§6 Malcev's theorem

The Wedderburn Splitting Theorem tells us how A is built up from its radical R and its semisimple part $^A/_R$. In the semi-direct sum $A = B \oplus R$ the radical part R is unique, and B is determined up to isomorphism. The exact extent to which B is unique is described by

6.1 (Malcev's Theorem) If A is a unital alternative algebra over a field of characteristic $\neq 2,3$ whose radical R is solvable and whose semisimple part A/R is separable, then any two Wedderburn splittings of A are conjugate: if $A = B_1 \oplus R = B_2 \oplus R$ for subalgebras B_i then B_1 and B_2 are conjugate under an inner automorphism (which is a product of conjugations).

This will follow from a more general conjugacy result

6.2 (Malcev's General Theorem) If A is a unital alternative algebra over a field of characteristic $\neq 2,3$ whose radical R is solvable and whose semisimple part A/R is separable, then for any separable unital subalgebra C and Wedderburn splitting $A = B \oplus R$ there is an inner automorphism (a product of conjugations) of A taking C to a subalgebra of B.

Proof. As in the Wedderburn Splitting Theorem, we prove this by induction on the index n of the solvable ideal R . If $n = o$ there is nothing to prove, $B = A$.

Now assume R has index 1, $D(R) = R^2 = 0$. If β,ρ are the projections of $A = B \oplus R$ on B,R respectively then any $x \in A$ may be written $x = \beta(x) + \rho(x)$. Since B is a subalgebra and R an ideal with $R^2 = 0$ we have $xy = \{\beta(x) + \rho(x)\}\{\beta(y) + \rho(y)\}$ $= \beta(x)\beta(y) + \beta(x)\rho(y) + \rho(x)\beta(y) = \beta(x)\beta(y) \oplus \{x\rho(y) + \rho(x)y\}$, so

$$(6.3) \qquad \beta(xy) = \beta(x)\beta(y), \qquad \rho(xy) = x\rho(y) + \rho(x)y .$$

Thus β is a homomorphism $A \to B$ and ρ a derivation $A \to R$.

The trouble is that β is not an automorphism of A. What we seek is a conjugation T on all of A which has the same effect on C that β does,

$$T(x) = \beta(x) \qquad (x \in C)$$

since then $T(C) = \beta(C) \subset B$ as desired. Offhand it might be hard to discover a T extending β , but if we observe $\beta(x) = x - \rho(x)$ we need only extend $I - \rho$, ie find T satisfying

$$(6.4) \qquad T(x) = x - \rho(x) \qquad (x \in C) .$$

The advantage of this formulation is that we know how to find automorphism $T \overset{\sim}{=} I - D$ infinitesimally generated by a derivation. D

The restriction D of ρ to C is a derivation $C \to R$, which because C is separable must be inner by the Derivation Theorem 4, and since the characteristic $\neq 2,3$ even intrinsically inner: $D = Ad_n + \Sigma D_{x_i,m_i}$ for $x_i \in C$, $m_i \in R$, and $n \in e\ Re \subset N(A)$ for $e \in C$ a strongly associative idempotent. Because $D_{x,m}$ is linear in x and C is spanned by invertible elements (see 0,00) we can assume all x_i are invertible. The inner derivation

$D = Ad_n + \Sigma D_{x_i,m_i}$ extends to all of A. By IV.3.35 the conjugation

$$T = T_y \Pi T_{x_i,y_i} \qquad (y = 1-n,\ y_i = 1 - m_i x_i)$$

on A has $T_x \equiv x - Dx$ modulo terms involving two or more factors from R and therefore belonging to $R^2 = 0$,

$$T(x) = x - D(x) = x - \rho(x) \qquad (x \in C).$$

By (6.4) this is precisely the T we've been looking for.

Assume we have found conjugations whenever the index is less than n, and suppose R has index n: $D^n(R) = 0$. In $\bar{A} = A/D^{n-1}(R)$ we have $\bar{C} \cong C$ separable and $\bar{A} = \bar{B} \oplus \bar{R}$ where $\bar{R}$ now has index $n-1$, $D^{n-1}(\bar{R}) = \overline{D^{n-1}(R)} = \bar{0}$. By the inductive hypothesis there is a conjugation

$$\bar{T} = T_{1-\bar{z}} \Pi T_{\overline{x_i},1-\overline{z_i}} \quad (\overline{x_i} \in \bar{C} \text{ invertible},$$

$\overline{z_i} \in \bar{R}$, $\bar{z} \in \bar{e}\bar{R}\bar{e}$ for $\bar{e} \in \bar{C}$ strongly associative (rel. $\bar{C}$) with $\bar{T}(\bar{C}) \subset \bar{B}$. We can identify $\bar{e},\overline{x_i}$ in $\bar{C}$ with e,x_i in C and lift $\bar{z} \in \bar{e}\bar{R}\bar{e}$, $\overline{z_i} \in \bar{R}$ to $z \in eRe$, $z_i \in R$. Then z,z_i are still nilpotent and $z \in eRe \subset eAe \subset N(A)$ is still nuclear by strong associativity of e (rel. C), so

$$T = T_{1-z} \Pi T_{x_i,1-z_i}$$

is a conjugation on A which induces $\bar{T}$ on $\bar{C}$. In particular, $\overline{T(C)} = \bar{T}(\bar{C}) \subset \bar{B}$ implies $C_o = T(C) \subset B + D^{n-1}(R) = A_o$. Now the radical $R_o = D^{n-1}(R)$ of A_o has index 1, $D(R_o) = D(D^{n-1}(R)) = D^n(R) = 0$. Therefore we can apply the index 1 case to $C_o \subset B + R_o$ to obtain a conjugation

$$T_o = T_{1-w} \prod T_{y_i, 1-w_i}$$

($y_i \in C_0$ invertible, $w_i \in R_0$, $w \in fR_0 f$ for $f \in C_0$ strongly associative) of A_o with $T_o(C_o) \subset B$. This extends to a conjugation on A since y_i are still invertible in A, $w_i \in R$, and $w \in fRf \subset fAf \subset N(A)$ for f strongly associative (rel. C_o). The composite T_oT is a conjugation on A with $T_oT(C) = T_o(C_o) \subset B$ as desired.

This completes the induction. ■

Malcev's Theorem can be thought of as the uniqueness theorem corresponding to the Wedderburn Splitting Theorem.

6.5 Corollary. Let A be a finite-dimensional alternative algebra over a field of characteristic $\neq 2,3$ with A/R separable. Then any separable subalgebra C of A is contained in a Wedderburn factor of A, $C \subset B$ for $A = B \oplus R$.

Proof. If $A = \tilde{B} \oplus R$ there is an automorphism T of A with $T(C) \subset \tilde{B}$, ie $C \subset T^{-1}(\tilde{B}) = B$ where $A = T^{-1}(A) = T^{-1}(\tilde{B}) \oplus T^{-1}(R)$ $= B \oplus R$. ■

6.6 Remark. In characteristic 0, if C is a semisimple subalgebra of $A = B \oplus R$ such that every derivation $C \to M$ is intrinsically inner, then C is conjugate to a subalgebra of B under an automorphism $T = \Pi \exp(D_i)$ for D_i nilpotent derivations of A. Indeed, in the proof once we know $C \xrightarrow{D} R$ is inner we can by hypothesis lift it to a derivation D of A , which is covered by the automorphism $\exp(D) = \Sigma\, D^k/k!$ (D is nilpotent and the characteristic is 0 !). Thus in characteristic 0 we can avoid the Infinitesimal Conjugation Lemma and use exponentials of derivations. ∎

Exercises

1. Show that Malcev's General Theorem follows if there exist conjugations moving C closer and closer to B , ie. if there exist $T^{(k)}$ with $T^{(k)}(C) \subset B + D^k(R)$.

2. Reduce the inductive construction of the successive approximations $T^{(k)}$ to the Lemma: if D is a separable subalgebra of an algebra $A = B \oplus S$ (B separable subalgebra, S a nil ideal) over a field of characteristic $\neq 2,3$ then there is a conjugate T(D) lying inside $B \oplus D(S)$.

3. Reduce the finding of T in #2 to finding T with $T(x) \equiv x - \sigma(x)$ mod D(S) for $x \in S$ ($\sigma : A \to S$ the projection on S). Show there is an intrinsically inner derivation D of A with $D(x) \equiv \sigma(x)$ mod D(S) for $x \in D$. Show there is a conjugation T of A with $T(x) \equiv x - \sigma(x)$ as desired.