§6 The Herstein-Kleinfeld-Osborn Theorem

Although nuclear involutions cannot be so closely characterized as ~~scalar~~ central involutions, we can precisely characterize those whose norms are all invertible or zero. Alternative algebras with such involutions ~~will be~~ are important ~~in Part 3~~ in coordinatizing Jordan algebras.

6.1 (Herstein-Kleinfeld-Osborn Theorem) If D is a semiprime unital alternative algebra with nuclear involution having an ample subspace D_o of the nucleus all of whose nonzero elements are invertible, then either

(I) D is a direct sum $\Delta \boxplus \Delta^*$ of anti-isomorphic associative division algebras under the exchange involution, and $D_o = \{\delta \oplus \delta^* \mid \delta \in \Delta\} = \mathrm{Sym}(D,*)$.

(II) D is an associative division ring Δ with involution and D_o an ample subspace Δ_o.

(III) D is a split quaternion algebra $M_2(\Omega)$ over its center Ω with standard involution and $D_o = \Omega 1$.

(IV) D is a Cayley algebra C over its center Ω with standard involution and $D_o = \Omega 1$.

Proof. We begin by showing D is actually $*$-simple, not merely semiprime. If $B \triangleleft D$ were a proper $*$-ideal it couldn't contain invertible elements, so $B \cap D_o = 0$. For $b \in B$ and $x \in D$ we have $n(b) = bb^*$, $t(bx) = bx + x^*b^* \in B \cap D_o$ since B is a $*$-ideal and D_o contains all norms and traces.

But

$$(6.2)\qquad n(z) = t(zD) = 0 \Longrightarrow Dz = zD \text{ is trivial } *\text{-ideal.}$$

We have $z^* = -z$ since $t(z) = 0$, so $zx = x^*z$ from $t(zx) = 0$. This shows $Dz = zD$ is $*$-invariant. It is a left ideal since $x(z^*y) + z(x^*y) = n(x,z)y = t(zx^*)y = 0$ shows $x(zy) = z(x^*y)$. Similarly it is right, and it is trivial because $(zD)(Dz) = zD^2z = z(zD^2) = -z(z^*D^2) = -n(z)D^2 = 0$. If D is semiprime this forces $z = 0$ for all $z \in B$ and $B = 0$.

First assume D is not simple, so it has a proper ideal Δ. Since $\Delta + \Delta^*$ is a nonzero $*$-ideal, by $*$-simplicity $\Delta + \Delta^* = D$; since $\Delta \cap \Delta^*$ is a $*$-ideal properly contained in D, $\Delta \cap \Delta^* = 0$ by $*$-simplicity. Thus $D = \Delta \boxplus \Delta^*$. Here every symmetric element $\delta \oplus \delta^*$ is a trace, so $D_o = H(D_\Delta, *)$. The fact that $\delta \oplus \delta^*$ for $\delta \neq 0$ is invertible and belongs to the nucleus of D forces δ to be invertible and in the nucleus of Δ, ie. Δ is an associative division algebra. This is Case I.

From now on we assume D is simple, not just $*$-simple. Then its center Ω is a field; if Ω_o denotes the $*$-center (the subfield of Ω fixed by $*$) then D may be regarded as an algebra with involution over Ω_o.

~~First suppose D is a division algebra. If D is associative we have case II. If D is not associative, its nucleus N and center Ω coincide by the Nucleus = Center Theorem III.1.10. Therefore the nuclear involution is actually a central involution, indeed all norms lie in Ω_o since they are symmetric and lie in $N(A) = \Omega$. Thus $*$ is a scalar involution over Ω_o~~

P If D is an associative division ring we have Case II, so we may suppose D is either not associative or not a division ring. If we can show * is a scalar involution on Ω_0, D will be a composition algebra [illegible] VII.[illegible], so by Hurwitz theorem VII.4.1 D is either a Cayley algebra (if not associative) or a split quaternion algebra (if not a division ring but simple and associative). In the quaternion or Cayley case [illegible] $\Omega_0 = C(D) \subset \Omega_0 \subset \Omega \subset N(D) = \Omega_0$ by VII.3.[illegible] since [illegible] $O_0 = \Omega_0 = \Omega$. Thus as soon as * is a scalar involution on Ω_0 we have Case III or IV of the theorem.

P To show * is a scalar involution we must show all norms lie in the *-center Ω_0. Since the norms are symmetric and lie in the nucleus by hypothesis, we need only show they commute with D. If D is a not-associative division ring, this is trivial: by the Nucleus = Center Theorem II.1.10 any nuclear element lies in the center. Henceforth we will assume D is simple but not a division ring.

If we define $n(x) = xx^*$, $m(x) = x^*x$ we have $n(x)x = xx^*x = xm(x)$, so $n(x), m(x) \in D_0$ are either both zero or both invertible. If they are both invertible x has left and right inverses, hence is invertible, while if they are both zero x cannot be invertible, so

(6.3) $\quad n(z) = 0 \iff m(z) = 0 \iff z$ is not invertible.

The relations $n(xy) = xn(y)x^*$, $m(xy) = y^*m(x)y$ (everything takes place in the associative subalgebra generated by x,y,N) show that if $n(z) = 0$ then $n(xz) = m(zx) = 0$ for any x. In view of (6.3) this becomes

(6.4) $\quad n(z) = 0 \Rightarrow n(xz) = n(zx) = 0 \qquad (x \in D)$.

Now since D is not a division algebra we have $n(z) = 0$ for some $z \neq 0$. By (6.2) $t(zx) \neq 0$ for some x, so ~~$e + f = 1$ for $e = zx\,t(zx)^{-1}$, $f = x^*z^*\,t(zx)^{-1}$. We have $n(e) = n(f) = 0$~~

Replacing zx by z we may assume $n(z) = 0$ but $t(z) \neq 0$. Note z and z^* commute [illegible], so $t(z)$ and $t(z)^{-1}$ commute with z, z^*. Then $1 = t(z)^{-1}z + t(z)^{-1}z^*$ [illegible] $= e + e^*$ where $zz^* = z^*z = 0$ implies $ee^* = $ [illegible] $= 0$, and [illegible] so $e^2 = e$ and $e^{*2} = e^*$ [illegible] by (2.4) since e, f involve a factor z, z^*; therefore $f = f1^* = f(e^* + f^*) = fe^* = (e + f)e^* = 1e^* = e^*$ and

$$1 = e + e^* = e_1 + e_2$$

where e_1, e_2 are orthogonal idempotents by $e_1 e_2 = ee^* = 0$, $e_2 e_1 = ff^* = 0$.

We are trying to prove traces and norms lie in Ω_o . Since they already are nuclear and symmetric, we need only show they commute with D. If they commute with the Peirce spaces D_{12}, D_{21} then (since they are nuclear) they will commute with the products $D_{11} = D_{12}D_{21}, D_{22} = D_{21}D_{12}$ by Interconnectivity VII.5.5 (remember D is simple), and therefore with all of $D = D_{11} + D_{12} + D_{21} + D_{22}$ as desired.

But any element $c \in D_o$ commutes with $D_{12} = eDe^*$ since in the associative subalgebra generated by e, x, and $D_o \subseteq N$ we have $c(exe^*) - (exe^*)c = cexe^* + ex^*e^*c = (e + e^*)cexe^* + ex^*e^*c(e + e^*)$ $= ecexe^* + ex^*e^*ce^* = e\{cex + x^*e^*c\}e^* = 0$ ~~since~~ because $et(x)e^*, e^*ce$, $et(cex)e^*$ all lie in $eD_oe^* = e^*D_oe \subseteq D_o \cap$ {elements of norm zero} $= 0$. Similarly c commutes with D_{21} . ∎

IX.6 Exercises

2.1 Show that any *-simple alternative algebra A is either simple or the direct sum $A = B \boxplus B^*$ of two anti-isomorphic simple algebras under the exchange involution. If * is nuclear, show B is associative.

2.2 If Z_ℓ, Z_r, Z are the elements without left, right, or two-sided inverses (A unital, * nuclear, nonzero elements of D_o invertible) show $Z = Z_r = Z_\ell$. Deduce $Z^* = Z, xZ \subset Z, Zx \subset Z$ for all $x \in D$. If $Z + Z \subseteq Z$ show Z is a proper *-ideal. If $Z + Z \not\subseteq Z$ show $e + e^* = 1$ for $e \in Z$.

2.3 If $1 = e_1 + e_2$ where $e_1^* = e_2$ (* nuclear, nonzero elements of D_o invertible) show $D_{11}^* = D_{22}, t(D_{21}) = 0$, and $[t(D), D_{12}] = 0$. If D is simple deduce $t(D) \subseteq C(D)$.

2.4 If D, * are as in ~~#3~~ Ex 3 show every norm $n(x) = xx^*$ of $x \in D = D_{11} + D_{12} + D_{21} + D_{22}$ is actually a trace $t(y)$ of $y \in D_{11}$.

2.5 If D, * are as in ~~#3~~ Ex 3 show D_{ii} are division algebras directly, and also by showing $D_{ii} = e_i D_o e_i$. Conclude from the Capacity Two Theorem that $D = M_2(\Delta)$ for an associative division algebra, or $D = \mathbb{C}$ is a Cayley algebra over Ω. Argue $\Delta = \Omega$ so $D = M_2(\Omega)$ is split quaternion. Show $\Omega = \Omega_o$ in these cases so * is standard.