§5 Splitting theorems

In this section we investigate how a general alternative algebra is built up out of its radical Rad(A) and its semisimple part A/Rad A. We hope to show that A is a semidirect product of its semisimple and radical parts, $A = B \oplus \text{Rad } A$ where B is a subalgebra (necessarily isomorphic under the canonical projection to A/Rad A).

We will need to make an assumption on each of the two pieces. Concerning the radical we will need to assume Rad A is solvable, which is weaker than assuming A is Artinian (by VI.0.0 an artinian algebra always has solvable radical). For the semisimple part we will have to make a stronger assumption than being artinian namely that Φ is a field and A/Rad A is separable over Φ. As in the associative case, the difficulty in lifting an $M_n(\Delta)$ or $\mathfrak{C}(\Delta)$ is not in lifting the matrix units, but rather in lifting Δ. Separability over a field Φ allows us to make a scalar extension which reduces us to the case when Φ is algebraically closed; but then the only choice for Δ is Φ itself, which lifts automatically. Thus by means of scalar extensions we can avoid explicitly trying to lift Δ.

5.1 (Wedderburn Splitting Theorem) If A is an alternative algebra over a field Φ whose radical R is solvable and whose semisimple part A/R is separable, then A is a semidirect product of its radical and its semisimple part: we have a Wedderburn splitting

$$A = B \oplus R \cong A/R \oplus R$$

for some separable subalgebra $B \cong A/R$.

Proof. The easy case is when Φ is algebraically closed. Then by 2.3 $\bar{A} = A/R$ is a direct sum $\bar{A}_1 \boxplus \cdots \boxplus \bar{A}_r$ of simple algebras which are either associative matrix algebras $M_n(\Phi)$ or split Cayley algebras $\mathbb{C}(\Phi)$. By Idempotent Lifting VII. 7.2, since R is a nil ideal the units $\overline{e_i}$ of $\overline{A_i}$ can be lifted to orthogonal idempotents $e_i \in A$; applying the Matrix Algebra Lifting Theorem VII.7.4 or the Cayley Algebra Lifting Theorem VII.7.6 to $e_i Ae_i$ (with $e_iAe_i/e_iAe_i \cap R \cong \overline{e_i}\,\bar{A}\,\overline{e_i}$) we can lift the matrix or Cayley algebra $\overline{A_i} = \overline{e_i}\,\bar{A}\,\overline{e_i}$ back to a subalgebra $B_i \subset e_iAe_i$ such that the restriction of the projection $e_i Ae_i \xrightarrow{\pi_i} \overline{e_i}\,\bar{A}\,\overline{e_i}$ to B_i is an isomorphism $B_i \to \overline{A_i}$. Then $B = B_1 \boxplus \cdots \boxplus B_r$ is a subalgebra (by the automatic orthogonality of the B_i , $B_iB_j \subset (e_iAe_i)(e_jAe_j) = A_{ii}A_{jj} = 0$ for $i \neq j$) such that the restriction of the projection $A \xrightarrow{\pi} \bar{A}$ to B is an isomorphism $B = \boxplus B_i \to \boxplus \overline{A_i} = \bar{A}$. Thus $A = B \oplus \mathrm{Ker}\,\pi = B \oplus R$ and we have our Wedderburn splitting.

So far R needed only to be nil. When we turn to the case of a non-algebraically-closed Φ we will assume R is solvable and prove the existence of a splitting by induction on the index of R. The case of index 0 is trivial: if $R = D^0(R) = 0$ we may take $B = A$. The hard part is the case of index 1,

$R^2 = D^1(R) = 0$. Suppose we have established the result for index 1 and all indices less than n . To establish the case of index n, $D^n(R) = 0$, we first use our inductive hypothesis to split $\bar{A} = A/D^{n-1}(R)$: since $\bar{A}/\bar{R} = (A/D^{n-1}(R))/(R/D^{n-1}(R)) \cong A/R$ is separable and $\bar{R}$ solvable of index n-1, $D^{n-1}(\bar{R}) = \overline{D^{n-1}(R)} = \bar{0}$, we have a splitting $\bar{A} = \bar{B} \oplus \bar{R}$. Let $C = \pi^{-1}(\bar{B})$ be the preimage of $\bar{B}$ in A . Then C is a subalgebra with radical $C \cap R = D^{n-1}(R)$ of index 1: certainly $D(D^{n-1}(R)) = D^n(R) = 0$, where $\bar{B} \cap \bar{R} = \bar{0}$ implies $C \cap R = D^{n-1}(R)$ by taking preimages, and this must be precisely the radical of C since it is solvable yet the quotient $C/D^{n-1}(R) \cong \bar{B} = A/R$ is separable. By the index 1 case we have a splitting $C = B \oplus D^{n-1}(R)$ for $B \cong \bar{B} = A/R$. Taking preimages of $\bar{A} = \bar{B} + \bar{R}$ gives $A = C + R$, hence $A = B + D^{n-1}(R) + R = B + R$, so by separability of B we have $B \cap R = 0$ and $A = B \oplus R$ is a Wedderburn splitting for A.

Thus everything is routine after we get by the hurdle of index 1. Assume now R has index 1, $R^2 = 0$. We first note that if Ω is the algebraic closure of Φ then $A_\Omega/R_\Omega \cong (A/R)_\Omega$ is still finite-dimensional semisimple if A/R is separable (since separability coincides with strict semisimplicity by VIII.0.00), where R_Ω is the radical of A_Ω since it is still trivial ($R_\Omega^2 = 0$) yet has semisimple quotient. Therefore by the algebraically closed case A_Ω has a splitting $A_\Omega = C_\Omega \oplus R_\Omega$. Under the isomorphism $C_\Omega \longrightarrow A_\Omega/R_\Omega = (A/R)_\Omega$ let C be the subalgebra of C_Ω mapping onto the subalgebra A/R of $(A/R)_\Omega$, $C \longrightarrow A/R$. If C were contained in A we'd be done, but unfortunately all we know is that it is contained in A_Ω . Nevertheless we can make use of it to obtain a splitting for A . Choose a bases $\{\omega_i\}$ for Ω/Φ with $\omega_o = 1$. This determines a decomposition $A_\Omega = A \otimes_\Phi (\oplus \Phi\omega_i) = \oplus(A \otimes_\Phi \Phi\omega_i) = \oplus A\omega_i$ and $R_\Omega = \oplus R\omega_i$. Let π_o be the projection of A_Ω on $A\omega_o$ determined by this decomposition, and let $B = \pi_o(C)$. Since π_o is not a homomorphism of algebras it is not at first clear B is a subalgebra. However, the $A\omega_i$ are invariant under multiplication by A and π_o is at least a homomorphism of A-bimodules,

(5.2) $\pi_o(ax) = a\pi_o(x), \ \pi_o(xa) = \pi_o(x)a \quad (a \in A, \ x \in A_\Omega)$.

To prove π_o is a homomorphism of algebras when restricted to C , consider any two elements $c, c' \in C$. Under the isomorphism $C \to A/R = \bar{A}$ let $c \to \bar{a}, \ c' \to \bar{a'}$; then $c \in A_\Omega$ and $a \in A$ project to the same element, so they differ by an element of the kernel R_Ω , and $c = a + r, \ c' = a' + r'$ for $r, r' \in R_\Omega$. Thus

$$\begin{aligned}
\pi_o(cc') &= \pi_o(\{a+r\}\{a'+r'\}) \\
&= \pi_o(aa'+ar'+ra') && (R_\Omega^2 = 0) \\
&= aa' + a\pi_o(r') + \pi_o(r)a' && (\text{by } (5.2)) \\
&= \{a+\pi_o(r)\}\{a'+\pi_o(r')\} && (\pi_o(R_\Omega) = R, \ R^2 = 0) \\
&= \pi_o(a+r) \ \pi_o(a'+r') && (\pi_o(a) = \pi_o(a\omega_o) = a) \\
&= \pi_o(c) \ \pi_o(c')
\end{aligned}$$

and $\pi_o : C \to B$ is an epiomorphism of algebras. It is an isomorphism because $\pi_o(c) = \pi_o(a+r) = a + \pi_o(r) = 0$ would imply $a = -\pi_o(r) \in R$, $c = a+r \in R+R_\Omega = R_\Omega$, and $C \cap R_\Omega \subset C_\Omega \cap R_\Omega = 0$.

Therefore B is a subalgebra of A isomorphic to A/R, $B \cong C \cong A/R$, and we have a Wedderburn splitting for the index 1 case. ∎

5.3 (Corollary) If A is a finite-dimensional alternative algebra over a perfect field (e.g. a finite field or a field of characteristic zero) then A has a Wedderburn splitting $A \cong A/R \oplus R$. ∎

We have already seen in the associative case that if A/R is not separable there need not be a Wedderburn splitting.

Problem Set: Alternate Approach

A Φ-algebra S is liftable if it is a semidirect summand every time it is a quotient by a solvable ideal, ie. if $S \cong A/R$ for R solvable then $A = B \oplus R$ for some subalgebra B of A (automatically isomorphic to S). In terms of short exact sequeness, every short exact sequence

$$0 \longrightarrow R \xrightarrow{\ i\ } A \underset{\sigma}{\overset{\pi}{\rightleftarrows}} S \longrightarrow 0$$

for solvable R splits: there is a homomorphism $S \xrightarrow{\sigma} A$ with $\pi \circ \sigma = 1_S$ (σ "lifts" S into A).

1. Show an associative matrix algebra $M_n(\Phi)$ or a Cayley matrix algebra $C(\Phi)$ is liftable.

   It is essential here that Φ is the ring of scalars; in general $M_n(\Omega)$ or $C(\Omega)$ are not liftable in the category of Φ-algebras.

2. Show a direct sum $\boxplus_i S_i$ of liftable unital alternative algebras is liftable. What about direct product? Conversely, show that if $S_1 \boxplus S_2$ is liftable so is each S_i.

3. Conclude that over an algebraically closed field Φ any finite-dimensional semisimple alternative algebra S is liftable.

4. If S is unital alternative, show S is liftable iff all $0 \to R \to A \xrightarrow{\pi} S \to 0$ split when A is unital.

5. If S is a nonassociative algebra over a field Φ such that some extension S_Ω is liftable as an Ω-algebra (in a category of

algebras where R^2 is an ideal whenever R is), show S itself is liftable as Φ-algebra. [Hint: given $A \xrightarrow{\pi} S$ with solvable kernel, construct a lift $S \xrightarrow{\sigma} A$ by successive approximations, finding linear liftings $S \xrightarrow{\sigma_n} A$ with $\pi \circ \sigma_n = 1_S$ and $\sigma_n(xy) \equiv \sigma_n(x)\sigma_n(y) \mod D^n(R)$.]

6. Alternately, prove #5 using elements instead of lifts. If $\{s_i\}$ is a bases for S/Φ with $s_i s_j = \Sigma \gamma_{ijk}\, s_k$ show there is a subalgebra of A isomorphic to S iff there are elements $\{x_i\}$ in A with $x_i x_j = \Sigma \gamma_{ijk} x_k$. Construct these by successive approximations, ie. finding $\{x_i^{(n)}\}$ with $x_i^{(n)} x_j^{(n)} \equiv \Sigma \gamma_{ijk} x_k^{(n)} \mod D^n(R)$.

7. Conclude that any separable alternative algebra S over a field Φ is liftable, and deduce the Wedderburn Splitting Theorem.

Problem Set on the Liberal

Let A be a finite-dimensional alternative algebra over a field Φ. If A/R is separable we know A has a Wedderburn decomposition $A \cong A/R \oplus R$. We want to obtain some sort of decomposition even if A/R is not separable.

1. Show that if L is <u>any</u> ideal of A such that A/L is separable (L need not be solvable!) then there is a subalgebra C of A such that A is the semi-direct product $A = C \oplus B$.

2. The <u>liberal</u> of A is the minimal ideal $L(A)$ such that A/L is separable. Show the liberal contains the radical, $L \supset R$; if $A \xrightarrow{\pi} A/R$ is the projection on the semisimple algebra A/R show L is the inverse image of the inseparable summands. Conclude the liberal of a semisimple algebra is just the direct sum of all inseparable summands, that $L(A) = R$ iff A/R is separable, $L(A) = 0$ iff A is separable and $L(A) = R(A)$ when Φ is algebraically closed. Conclude we always have a decomposition $A = C \oplus L(A)$ for separable C.

3. Show $L(A) \supset A \cap R(A_\Omega)$ if Ω is the algebraic closure of Φ. Show $A/A \cap R(A_\Omega)$ is isomorphic to a Φ-subalgebra $\bar{B}$ of $\bar{A} = A_\Omega/R(A_\Omega)$ with $\Omega\bar{B} = \bar{A}$; conclude $A/A \cap R(A_\Omega)$ is semisimple. Give examples where $L(A) > A \cap R(A_\Omega)$ and $L(A)_\Omega > L(A_\Omega)$.