§4 Planes from rings

We have seen that we can construct affine planes from associative division rings. Actually, we can get by with much weaker algebraic structures. In Example 2.4 we built $\mathrm{Aff}(\Delta)$ out of ordered pairs (x,y) from Δ , with lines of the form $[a]$ or $[m,b]$. By (2.5) the incidence structure was given by $(x,y)I[a] \Longleftrightarrow x = a$ and $(x,y)I[m,b] \Longleftrightarrow y = xm + b$. Thus the addition and multiplication of Δ occur only in the form $xm + b$.

This suggests the following construction. Let R be any set with a ternary composition $F\colon R \times R \times R \longrightarrow R$ (thinking of $F(x,m,b) = xm + b$ in our example). An <u>isomorphism</u> $R \xrightarrow{\sigma} R$ of ternary systems is a bijective map preserving what little algebraic structure we have, namely the ternary composition

$$\sigma(F(x,m,b)) = \tilde{F}(\sigma(x), \sigma(m), \sigma(b)).$$

The <u>plane associated with the ternary system</u> (R,F) is $\mathrm{Aff}(R,F) = (P,L,I)$ where

$$(4.1)\quad \begin{cases} P(R,F) = R \times R = \{(x,y) \mid x,y \in R\} \\ L(R,F) = R \cup R \times R = \{[a],[m,b] \mid a,m,b \in R\} \\ I(R,F) \text{ is defined by} \\ \qquad (x,y)\ I\ [a] \Longleftrightarrow x = a \\ \qquad (x,y)\ I\ [m,b] \Longleftrightarrow y = F(x,m,b). \end{cases}$$

We say $\mathrm{Aff}(R,F)$ satisfies the <u>Parallel Criterion</u> if

$$(4.2)\qquad [m,b] \parallel [m',b'] \Longleftrightarrow m = m'.$$

This insures that the parameter m measures slope (as in the example of $\mathrm{Aff}(\Delta)$).

We say (R,F) is a <u>ternary ring</u> if it satisfies the axioms

(TR. I) for any $x,m,y \in R$ the equation

$F(x,m,b) = y$ has a unique solution b

(TR. II) for any $x,x',y,y' \in R$ the pair of equations

(4.3) $F(x,m,b) = y$, $F(x',m,b) = y'$ has a unique solution m,b if $x \neq x'$

(TR. III) for any $b,b',m,m' \in R$ the equation

$F(x,m,b) = F(x,m',b')$ has a unique solution x if $m \neq m'$.

These three rather strange axioms are precisely what we need to guarantee the plane Aff(R,F) is affine. We consistently write $F(x,m,b) = y$ even though at first glance all the variables x,m,b,y from R are on the same footing and could be denoted by any letter we choose; the reason is that x,y will be the x and y coordinates of points, m will be slope of a line and b the y-intercept of a line. To keep these roles straight we distinguish the variables by using different sorts of letters.

4.4 (Plane Construction Theorem) The plane Aff(R) constructed from a ternary system (R,F) is an affine plane satisfying the Parallel Criterion 4.2 iff (F,R) is a ternary ring with at least 2 distinct elements.

Proof. Consider the axiom Aff I that there be a unique line through any two points $P = (x,y)$, $P' = (x',y')$. If $x = x'$ then there is a unique line L of the form [a] incident to both points (namely $a = x = x'$), and no line of the form [m,b] since $y = F(x,m,b)$, $y' = F(x',m,b)$ for $x = x'$ forces $y = y'$, $P = P'$ by single-valuedness of F. If $x \neq x'$ there is no L = [a] on both points, and the condition that there be a unique L = [m,b] on both is that $y = F(x,m,b)$, $y' = F(x',m,b)$ have a unique solution m,b for $x \neq x'$. Thus Aff I is equivalent to TR. II.

The axiom that two lines L,L' intersect in exactly 1 or 0 points involves three cases. If L = [a], L' = [a'] for $a \neq a'$ the lines do not intersect, since if (x,y) were on L and L' we would have $a = x = a'$. If L = [a], L' = [m,b] then (x,y) is on L and L' iff $x = a$ and $y = F(x,m,b)$, so the unique point of intersection is $(a,F(a,m,b))$. If L = [m,b], L' = [m',b'] then the points of intersection $P = (x,y)$ are the solutions of the equations $y = F(x,m,b) = F(x,m',b')$. The Parallel Criterion is that no solution exist (the lines are parallel) iff $m = m'$. Thus Aff II and the Parallel Criterion together are equivalent to the condition that $F(x,m,b) = F(x,m',b')$ have a unique solution if $m \neq m'$ and no solution if $m = m'$ $(b \neq b')$, ie. to TR. III and the uniqueness part of TR. I.

The axiom Aff III that through each $P = (x,y)$ there is a unique line L' parallel to L breaks into two cases. If L = [a] we saw the only lines parallel to L are the L' = [a'], and there is only one of these incident to P (namely a' = x). If L = [m,b] we saw the only lines parallel to L are the L' = [m,b'], which is on (x,y) iff y = F(x,m,b'), so Aff III is equivalent to the existence of unique solutions b' of equations y = F(x,m,b'). This is just TR. I.

Thus we have a pre-affine plane iff (R,F) is a ternary ring. The axiom Aff IV that a 3-point exist amounts to the condition that $|R| \geq 2$: if R = {r} contains only one element then the plane contains only one point (r,r), while if R contains $r \neq s$ then already (r,r), (r,s), (s,s) form a 3-point (they are not all on a common line L since if they are on L = [a] we would have r = a = s, and if they are on L = [m,b] then r = F(r,m,b) = s). ∎

This construction is <u>functorial</u> in the sense that any isomomorphism $\sigma: R \to \tilde{R}$ induces an isomorphism $Aff(\sigma): Aff(R) \to Aff(\tilde{R})$ defined by

$$(x,y) \longrightarrow (\sigma(x), \sigma(y))$$
$$[m,b] \to [\sigma(m), \sigma(b)]$$
$$[a] \longrightarrow [\sigma(a)].$$

How can this help but be an isomorphism of planes? Aff(R) is built up from R using only the ternary structure of R, so any map preserving this ternary structure must also preserve the derived geometric structure. For skeptics: $Aff(\sigma)$ certainly is bijective from points to points and lines to lines, and preserves incidence since

$(x,y)\ I\ [a] \Longleftrightarrow x = a \Longleftrightarrow \sigma(x) = \sigma(a) \Longleftrightarrow (\sigma(x), \sigma(y))\ \tilde{I}[\sigma(a)]$

and $(x,y)\ I[m,b] \Longleftrightarrow F(x,m,b) = y \Longleftrightarrow$

$\tilde{F}(\sigma x,\sigma m,\sigma b) = \sigma(F(x,m,b)) = \sigma y \Longleftrightarrow (\sigma x,\sigma y)\ \tilde{I}[\sigma m,\sigma b]$. The other requirements for a functor are trivially met: if $\sigma = 1$ is the identity map from R to R by its very definition $Aff(\sigma) = 1$ is the identity map on Aff(R), and similarily if $R \xrightarrow{\sigma} R \xrightarrow{\tau} R$ then by definition $Aff(\tau) \circ Aff(\sigma) = Aff(\tau\circ\sigma)$. Thus we have a functor

$$\text{Ternary rings} \xrightarrow{Aff} \text{Affine planes.}$$

In short, we can construct affine planes in a natural way from ternary rings.

We noticed before that we could construct ternary systems out of rings. Indeed, if $(R,+,\cdot)$ consists of a set R with two binary operations + and $\cdot$ we can form a ternary composition

$$F(x,m,b) = x\cdot m + b .$$

The conditions on $(R,+,\cdot)$ in order that (R,F) be ternary are

(4.5)
- (R.I) an equation $x\cdot m + b = y$ has unique solution b
- (R.II) a pair of equations $x\cdot m + b = y$, $x'\cdot m + b = y'$ for $x \neq x'$ have a unique solution m,b
- (R.III) an equation $x\cdot m + b = x\cdot m' + b'$ for $m \neq m'$ has a unique solution x.

Clearly the construction of (R,F) from $(R,+,\cdot)$ is functorial. In the next section we will investigate when $(R,+,\cdot)$ can be recovered from (R,F).

Exercise

4.1 A ternary ring is rigid in the same way a division ring is rigid: a homomorphism must be zero or injective. Define <u>homomorphism</u> of ternary rings and show that if $\sigma: R \to \tilde{R}$ is a homomorphism of ternary rings then either

(i) $\sigma(R) = \{ r_o \}$

(ii) σ is injective.

The fact that ternary rings admit essentially only isomorphisms explains why planes admit essentially only isomorphisms:

4.2 A <u>zero element</u> 0 and <u>unit element</u> 1 of a ternary ring are elements satisfying $F(0,x,b) = F(x,0,b) = b$, $F(1,m,0) = F(m,1,0) = m$ for all x,m,b. Show the zero and unit of a ternary ring are unique if they exist.

4.3 An <u>isotope</u> $(\tilde{R},\tilde{F})$ of a ternary ring (R,F) is defined by $\tilde{R} = R$, $\tilde{F}(x,m,b) = F(\sigma(x), \rho(m), \tau(b))$ for bijective $\rho,\sigma,\tau : R \to R$. Show every such isotope is again a ternary ring. Does every ternary ring have an isotope which is unital: if $r \neq s$ is there an isotope where $r = 0$, $s = 1$?

4.4 Show that if the plane associated with (R,F) is affine, so is the plane associated with $(R,\tilde{F})$ for any map $\tilde{F}(x,y,z) = F(x,\sigma(y,z),\tau(y,z))$ where $(y,z) \to (\sigma(y,z),\tau(y,z))$ is a bijection of $R \times R$ onto itself. Conclude that Aff(R,F) does not in general satisfy the Parallel Criterion.

4.5 Given an arbitrary $\tilde{F}$ such that Aff(R,$\tilde{F}$) is affine, show there is F with $\tilde{F}(x,y,z) = F(x,\sigma(y,z),\tau(y,z))$ such that Aff(R,F) does satisfy the Parallel Criterion. Thus we can always normalize F so that m measures slope.

4.6 In Aff(R,F) where R is unital (as in #2) show (i) $[x] \wedge [m,b] = (x, F(x,m,b))$, (ii) $[m,b] = (0,b) \vee ((0,0) \vee (1,m))$ (iii) $[1,0] \wedge [0,y] = (y,y)$, $[x] \wedge [1,0] = (x,x)$, $[0] \wedge [m,b] = (0,b)$, $[x] \wedge [0,b] = (x,b)$, (iv) $(x,y) \vee [0,0] = [0,y]$, $(x,y) \vee [0] = [x]$.