Appendix IV

Left Moufang Algebras

§1. The variety of left Moufang algebras

Rather than the variety of left alternative algebras, it is the variety of left Moufang algebras which arises naturally from geometrical contexts and which has a more satisfactory structure theory. In characteristic $\neq 2$ all left alternative algebras are left Moufang, but in characteristic 2 the left Moufang law provides just enough of an additional handle to make things manageable. In this section we collect a few of the most basic facts about this variety.

A **left Moufang** algebra is one satisfying both the left Moufang and left alternative laws

(1.1) $\{x(yx)\}z = x\{y(xz)\}$ (left Moufang)

(1.2) $x^2z = x(xz)$ (left alternative) .

In terms of operators the axioms are

(1.1 op) $L_{x(yx)} = L_x L_y L_x$

(1.2 op) $L_{x^2} = L_x^2$

while in terms of associators they become

(1.1 a) $[x,yx,z] + x[y,x,z] = 0$ (left bumping)

(1.2 a) $[x,x,z] = 0$

(note the left side of (1.1a) reduces to $\{x(yx)\}z - x\{(yx)z\} + x\{(yx)z\} - x\{y(xz)\} = \{x(yx)\}z - x\{y(xz)\}$).

Since the defining identities are quadratic they can be linearized; in particular, the linearization of (1.2a) will prove useful,

(1.2 a') $[x,y,z] + [y,x,z] = 0.$

You should think of the left Moufang law (1.1) as a strengthening of the left alternative law (1.2). Indeed, for unital algebras (1.2) is a special case of (1.1), obtained by setting $y = 1$. Thus for unital algebras (1.2) is superfluous, and (1.1) alone is the defining identity.

From this we see that if we want a variety closed under adjunction of unit elements, we are forced to require (1.2) in addition to (1.1): if the unital hull $\hat{A}$ is to continue to satisfy (1.1) it will have to satisfy (1.2) as well, so A too must satisfy (1.2) from the start. And, in fact, our variety is closed under adjunction:

1.3 (Adjunction Proposition) If A is left alternative or left Moufang, so is its unital hull $\hat{A}$.

Proof. For any elements $x = \alpha 1+a$, $y = \beta 1+b$, $z = \gamma 1+c$ in $\hat{A} = \Phi 1+A$ we have $[x,x,z] = [\alpha 1+a,\alpha 1+a,\gamma 1+c] = [a,a,c] = 0$ since 1 is in the nucleus and A satisfies (1.2a). Thus $\hat{A}$ inherits left alternativity. Similarly, for left Moufang in associator form (1.1a) we have $[x,yx,z] + x[y,x,z] = [\alpha 1+a,(\beta 1+b)(\alpha 1+a),\gamma 1+c]$ $+ (\alpha 1+a)[\beta 1+b,\alpha 1+a,\gamma 1+c] = [a,\beta a+\alpha b+ba,c] + (\alpha 1+a)[b,a,c] = \beta[a,a,c]$ $+ \alpha\{[a,b,c]+[b,a,c]\} + \{[a,ba,c]+a[b,a,c]\} = 0$ by (1.2a), (1.2a'), and (1.1a). Thus $\hat{A}$ inherits left Moufangitivity. ■

Since the defining axioms (1.1), (1.2) are quadratic they remain valid in all extensions, so the scalar extension A_Ω of a left alternative or Moufang algebra A remains left alternative or Moufang.

If A is unital then $L_1 = I$ and the left regular representation $x \to L_x$ of A in End(A) is injective ($L_x = 0 \Rightarrow x = L_x 1 = 0$). This need not hold if A is not unital (consider a trivial A!), but since any A is imbedded in a unital algebra A we have an injection $A \to \hat{A} \to \mathrm{End}(\hat{A})$. Furthermore, the left Moufang laws (1.1), (1.2) are just the conditions that $x \to L_x$ be a homomorphism of quadratic algebras, i.e., preserve the compositions $x^2 = xx$, $V_x y = x \circ y = xy+yx$, and $U_x y = x(yx)$:

$$L_{x^2} = L_x^2,\ L_{x \circ y} = L_x \circ L_y,\ L_{U(x)y} = L_x L_y L_x.$$

Since $2U_x = V_x^2 - V_{x^2}$ can be built out of squares and circles, if $x \mapsto L_x$ preserves squares it will also preserve $2U_x$. This is why left alternativity implies left Moufangitivity in characteristic $\neq 2$.

1.4 (Equivalence Theorem) Any left alternative algebra A satisfies

$$2L_{U(x)y} = 2L_x L_y L_x,$$

so if 2 is injective or surjective on A then A is actually left Moufang.

Proof. By (1.1 op) and its linearization

$$\begin{aligned}
&L_{2x(yx)} - 2L_x L_y L_x \\
&= L(\{x(yx+xy)-x(xy)\}+\{x(yx)+y(xx)-yx^2\}) \\
&\quad -\{L_x(L_y L_x + L_x L_y)+(L_y L_x + L_x L_y)L_x - (L_x^2 L_y + L_y L_x^2)\} \\
&= L(\{x(yx+xy)-x^2 y\}+\{(xy)x+(yx)x-yx^2\}) \\
&\quad -\{L_x \circ (L_x \circ L_y) - L_x^2 \circ L_y\} \\
&= L(x \circ (x \circ y) - x^2 \circ y) - \{L_{x \circ (x \circ y)} - L_{x^2 \circ y}\} \\
&= 0 \quad .
\end{aligned}$$

If 2 is injective we can cancel it from the above relation $2L_x L_y L_x z = 2L_{U(x)y} z$ to get the left Moufang law $L_x L_y L_x z = L_{U(x)y} z$ for all x,y,z. If 2 is surjective, every $z \in A$ has the form $z = 2w$, so applying the above to w gives $L_x L_y L_x z = 2L_x L_y L_x w = 2L_{U(x)y} w = L_{U(x)y} z$. ∎

The fact that $x \to \hat{L}_x$ is a monomorphism $A \to \mathrm{End}(\hat{A})$ of quadratic algebras allows us to carry back to A any quadratic identities which hold in $\mathrm{End}(\hat{A})$. For example, we have

$$(1.5) \qquad U_{x(yx)} = U_x U_y U_x \qquad \text{(Fundamental Formula)}$$

since $\hat{L}_{U(U(x)y)z} = \hat{L}_{U(x)y}\hat{L}_z\hat{L}_{U(x)y} = (\hat{L}_x\hat{L}_y\hat{L}_x)\hat{L}_z(\hat{L}_x\hat{L}_y\hat{L}_x) = \hat{L}_{U(x)U(y)U(x)z}$.

An **inner ideal** is a subspace B of A such that $U_B A \subset B$. The Fundamental Formula shows that each space $U_x A$ is an inner ideal, called the **principal inner ideal** determined by x: from (1.5) $U_{U(x)A}A = U_x(U_A U_x A) \subset U_x A$. As in the alternative case, these inner ideals play a more important role than the one-sided ideals.

Another consequence of the fact that $x \to \hat{L}_x$ is a monomorphism of quadratic algebras is

$$(1.6) \qquad L_{x^n} = L_x^n, \quad U_{x^n} = U_x^n$$

which also could have been proven directly from the axioms. This has as immediate consequence

1.7 (Power-Associativity Theorem) A left Moufang algebra is strictly power-associative,

$$x^n x^m = x^{n+m}$$

where $x^n = L_x^n 1$ (i.e., $L_x^{n-1}x$).

Proof. $x^n x^m = L_{x^n}L_{x^m}1 = L_x^n L_x^m = L_x^{n+m}1 = x^{n+m}$ by (1.6), and this remains valid in all scalar extensions A_Ω since they all remain left Moufang. ■

This allows us to talk about **idempotent** elements ($e^2 = e$) and **nilpotent** elements ($z^n = 0$) in the accustomed way.

1.8 Remark. Most well-behaved left alternative algebras turn out to be alternative. In establishing alternativity, it will be important to observe that A LEFT ALTERNATIVE OR LEFT MOUFANG ALGEBRA IS ACTUALLY ALTERNATIVE IF AND ONLY IF IT IS FLEXIBLE, $[x,y,x] = 0$, since then right alternativity follows from the linearization (1.2a'): $[y,x,x] = -[x,y,x] = 0$.

Exercises AIV.1

We can base the study of left Moufang algebras on the products x^2 and x^3 rather than x^2 and $U_x y$. Clearly $x^3 = U_x x$ can be defined in terms of U's, and conversely we can recover $U_x y$ from linearizing x^3.

1.1 Show that A is left Moufang iff it strictly satisfies

(1.1) $[x,x,z] = 0$

(1.$\hat{2}$) $[x^2,x,z] = 0,$

or in operator notation

(1.1 op) $L_{x^2} = L_x^2$

(1.$\hat{2}$ op) $L_{x^3} = L_x^3 \quad (x^3 = x^2 x = xx^2).$

1.2 Show that if 2 is injective or surjective then (1.1) implies (1.$\hat{2}$).

1.3 Linearize (1.$\hat{2}$) and (1.2) to show

$$(R_{xz} - R_z R_x)V_x = [R_z, L_{x^2}]$$

$$[R_z, L_x L_y] = R_{y(xz)} - R_z R_{yx}$$

1.4 Show L_x, U_x are idempotent (resp. nilpotent) if x is idempotent (resp. nilpotent).

1.5 Establish (1.5) and (1.6) directly from the axioms for a left Moufang algebra. Show $U_x V_{y,x} = V_{x,y} U_x = U_{U(x)y,x}$ where $V_{x,y}(z) = U_{x,z}y = \{U_{x+z} - U_x - U_z\}y = x(yz)+z(yx)$.

1.6 Define the Jordan powers of x by $x^0 = 1$, $x^1 = x$, $x^2 = xx$, and recursively $x^{n+2} = U_x x^n$. Show that $L_{x^n} = L_x^n$ in a left Moufang algebra, hence the Jordan power x^n coincides with the usual left power (defined by $x^{n+1} = xx^n$).

1.7 Prove the following are equivalent for a left alternative algebra A: (i) A is left Moufang, (ii) A satisfies left bumping $[x,yx,z]+x[y,x,z] \equiv 0$, (iii) $[x,yx,x]+x[y,x,x] \equiv 0$ holds strictly, (iv) $[x,x^2,z] = 0$ holds strictly, (v) $U_xV_x = V_xU_x$ holds strictly, (vi) $V_{x,x} = V_{x^2}$, $U_xV_x = V_xU_x$, $U_x(x^2) = (x^2)^2$, $(x^3)^2 = (x^2)^3$, $U_{x^2} = U_x^2$, $U_{x^3} = U_x^3$ hold strictly.

1.8 Prove that $\Phi x+U_xA$ is also an inner ideal in A, which automatically contains x.

AIV.1.1 Problem Set on Operator Nilpotence

If z is a nilpotent element in a left Moufang algebra, L_z and U_z are easily seen to be nilpotent operators. Because $R_{x^n} \neq R_x^n$ in general, it is not so clear that R_z is nilpotent.

1. From the associator identities (1.1a), (1.2a') derive the operator identities

(*) $$R_y R_x - R_{xy} = [L_x, R_y]$$

(**) $$[L_x, R_y] R_x = L_x [L_x, R_y].$$

2. Show that in any left Moufang algebra we have

$$R_x^k = R_{x^k} + \sum_{j=1}^{k-1} L_{x^{k-1-j}} [L_x, R_{x^j}].$$

3. Conclude that if $z^n = 0$ in a left Moufang algebra then $R_z^{2n} = 0$.

4. Alternately, if $z^n = 0$ use (*), (**) to show

$$R_z^{n+2m} = R_{z^2}^m R_z^n \qquad (z^n = 0).$$

Induct on the index of nilpotency to deduce that if $z^n = 0$ then some $R_z^{n+2m} = 0$.

5. Still a different proof proceeds to establish

$$R_z^{2n} = -\sum_{i=1}^{n-1} R_z^{n+i} L_z^{n-i} \qquad (z^n = 0)$$

and then by induction

$$R_z^{2n \cdot m} = \sum_{k=1}^{n} P_k^{(m)} L_z^m Q_k^{(m)} \qquad (z^n = 0)$$

where the $P_k^{(m)}$ are polynomials in R_z, the $Q_k^{(m)}$ in L_z. Conclude $R_z^{2nn} = 0$ if $z^n = 0$.