§3. Division rings

Skornyakov's Theorem asserts that all left Moufang division rings are alternative, hence associative or Cayley algebras. This has important geometric consequences for which as yet no purely geometric proof is known. The algebraic proof is an application of Micheev's Identity, which says the fourth power of any associator [x,y,x] vanishes in a left Moufang algebra.

As in the case of alternative division algebras, we will be able to classify all left Moufang division algebras. Since we already know that any alternative division algebra is either associative or a Cayley algebra, all we need do is show a left Moufang division algebra is actually alternative (that is, by 1.8 we must establish flexibility). In the next section we will show that left alternativity is not sufficient - there exist left alternative division rings which are not left Moufang (much less alternative); of course, this can happen only in characteristic 2 in view of the Equivalence Theorem 1.4.

Let us introduce two operators

(3.1) $$A_{x,y} = L_{xy} - L_x L_y, \quad B_{x,y} = L_{xy} - L_y L_x.$$

Clearly left alternativity gives

(3.2) $$A_{x,x} = B_{x,x} = 0$$

(3.2') $$A_{x,y} + A_{y,x} = B_{x,y} + B_{y,x} = 0.$$

In the presence of the left Moufang Law we will establish

(3.3) $$B_{x,y} L_x A_{x,y} = 0$$

(3.3') $$B_{x,y} A_{x,y} = 0$$

(3.4) $$A_{x,y} L_x B_{x,y} = L_{A(x,y)x[x,y]}$$

(3.4') $$A_{x,y}B_{x,y} = L_{A(x,y)[x,y]}$$

(3.5) $$L_{[x,y,x]} = A_{xy,x}+A_{x,y}L_x = B_{xy,x}+L_xB_{x,y}$$

(3.6) $$L_{[x,y,x]}A_{x,y} = -B_{x,y}L_{[x,y,x]}$$

(3.7) $$L_{[x,y,x]}A_{xy,x} = -L_xL_{[x,y,x]}A_{x,y} \quad .$$

Observe that (3.3') and (3.4') follow from (3.3) and (3.4) in the left Moufang algebra $\hat{A}$ by linearizing $x \to x,1$ (since $A_{x,y}$, $B_{x,y}$, and $[x,y]$ all vanish when $x = 1$). Therefore we must check only (3.3), (3.4), (3.5), (3.6), (3.7).

For (3.3) we compute

$$\begin{aligned} B_{x,y}L_xA_{x,y} &= \{L_{xy}-L_yL_x\}L_x\{L_{xy}-L_xL_y\} \\ &= L_{xy}L_xL_{xy}-L_yL_x^2L_{xy}-L_{xy}L_x^2L_y+L_yL_x^3L_y \\ &= L_{(xy)\{x(xy)\}}-\{L_yL_{x^2}L_{xy}+L_{xy}L_{x^2}L_y\}+L_yL_{x^3}L_y \\ &= L_{(xy)(x^2y)}-\{L_{y(x^2(xy))}+L_{(xy)(x^2y)}\}+L_{y(x^3y)} \\ &= 0 \end{aligned}$$

by repeated use of left Moufangitivity (recall $x^2(xy) = x^3y$ by (1.6)).

(3.4) is similar but messier: $A_{x,y}L_xB_{x,y} = \{L_{xy}-L_xL_y\}L_x\{L_{xy}-L_yL_x\}$ $= L_{xy}L_xL_{xy}-L_xL_yL_xL_{xy}-L_{xy}L_xL_yL_x+L_xL_yL_xL_yL_x = L_{(xy)\{x(xy)\}}-L_{x(yx)}L_{xy}-L_{xy}L_{x(yx)}+$ $+L_xL_{y(xy)}L_x = L_{(xy)\{x(xy)\}}-L_{\{x(yx)\}(xy)}-L_{(xy)\{x(yx)\}}+L_{x\{y(xy)\cdot x\}}$ $= L(\{xy\}\{x(xy)\}-x\{y\{x(xy)\}\}-(xy)\{x(yx)\}+x\{y\{x(yx)\}\}) = L(\{xy\}x[x,y]+x\{y(x[y,x])\})$ $= L(A_{x,y}x[x,y])$.

For (3.5) note $L_{[x,y,x]} = L_{(xy)x}-L_{x(yx)} = L_{(xy)x}-L_xL_yL_x$ by left Moufang, while $A_{xy,x}+A_{x,y}L_x = \{L_{(xy)x}-L_{xy}L_x\}+\{L_{xy}-L_xL_y\}L_x = L_{(xy)x}-L_xL_yL_x$ $= \{L_{(xy)x}-L_xL_{xy}\}+L_x\{L_{xy}-L_yL_x\} = B_{xy,x}+L_xB_{x,y}$ straight from the definitions (3.1).

For (3.6) multiply the first and third terms in (3.5) on the right by $A_{x,y}$ to get $L_{[x,y,x]}A_{x,y} = B_{xy,x}A_{x,y}$ by (3.3'), next observe $B_{xy,x}A_{x,y} = -B_{x,xy}A_{x,y}$ (by (3.2')) $= +B_{x,y}A_{x,xy}$ (linearized (3.3')) $= -B_{x,y}A_{xy,x}$, then multiply the first and second terms in (3.5) on the left by $B_{x,y}$ to see $-B_{x,y}L_{[x,y,x]} = -B_{x,y}A_{xy,x}$. For (3.7), multiply these by L_x to get $L_xL_{[x,y,x]}A_{x,y} = -L_xB_{x,y}A_{xy,x}$, then multiply the first and third terms in (3.5) on the right by $A_{xy,x}$ to see $L_{[x,y,x]}A_{x,y,x} = L_xB_{x,y}A_{xy,x}$. ∎

Once we have these computations out of the way we are ready to establish an important identity for left Moufang algebras: we will show the fourth power of any associator $[x,y,x]$ is zero. This striking result of Micheev resembles the Fourth Power Theorem for alternative algebras.

3.8 (Micheev's Identity) In a left Moufang algebra the associator $[x,y,x]$ of any two elements x,y is nilpotent,

$$[x,y,x]^4 = 0.$$

Proof. If we write $p = [x,y,x]$ we have

$$\begin{aligned}
L_p^3A_{x,y} &= L_pL_p\{L_pA_{x,y}\} = L_p\{A_{xy,x}+A_{x,y}L_x\}\{-B_{x,y}L_p\} && \text{(by (3.5),(3.6))}\\
&= L_xL_pA_{x,y}B_{x,y}L_p - L_p\{A_{x,y}L_xB_{x,y}\}L_p && \text{(by (3.7))}\\
&= L_xL_pL_cL_p - L_pL_dL_p && \text{(by (3.4'),(3.4))}\\
&= L_xL_{p(cp)} - L_{p(dp)} = 0 && \text{(left Moufang)}
\end{aligned}$$

since by (3.4), (3.4') the elements $c = A_{x,y}[x,y]$, $d = A_{x,y}x[x,y]$ satisfy $cp = L_cp = \{A_{x,y}B_{x,y}\}\{A_{x,y}(x)\} = 0$ and $dp = L_dp = \{A_{x,y}L_xB_{x,y}\}\{A_{x,y}x\} = 0$ by orthogonality (3.3'). Thus $p^4 = L_p^3p = L_p^3A_{x,y}x = 0$ for any $p = [x,y,x]$. ∎

If all the nilpotent $[x,y,x]$ vanish we get flexibility $[x,y,x] = 0$, hence alternativity:

3.9 (Kleinfeld-Micheev Theorem). A left Moufang algebra without nilpotent elements (even without nilpotent associators [x,y,x]) is alternative. ■

In particular

3.10 (Skornyakov's Theorem). A left Moufang division algebra is alternative, therefore associative or a Cayley division algebra. ■

Since a division ring with left inverse property is left Moufang by 2.8, we have the geometrically significant

3.11 (Left Inverse Theorem). A nonassociative division ring with left inverse property is alternative, therefore associative or a Cayley algebra. ■

As mentioned previously, a projective plane with enough translations is coordinatized by a division ring with left inverse property.

AIV.3 Exercises

3.1 Derive (3.4) from (3.3) using the relation

(*) $$A_{x,y}+B_{x,y} = L_{[x,y]}.$$

3.2 Use $[x,y,z] = xy\circ z - z(xy) - U_{x,z}y + z(yx)$ to obtain

(**) $$L_{A(x,y)z} = A_{x,y}L_z + L_z B_{x,y} - L_{z[xy]}$$

Deduce (3.4') from (3.3') using (*) and (**).

AIV.3.1 Problem Set on Property (K)

We want to obtain Kleinfeld's original result (before Micheev's result was known).

Let us consider left Moufang algebras with the following property K (as in Kleinfeld):

(K) If z has the form (i) x, (ii) $[x,y]$, (iii) $x[x,y]$, or (iv) $x^2[x,y]$ then $[x,y,z]^2 = 0$ implies $[x,y,z] = 0$.

Clearly (K) holds if A is a division algebra, or has no zero divisors, or has no nilpotent elements, or even if it has no nilpotent associators $[x,y,p(x,y)]$ involving only 2 variables x,y (which associators are therefore zero in all alternative algebras).

3.1 Show $L_{[x,y,z]}A_{x,y} = 0$ for $z = x[x,y]$ or $z = [x,y]$. Deduce $[x,y,z]^2 = 0$. Conclude $[x,y,z] = 0$ for $z = x[x,y]$ or $z = [x,y]$ if (Kiii) or (Kii) hold. When A satisfies (K) show we can strengthen (3.4), (3.4') to read

(3.4K) $$A_{x,y}L_xB_{x,y} = 0$$

(3.4'K) $$A_{x,y}B_{x,y} = 0.$$

4.2 Show $L_p^2 = A_{x,y}L_x^2B_{x,y}$ for any associator $p = [x,y,x]$ in a left Moufang algebra satisfying (K). Deduce $p^3 = 0$ and $(p^2)^2 = 0$.

4.3 Show $p^2 = [x,y,z]$ for $z = x^2[x,y]$. Deduce $p^2 = 0$ when (Kiv) holds and then $p = 0$ when (Ki) holds.

4.4 Prove the <u>Property (K) Theorem:</u> A left Moufang algebra is alternative iff it has property (K).

4.5 Derive Skornyakov's Theorem and the Kleinfeld-Micheev Theorem.

AIV.3.2 Problem Set on Jordan Homomorphisms

Let $F: A \to D$ be a **Jordan homomorphism** of the left Moufang algebra A into the associative algebra D, in the sense that F is a linear map satisfying

$$F(x^2) = F(x)^2 \qquad F(x(yx)) = F(x)F(y)F(x)$$

(for example, $D = \mathrm{End}\, A$ and $F(x) = L_x$). Introduce the abbreviations

$$x^y = F(xy) - F(x)F(y) \qquad x_y = F(xy) - F(y)F(x).$$

Thus F is a homomorphism iff all $x^y = 0$, and an antihomomorphism iff all $x_y = 0$.

1. Prove $x^x = x_x = 0$; linearize. Show $x^y - y_x = F([xy])$, $x^y - x_y = [F(y), F(x)]$.

2. Show $x_y x^y = 0$, $x^y x_y = F([x,y,[x,y]])$.

3. Show $x^{yx} = F(x)x^y$, $x_{yx} = x_y F(x)$.

4. Show $x_y F(x) x^y = 0$ (when $1/2 \in \Phi$ derive this immediately from #2, #3).

5. Show $F([y,x,x]) = x^{xy} - x^y F(x) = x_{xy} - F(x)x_y$.

6. Show $x \circ y = 0 \Rightarrow F([y,x,x])x^y = 0$.

7. Show $x^y F(z) + F(z)x_y = F([x,y,z] + z[x,y])$, $x_y F(z) + F(z)x^y = F(-[x,y,z] + [x,y]z)$.

8. Show $x_y F([x,y,z]) + F([x,y,z])x^y = 0$.

9. Show $x^y F(z) x_y + x_y F(z) x^y = F([x,y,z[x,y]])$.

10. Show $x^y F(x) x_y = F([x,y,x[x,y])$ and $F([x,y,x[x,y])x^y = 0$.

11. When $F(x) = L_x$ show $x^y(z) = [x,y,z]$, $x_y(z) = [x,y]z + [x,y,z]$, $x_y(1) = [x,y]$. Simplify notation in #2, #7, #8, #9, #10.