§4. Bruck's examples

We know left alternative and left Moufang are the same in characteristic $\neq 2$. In this section we construct examples of left alternative division rings of characteristic 2 which are not left Moufang, much less alternative.

We begin with any unital commutative associative Φ-algebra Ω of characteristic 2, and any Φ-linear mapping s of Ω into itself. We define an algebra

(4.1) $$A(\Omega,s) = \Omega 1 \oplus \Omega u$$

$$xy = (\alpha 1 \oplus \beta u)(\gamma 1 \oplus \delta u) = (\alpha\gamma + s(\beta)\delta)1 \oplus (\alpha\delta + \beta\gamma)u.$$

This is like the Cayley-Dickson formula, except we use $\beta\gamma$ instead of the expected $\beta s(\gamma)$. For fixed α,β this expression is linear in γ and δ, so L_x is Ω-linear (though R_y is only Φ-linear, due to the presence of the $s(\beta)$).

4.2 Lemma. $A(\Omega,s)$ is always left alternative, but is left Moufang iff $s(q(x)\omega) = q(x)s(\omega)$ for all $\omega,\alpha,\beta \in \Omega$ where $q(x) = \alpha^2+s(\beta)$. If Ω contains no nilpotent elements, the left Moufang condition is that $s = L_\sigma$ for $\sigma = s(1)$.

Proof. To check left alternativity $L_{x^2} = L_x^2$, note that for $x = \alpha 1+\beta u$ we have $x^2 = (\alpha^2+s(\beta)\beta)1+(2\alpha\beta)u = q(x)1$, therefore $L_{x^2} = q(x)I$ since $L_{\omega 1} = \omega I$ (BEWARE: $L_{\omega u} \neq \omega L_u$ since R_y is not Ω-linear!), and $L_x^2 = (\alpha I+L_{\beta u})^2 = \alpha^2 I+L_{\beta u}^2 = q(x)I$ since we are in characteristic 2. (Note

$$L_{\beta u} = \begin{pmatrix} 0 & s(\beta) \\ \beta & 0 \end{pmatrix}, \quad L_{\beta u}^2 = \begin{pmatrix} s(\beta)\beta & 0 \\ 0 & \beta s(\beta) \end{pmatrix}$$ relative to the obvious Ω-basis for A).

Thus left alternativity is automatic.

However, the left Moufang axiom $L_{x(yx)} = L_x L_y L_x$ will only be satisfied for certain kinds of s. Indeed, $x^2 = q(x)1$ yields $x \circ y = q(x,y)1$ for $q(x,y) = q(x+y)-q(x)-q(y) = 2\alpha\gamma+s(\beta)\delta+s(\delta)\beta = s(\beta)\delta+s(\delta)\beta$, so $L_{x(yx)} = L_{x(xy+yx)}-L_{x^2y}$ (left alternativity) $= L_{q(x,y)x}-L_{q(x)y}$ $= q(x,y)\alpha I+L_{q(x,y)\beta u}-q(x)\gamma I-L_{q(x)\delta u}$. On the other hand $L_x L_y L_x = L_x(L_y \circ L_x)-L_x^2 L_y$ $= q(x,y)L_x-q(x)L_y = q(x,y)\alpha I+q(x,y)L_{\beta u}-q(x)\gamma I-q(x)L_{\delta u}$. Thus the axiom reduces to $L_{q(x)\delta u} = q(x)L_{\delta u}$ (and its linearization $L_{q(x,y)\beta u} = q(x,y)L_{\beta u}$). Since $L_{\omega u} \cong \begin{pmatrix} 0 & s(\omega) \\ \omega & 0 \end{pmatrix}$ the condition becomes $s(q(x)\delta) = q(x)s(\delta)$ (and its linearization $s(q(x,y)\delta) = q(x,y)s(\delta)$).

In particular s commutes with $q(\omega 1) = \omega^2$, with $q(\omega u) = \omega s(\omega)$, with $q(u) = s(1) = \sigma$, and with $q(u,\omega u) = \sigma\omega+s(\omega)$, so that $\{\sigma\omega+s(\omega)\}s(\omega)$ $= s(\{\sigma\omega+s(\omega)\}\omega) = s(\{\omega^2\sigma+\omega s(\omega)\}1) = \{\omega^2\sigma+\omega s(\omega)\}s(1) = \omega^2\sigma^2+\sigma\omega s(\omega)$. Comparing gives $s(\omega)^2 = \omega^2\sigma^2$. In characteristic 2 this implies $\{s(\omega)-\sigma\omega\}^2 = 0$, so if Ω has no nilpotent elements $s(\omega) = \sigma\omega$ for all ω. ∎

4.3 Lemma. $A(\Omega,\sigma)$ is a left alternative division algebra iff (i) Ω is a field, (ii) $s-L_{\omega^2}$ is bijective on Ω for all ω.

Proof. Certainly Ω must be a field: it is a commutative associative subalgebra of A, and if $\alpha+\beta u$ is an inverse in A of $\omega\in\Omega$ then α is an inverse of ω in Ω.

Assume from now on Ω is a field. "Division algebra" means all L_x, R_x for $x \neq 0$ are bijective. Now L_x is bijective iff $L_x^2 = q(x)I$ is bijective, so the condition that all L_x for $x \neq 0$ be bijective is that $q(x) \neq 0$ for $x \neq 0$. In particular, this is satisfied when (i) and (ii) hold: clearly $q(x) = \alpha^2+\beta s(\beta)$ $= 0$ is impossible for $\beta = 0$ (since then $\alpha \neq 0$), while if $\beta \neq 0$ it would imply $s(\beta) = -\alpha^2/\beta = (\alpha/\beta)^2\beta$ (characteristic 2!) and therefore $s-L_{\omega^2}$ is not bijective for $\omega = \alpha/\beta$ since it kills β.

Turning now to the R's, if $x = \alpha+\beta u$ has $\beta = 0$ then $x = \alpha 1$ and $R_x = \alpha I$ is clearly bijective. If $\beta \neq 0$ then $x = \beta(\beta^{-1}\alpha 1+u) = \beta y$, so $R_x = \beta R_y$ is bijective iff R_y is for $y = \omega 1+u$. But

$$R_y = \omega I+R_u \cong \begin{pmatrix} \omega & s \\ 1 & \omega \end{pmatrix} \quad \text{has} \quad R_y^2 \cong \begin{pmatrix} \omega^2+s & \omega s+s\omega \\ 2\omega & s+\omega^2 \end{pmatrix} = \begin{pmatrix} s-\omega^2 & \omega s+s\omega \\ 0 & s-\omega^2 \end{pmatrix}$$

in characteristic 2, which is invertible iff $s-L_{\omega^2}$ is invertible. ■

4.4 Theorem. If Ω is a field of characteristic 2, with nontrivial involution $*$, and $\alpha = \alpha^*$ is a symmetric nonsquare in Ω, then $A(\Omega,s)$ for $s = *+L_\alpha$ is a left alternative division algebra which is not left Moufang.

Proof. Such an algebra is not left Moufang by 3.2 because $s \neq L_\sigma$: $\sigma = s(1) = 1^*+\alpha = 1+\alpha, L_\sigma = I+L_\sigma \neq *+L_\sigma = s$ since $* \neq I$ is nontrivial by hypothesis.

Since Ω is a field, $A(\Omega,s)$ will be a division algebra by 4.3 as soon as all $s-L_{\omega^2} = *+L_{\alpha-\omega^2}$ are bijective, i.e., $*+L_\delta$ is bijective for all $\delta = \alpha-\omega^2$. Now $(*+L_\delta)(*-L_{\delta^*}) = (*-L_{\delta^*})(*+L_\delta) = I-L_{\delta\delta^*} = L_{1-\delta\delta^*}$ (recall $*L_\omega = L_{\omega^*}*$) is invertible since $1-\delta\delta^*$ is nonzero in Ω no matter what $\delta = \alpha-\omega^2$ we choose: if ω is symmetric, $\omega^* = \omega$, then $\delta^* = \delta$ since $\alpha^* = \alpha$ by hypothesis, so $\delta\delta^* = 1 \Rightarrow \delta^2 = 1 \Rightarrow \delta = 1 \Rightarrow 1 = \alpha-\omega^2 \Rightarrow \alpha = 1+\omega^2 = (1+\omega)^2 \in \Omega^2$ (heavily using characteristic 2), contrary to our choice of α as a nonsquare, while if Ω is nonsymmetric, $\omega+\omega^* \neq 0$, then $1 = \delta\delta^* = \{\alpha-\omega^2\}\{\alpha-\omega^{*2}\} = \alpha^2+(\omega\omega^*)^2-\alpha(\omega+\omega^*)^2$ would imply $\alpha(\omega+\omega^*)^2 \in \Omega^2$, hence $\alpha \in \Omega^2$ (recall $\omega+\omega^* \neq 0$), again contrary to choice. Thus $1-\delta\delta^* \neq 0$ is invertible in all cases, so all $s-L_{\omega^2} = *+L_\delta$ are bijective. ■

4.5 Example. As a specific example, take $\Omega = \Phi(x)$ for any field Φ of characteristic 2. Then $f(x)^* = f(\frac{1}{x})$ is a nontrivial involution on Ω, $\alpha = x+1/x$ is symmetric but a nonsquare (since $\alpha = x(1+1/x)^2$ and x is a nonsquare by $\Omega^2 = \Phi^2(x^2)$).

This particular Ω and $\sigma = *+L_\alpha$ lead to a left alternative division ring $A(\Omega,\sigma)$ which is not left Moufang. ■

AIV.4 Exercises

4.1 Show $\Omega \xrightarrow{s} \Omega$ satisfies $s(q(x)\omega) = q(x)s(\omega)$ for all $q(x) = \alpha^2+\beta s(\beta)$ iff (i) $s(\alpha^2\omega) = \alpha^2 s(\omega)$, (ii) $s(s(\omega)) = \sigma^2\omega$, (iii) $\{s(\omega)-\sigma\omega\}^2 = 0$.

4.2 Show directly R_x is invertible for $x = \beta u$ iff s is bijective; if $\alpha \neq 0$ show $x = \alpha y$, where $R_y = I+R_{\gamma u}$ is bijective iff $s+L_\gamma^{-2}$ is bijective.