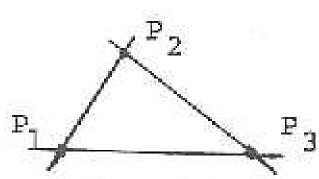
§2 Affine planes

More familiar to us are the affine planes. The ordinary real x-y plane of calculus is an example of an affine plane. To be axiomatic about it, an <u>affine plane</u> $\Pi_a = (P,L,I)$ is a plane satisfying the axioms

(2.1)

- (Aff I) any two distinct points are incident to a unique line
- (Aff II) any two distinct lines are either parallel or are incident to a unique point
- (Aff III) given a point P and a line L, there is a unique line incident to P and parallel to L
- (Aff IV) there exists a 3-point.

Here two lines are <u>parallel</u>, written $L \parallel L'$, if either $L = L'$ or there is no point incident to both (their intersection is everything or nothing). A <u>3-point</u> consists of three non collinear points $\{P_1, P_2, P_3\}$.

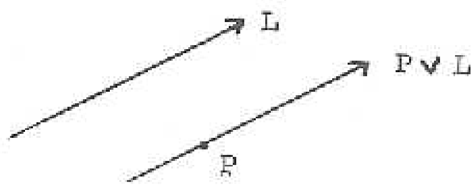


3-point

Because all points are alike but lines come in two varieties (parallel or intersecting), affine planes lose the duality enjoyed by projective planes.

As in the projective case, we write $P \vee Q$ for the unique line on $P \neq Q$, $L \wedge L'$ for the unique point of intersection of

L and L' if $L \not\parallel L'$ (so before writing $L \wedge L'$ we must check that L, L' are not only distinct but also non-parallel). The unique line through P parallel to L guaranteed by Aff III will be denoted by $P \vee L$:



The line $P \vee L$

Note that if P lies on L already then $P \vee L$ is just L.

In the next section we will see a general method of obtaining affine planes from projective ones by deleting a "line at infinity;" parallelism in the resulting plane means the lines intersect on this deleted line, which no longer exists within the affine plane but only has an "ideal" existence in the surrounding projective plane. Thus the "points at infinity" are constructed from the affine plane as points of intersection of parallel lines, so are in 1-1 correspondence with parallel classes of lines.

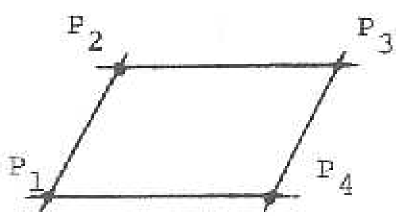
Let us note that parallelism is indeed an equivalence relation, so that

$$\mathcal{L} = \bigcup \parallel(L)$$

breaks up into a disjoint union of parallel classes (the <u>parallel class</u> of L is denoted $\parallel(L)$). We have reflexivity $L \parallel L$

by definition, as well as symmetry $L \| M \Longleftrightarrow M \| L$, while relexivity $L \| M$, $M \| N \Longrightarrow L \| N$ follows since if $L \nparallel N$ then $L \neq N$, $N \neq M$, $M \neq L$ so by Aff II L and N intersect at a point P, whereupon L and N would be distinct lines on P parallel to M , which would contradict uniqueness Aff III.

Affine planes not only contain 3-points, they contain 4-points. An ordered set $\{P_1, P_2, P_3, P_4\}$ of distinct points in an affine plane is called a <u>parallelogram</u> if they are not collinear and $P_1 \vee P_2 \| P_3 \vee P_4$, $P_2 \vee P_3 \| P_4 \vee P_1$.



Parallelogram

A parallelogram is automatically a 4-point. Since the parallelogram condition is invariant under cyclic permutation of P_1, P_2, P_3, P_4, it suffices to check P_1, P_2, P_3 are not collinear. If they were collinear, $P_1 \vee P_2 = P_2 \vee P_3$ would imply $P_2 \vee P_3 = P_4 \vee P_1$ (since they are parallel and pass through P_1), all four points would be collinear, contrary to hypothesis.

2.2 (Parallelogram Lemma) Any 3 noncollinear points P_1, P_2, P_3 uniquely determine a 4th point $P_4 = \{P_1 \vee (P_2 \vee P_3)\} \wedge \{P_3 \vee (P_1 \vee P_2)\}$ such that $\{P_1, P_2, P_3, P_4\}$ is a parallelogram.

Proof. Any P_4 must lie on $P_3 \vee (P_1 \vee P_2)$ if $P_3 \vee P_4 \| P_1 \vee P_2$, similarly it must lie on $P_1 \vee (P_2 \vee P_3)$ if $P_4 \vee P_1 \| P_2 \vee P_3$. These lines intersect in a single point since they are non-parallel: if $P_2 \vee P_3 \| P_1 \vee P_2$ then $P_2 \vee P_3 = P_1 \vee P_2$ and P_1, P_2, P_3 are collinear, which we disproved above. Thus P_4 can only be $\{P_1 \vee (P_2 \vee P_3)\} \wedge \{P_3 \vee (P_1 \vee P_2)\}$, and this point works: P_1 is off $P_2 \vee P_3$, so $P_1 \vee (P_2 \vee P_3)$ is disjoint from $P_2 \vee P_3$, and P_4 is off $P_2 \vee P_3$; similarly it is off $P_1 \vee P_2$. Then $P_4 \vee P_3 = P_3 \vee (P_1 \vee P_2) \| P_1 \vee P_2$ and $P_4 \vee P_1 = P_1 \vee (P_2 \vee P_3) \| P_2 \vee P_3$, so $\{P_1, P_2, P_3, P_4\}$ is a parallelogram. ∎

Besides constructing affine planes from projective planes, it is quite easy to build them from scratch.

2.3 Example (Vector space planes). If V is a 2-dimensional left vector space over an associative division ring Δ then we obtain an affine plane Aff(V) (the plane of the vector space V) by taking the elements of V as points and the 1-dimensional affine subspaces (=translates of 1-dimensional linear subspaces) as lines. Thus a line has the form of a coset $L_{P,Q} = P + \Delta Q$ for P, $Q \neq 0$ points in V. Two such lines $L_{P,Q}$ and $L_{P',Q'}$ are parallel iff they are translates of the same subspace $\Delta Q = \Delta Q'$ (ie. Q, Q' are dependent), and coincide iff $P - P' \in \Delta Q = \Delta Q'$. Two non-parallel lines $L_{P,Q}$ and $L_{P',Q'}$ have unique intersection $R = P + tQ = P' + t'Q'$ where t, t' are the unique solutions to $P - P' = t'Q' - tQ$ (since Q, Q'

form a basis for V). The unique line through P parallel to $L = L_{P',Q'}$ is $L_{P,Q'}$. This establishes Aff II and Aff III. For Aff I note that if P,P' are distinct then $P - P' = Q \neq 0$ so both lie on $L_{P,Q} = L_{P',Q}$. For Aff IV note that if $\{X,Y\}$ is a basis for V then $P_1 = 0$, $P_2 = X$, $P_3 = Y$ are non-collinear.

Note that points P,Q,R are collinear iff the vectors $P - Q$, $Q - R$, $R - P$ all lie in a 1-dimensional subspace of V (ie. $P \equiv Q \equiv R$ modulo a subspace W of V).

Any semilinear isomorphism $V \xrightarrow{\sigma} \tilde{V}$ of vector spaces over $\Delta, \tilde{\Delta}$ induces an isomorphism of planes $\mathrm{Aff}(V) \xrightarrow{\mathrm{Aff}(\sigma)} \mathrm{Aff}(\tilde{V})$ by $P \to \sigma(P)$, $L = P + \Delta Q \to \sigma(P) + \tilde{\Delta}\sigma(Q)$. (An easy way to see this is to note σ preserves collinearity since it preserves dependence). This gives us a functor

$$\text{Vector spaces} \xrightarrow{\mathrm{Aff}} \text{Affine planes}$$

from the category of 2-dimensional vector spaces with semi-linear isomorphism to the category of affine planes. ■

2.4 Example (Division ring planes) With every division ring Δ we can construct a canonical 2-dimensional vector space $V_a(\Delta) = \Delta^2$. Here isomorphisms $\Delta \xrightarrow{\sigma} \tilde{\Delta}$ induce semilinear isomorphisms $V_a(\Delta) \xrightarrow{V_a(\sigma)} V_a(\tilde{\Delta})$ by $(x,y) \to (\sigma(x), \sigma(y))$. Thus we have a functor from division rings to vector spaces; composing it with the previous functor gives a functor from division ring Δ to affine planes

$$\mathrm{Aff}(\Delta)$$

(<u>the affine plane of Δ</u>). Here points are ordered pairs $P = (x,y)$ $(x,y \in \Delta)$ and lines are represented in parametric form $x = a + tc$, $y = b + td$ for $(c,d) \neq (0,0)$. If $c = 0$ this is just the set of points (x,y) with $x = a$, otherwise we can solve for $t = (x - a)c^{-1}$ to write $y = b + (x - a)c^{-1}d = xm + b$ $(m = c^{-1}d,$ $b = b - ac^{-1}d)$. The expressions $L = [a]$ or $L = [m,b]$ uniquely determine the line. The incidence structure is determined by

(2.5) $$(x,y) \ I \ [a] \Longleftrightarrow x = a$$
$$(x,y) \ I \ [m,b] \Longleftrightarrow y = xm + b \ .$$

Later we will see that we can introduce similar sorts of coordinates in an arbitrary affine plane.

Since every 2-dimensional left vector space V over Δ is (non-canonically) isomorphic to $V_a(\Delta)$ relative to a choice of basis, all vector space planes are isomorphic to division ring palnes, $\mathrm{Aff}(V) \cong \mathrm{Aff}(V_a(\Delta)) = \mathrm{Aff}(\Delta)$. ∎

Exercise

2.1 If Π_a is a finite affine plane such that one line L contains exactly n points, show

(i) every line contains n points

(ii) there are n^2 points in all

(iii) through each point there are n+1 lines

(iv) there are n^2+n lines in all

(v) there are n lines parallel to any given line.

2.2 Show all affine planes are concrete.

2.3 Show a bijection $P \xrightarrow{\sigma} \tilde{P}$ induces an isomorphism of affine planes iff it preserves collinearity: P_1, P_2, P_3 collinear $\Longleftrightarrow \sigma(P_1), \sigma(P_2), \sigma(P_3)$ collinear.

2.4 Show any isomorphism $\Pi_a \xrightarrow{\sigma} \tilde{\Pi}_a$ preserves parallelism, $L || M \Longleftrightarrow \sigma(L) || \sigma(M)$. If a bijection $L \xrightarrow{\sigma} \tilde{L}$ is given which preserves parallelism, show σ extends to an isomorphism of affine planes iff $L \wedge L_1 = L \wedge L_2 \Longrightarrow \sigma(L) \wedge \sigma(L_1) = \sigma(L) \wedge \sigma(L_2)$. Give an example where this condition is not met.

2.5 Define <u>homomorphism</u> of affine planes. Show that a homomorphism $\Pi_a \xrightarrow{\sigma} \tilde{\Pi}_a$ is either injective (isomorphism into) or else collapses Π_a to a point, $\sigma(P_a) = \{\tilde{P}\}$ and $\sigma(L_a) \subset L_a(\tilde{P})$, or collapses it to a line, $\sigma(L) = \{\tilde{L}\}$ and $\sigma(P_a) \subset \tilde{P}_a(\tilde{L})$.