5) \qquad \alpha+\beta\gamma = \alpha'+\beta'\gamma' = 0.$$

Taking traces of (5.4) yields $t(xy) = 2\alpha+\beta t(x)+\gamma t(y) = 2\alpha-\{t(x)-\gamma\}\{t(y)-\beta\}$ $+t(x)t(y)+\beta\gamma = \alpha+\beta\gamma+t(x)t(y)+\alpha-\beta'\gamma' = t(x)t(y)+\alpha+\alpha'$ (by (5.5)) $= t(x)t(y)-n(x,y)$ as required in (5.3).

Now suppose xy is independent of $1,x,y$. We have the usual U-formula

$$(5.6) \qquad U_a b = n(a,b^*)a-n(a)b^* \in \Phi 1+\Phi a+\Phi b$$

since $a(ba) = a(a\circ b)-a(ab) = a\{t(a)b+t(b)a-n(a,b)1\}-a^2b = t(b)a^2-n(a,b)a+n(a)b$ $= \{t(a)t(b)-n(a,b)\}a+n(a)\{b-t(a)1\}$ by (5.1) and left alternativity. Then $0 = y\{x^2y\}-y\{x(xy)\} = U_y x^2-U_{y,xy}x+(xy)\{xy\} \in \{\Phi 1+\Phi y+\Phi x^2\}-\{n(xy,x^*)y+n(y,x^*)xy$ $-n(y,xy)x^*\}+\{t(xy)xy-n(xy)1\} \subset \{t(xy)-n(x^*,y)\}xy+\Phi 1+\Phi x+\Phi y$, so by independence the coefficient $f(x,y)$ of xy must be zero.

Thus $f(x,y)$ vanishes whether xy is dependent or independent of $1,x,y$ and by (5.3) A is flexible. ∎

AIV.5 Exercises

5.1 Show that a left alternative degree 2 algebra over an arbitrary ring of scalars Φ is left Moufang.

5.2 Show that if A is left alternative of degree 2 so is any isotope $A^{(u)}$, with $t^{(u)}(x) = n(u^*,x)$ and $n^{(u)}(x) = n(u)n(x)$.

5.3 If $n(x,y)$ vanishes identically on A of degree 2, show A is commutative of characteristic 2; otherwise show (over a field) some isotope $A^{(u)}$ has nonzero trace $t^{(u)} \neq 0$.

5.4 If A is degree 2 over an algebraically closed field Φ with <u>nondegenerate</u> norm form $n(x,y)$, show either $A = \Phi 1$ or A contains a proper idempotent $e \neq 0,1$.

5.5 If A of degree 2 over an algebraically closed field Φ contains a proper idempotent $e_0 \neq 0$, show for each x there are infinitely many $\lambda \in \Phi$ with $y = x+\lambda e_0$ separable, so if $[y,A,y] = 0$ for all separable y then $[x,A,x] = 0$ for all x. If $y = \alpha e+\beta(1-e)$ is separable, show $[y,A,y] = 0$ if $[e,A,e] = 0$ for the idempotent e. Conclude that if $[e,A,e] = 0$ for all idempotents e then A is alternative.

5.6 Show $[e,x,e] = 0$ for any x and any idempotent e in a degree 2 left alternative algebra.