§4 Structure of derivations

4.2 We want to classify the derivations of a separable alternative algebra over a field Φ by proving they are all inner: for any bimodule M

$$\mathrm{Der}(A,M) = \mathrm{Inder}(A,M) \quad , \quad \mathrm{Outder}(A,M) = 0 \; .$$

Let us recall our nomenclature for the varied degrees of perfection that a derivation may attain. A derivation from A to M is inner if it has the form

$$D = Ad_m + \Sigma\, A_{x_i,m_i}$$

for suitable $m, m_i \in M$ and $x_i \in A$ such that

$$3m + \Sigma [x_i, m_i] \in N(M) \qquad \text{(inner)}.$$

It is strictly inner or f-strictly inner for a strongly associative idempotent f relative to A if

$$3m + \Sigma [x_i, m_i] = 0 \qquad \text{(strictly inner)}$$

$$3m + \Sigma [x_i, m_i] \in fMf \qquad \text{(f-strictly inner)}.$$

In both these cases $3m + \Sigma [x_i, m_i]$ remains nuclear (so the derivation extends to an inner derivation) from any larger algebra $\tilde{A} \supset A$ to any larger bimodule $\tilde{M} \supset M$. If $f = 0$ then "f-strictly inner" reduces to "strictly inner." In characteristic $\neq 3$ any strictly inner derivation is standard,

$$D = \Sigma\, D_{y_i, n_i} \; .$$

It's not true that a derivation kills idempotents, the way it kills units, but it can be persuaded to after a little straightening-out.

4.1 (Idempotent-Killing Lemma) If $D: A \to M$ is a derivation of an alternative algebra A into a bimodule M, where $e_1,\cdots,e_n$ are pairwise orthogonal idempotents in A, then

$$\tilde{D} = D + \Sigma_{i\le j} D_{e_i,D(e_j)}$$

is a derivation $\tilde{D}: A \to M$ which kills all e_i, $\tilde{D}(e_i) = 0$, and consequently maps Peirce spaces $\tilde{D}(A_{ij}) \subset M_{ij}$.

Proof. To see $\tilde{D}$ kills e_k, simply compute $\tilde{D}(e_k) = D(e_k) + \Sigma_{i\le j} D_{e_i,D(e_j)} e_k = D(e_k) + \Sigma_{i\le j} [[e_i,D(e_j)],e_k] + 3\Sigma_{i\le j} [e_i,D(e_j),e_k]$. Of the three parts to the expression for $\tilde{D}(e_k)$, the first is $D(e_k)$, the last vanishes since $[e_i,M,e_j] = 0$, and the middle is $-D(e_k)$ since $\Sigma_{i\le j}\{(e_iD(e_j) - D(e_j)e_i)e_k - e_k(e_iD(e_j) - D(e_j)e_i)\}$ $= \Sigma_{i\le j}\{e_iD(e_j)e_k + e_kD(e_j)e_i\} - \Sigma_{i\le j}\{D(e_j)e_ie_k + e_ke_iD(e_j)\}$ (by the above associativity) $= \Sigma_{i\le j}\{D(e_ie_j) - D(e_i)e_j\}e_k + e_k\{D(e_je_i) - e_jD(e_i)\} - \Sigma_{i\le j}\{D(e_j)e_ie_k + e_ke_iD(e_j)\}$ (as $D(xy) = D(x)y + xD(y)$) $= \Sigma_i\{D(e_i)e_k + e_kD(e_i)\}$ $-\Sigma_{i\le j=k}\{D(e_i)e_k + e_kD(e_i)\} - \Sigma_{k=i<j}\{D(e_j)e_k + e_kD(e_j)\}$ (by orthogonality of the e's) $= -\{D(e_k)e_k + e_kD(e_k)\} = -D(e_k^2) = -D(e_k)$. Thus the three expressions for $\tilde{D}(e_k)$ add up to zero.

Once $\tilde{D}$ kills all e_k we have $\tilde{D}(A_{ij}) = \tilde{D}(e_iAe_j)$ $= e_i\tilde{D}(A)e_j \subset e_iMe_j = M_{ij}$. ∎

This generalizes IV.5.5 to the case of several idempotents.

Using this we can reduce derivations from a direct sum to derivations on the individual pieces.

4.2 (Algebra Sum Lemma) Let $A = A_1 \boxplus \cdots \boxplus A_n$ be a direct sum of unital algebras A_i with units e_i, M an A-bimodule. If all derivations $A_i \to e_i M e_i$ are inner (resp. strictly, f_i-strictly inner for $f_i \in A_i$) then all derivations $A \to M$ are inner (resp. strictly, f-strictly inner for $f = \Sigma f_i \in A$).

Proof. The Peirce decompositions of A, M are $A = \overset{n}{\underset{i=1}{\oplus}} A_{ii}$ $(A_{ii} = A_i)$, $M = \oplus_{i,j=0}^{n} M_{ij}$. Replacing D by $D = \tilde{D} + \Sigma_{i\leq j} D_{e_i,D(e_j)}$ we may assume D kills all e_i by the Idempotent Killing Lemma 4.1. (Note D will be inner, strictly inner, or f-strictly inner if $\tilde{D}$ is since we are adding on only standard derivations). Once $D(e_i) = 0$ we have $D(A_{ii}) = D(e_i A e_i) = e_i D(A) e_i \subset M_{ii}$.

The restriction D_i of D to A_i is a derivation $A_i \to M_{ii}$, which by hypothesis is inner (resp. strictly or f_i-strictly inner): $D_i = \mathrm{Ad}_{m_i} + \Sigma\, \Lambda_{x_{i\alpha},m_{i\alpha}}$ for $m_i, m_{i\alpha} \in M_{ii}$, $x_{i\alpha} \in A_{ii}$ with $3m_i + \Sigma[x_{i\alpha}, m_{i\alpha}] \in N(M_{ii})$ (resp. $= 0$ or in fMf). Since $m_i, m_{i\alpha}$, $x_{i\alpha}$ in M_{ii}, A_{ii} kill A_{jj} by Peirce orthogonality, D_i extends to a derivation on all of A with $D_i(A_{jj}) = 0$ for $j \neq i$. Therefore D coincides with ΣD_i on each A_k, so $D = \Sigma D_i = \Sigma\, \mathrm{Ad}_{m_i} + \Sigma_{i,\alpha} \Lambda_{x_{i\alpha},m_{i\alpha}}$ where $3\Sigma m_i + \Sigma[x_{i\alpha}, m_{i\alpha}] \in \Sigma N(m_{ii}) \subset N(M)$ (resp. $= 0$ or in $\Sigma f_i M_{ii} f_i \subset fMf$) and D

is inner (resp. strictly inner or f-strictly inner). (Recall $f = \Sigma f_i$ is strongly associative relative to A if the f_i are relative to A_i). ■

Note the non-unital bimodules take care of themselves; see also exercise 4.4.

We can also break M into pieces.

4.3 (Module Sum Lemma) If A is finitely spanned over Φ and $M = \oplus M_\alpha$ is a direct sum of bimodules such that all derivations $A \to M_\alpha$ are inner (resp. strictly or f-strictly inner), then all derivations $A \to M$ are inner (resp. strictly or f-strictly inner).

Proof. If $\pi_\alpha : M \to M_\alpha$ is the projection on M_α associated with the direct sum decomposition, the composition $D_\alpha = \pi_\alpha \circ D$ is a derivation $A \to M_\alpha$. By hypothesis D_α is inner, $D_\alpha = Ad_{m_\alpha} + \Sigma A_{x_{i\alpha}, m_{i\alpha}}$ for $m_\alpha, m_{i\alpha} \in M_\alpha$. Furthermore, the hypothesis that A is finitely spanned guarantees only finitely many D_α are nonzero: each D(a) has components in only finitely many summands M_α, so if $a_1, \ldots, a_r$ span A and involve only $M_{\alpha_1}, \ldots, M_{\alpha_s}$ then $D(A) = D(\Sigma \Phi a_i) = \Sigma \Phi D(a_i) \subset \Sigma M_{\alpha_j}$. Thus $D = Ad_m + \Sigma_{i,\alpha} A_{x_i, m_{i\alpha}}$ ($m = \Sigma m_\alpha$), where all sums are actually finite. If $3m_\alpha + \sum_i [x_{i\alpha}, m_{i\alpha}]$ belongs to $N(M_\alpha)$, 0, or $fM_\alpha f$ then $3m + \sum_{i,\alpha} [x_{i\alpha}, m_{i\alpha}]$ belongs to $N(M)$, 0, or fMf. ■

Finally, it suffices to prove all derivations of a scalar extension are inner.

4.4 (Linearity Lemma) If all derivations $A_\Omega \to M_\Omega$ are inner (resp. strictly inner or f-strictly inner for $f \in A$) where $\Omega \supset \Phi$ is an extension such that $\Omega = \Phi 1 \oplus \Omega_o$ as Φ-module, then all derivations $A \to M$ are inner (resp. strictly or f-strictly inner).

Proof. Let $D: A \to M$ be a derivation; then $D_\Omega = D \otimes 1$: $A_\Omega = A \otimes_\Phi \Omega \to M \otimes_\Phi \Omega = M_\Omega$ is also a derivation. If we assume all such are inner we have $D_\Omega = Ad_m + \Sigma A_{x_i,m_i}$ for $m, m_i \in M_\Omega$ and $x_i \in A_\Omega$; using linearity $A_{\Sigma\omega_i y_i, m} = \Sigma A_{y_i, \omega_i m}$ we can assume $x_i \in A = A \otimes 1 \subset A_\Omega$. Let $\Omega = \Phi 1 \oplus \Omega_o$, so $M_\Omega = (M \otimes \Phi 1) \oplus (M \otimes \Omega_o) = M \oplus M_o$, and write $m = n \oplus n_o$, $m_i = n_i \oplus n_{io}$ for $n, n_i \in M$ and $n_o, n_{io} \in M_o$. Since M and M_o are A-bimodules, applying $D_\Omega = D \otimes 1$ to $a \otimes 1 \in A \otimes 1$ yields $D(a) \otimes 1 = \{[n,a] + \Sigma[x_i,n_i,a]\} \otimes 1 + \{[n_o,a] + \Sigma[x_i,n_{io},a]\}$. Identifying components in M gives $D(a) = [n,a] + \Sigma[x_i,n_i,a]$, and

$$D = Ad_n + \Sigma A_{x_i,n_i} .$$

The conditions that $3m + \Sigma[x_i,m_i] = \{3n + \Sigma[x_i,n_i]\} \oplus \{3n_o + \Sigma[x_i,n_{io}]\}$ belongs to $N(M_\Omega) = N(M) \oplus N(M_o)$ (resp. to 0 or $fM_\Omega f$) implies $3n + \Sigma[x_i,n_i]$ belongs to $N(M)$, to 0, or to fMf. Therefore D is inner, strictly inner, or f-strictly inner if D_Ω is. ∎

Using these reductions and our previous knowledge of derivations of the basic bimodules, we are ready to tackle an arbitrary separable algebra. As a preliminary bit of notation, if A is separable over Φ it is a direct sum $A = A_1 \boxplus \cdots \boxplus A_n$ of algebras A_i simple over their separable centers Γ_i with units e_i ; the indicator of A is the sum $f(A) = \Sigma e_i$ of the units e_j corresponding to the simple summands A_j whose degree is divisible by the characteristic p (ie which become matrix algebras $A_j \otimes_{\Gamma_j} \Omega \cong M_k(\Omega)$ over the algebraic closure Ω of degree k divisible by p). Note that $f(A) = f(A_\Omega)$ remains the same under scalar extension, since $A_j \otimes_\Phi \Omega \cong \boxplus A_{j\alpha}$ for $A_{j\alpha} \cong A_j \otimes_{\Gamma_j} \Omega$ by Separable Decomposition 0,00 , and therefore if A_j contributes its unit e_j to f(A) then the $A_{j\alpha}$ contribute their units $\Sigma e_{j\alpha} = e_j \otimes 1$ to $f(A_\Omega)$ so that $f(A_\Omega) = f(A) \otimes 1$. Note also that $f(\boxplus A_i) = \boxplus f(A_i)$, and $f(A) = 0$ in characteristic 0.

4.5 (Derivation Theorem) Any derivation D of a separable alternative algebra A over a field Φ into a bimodule M is inner:

$$D = Ad_m + \Sigma A_{x_i,m_i} \quad (3m + \Sigma[x_i,m_i] \in N(M)).$$

If the characteristic $\neq 2$ (or the characteristic is 2 but A has no quaternion summands) then D is f-strictly inner, where the indicator f(A) is strongly associative.

If the characteristic $\neq 2,3$ D is f-standard

$$D = Ad_m + \Sigma D_{y_i,n_i} \quad (m \in fMf)$$

and in characteristic 0 D is standard

$$D = \Sigma D_{y_i,n_i} .$$

Proof. The last remarks follow from f-strict innerness: when $1/3 \in \Phi$ any $Ad_m + \Sigma A_{x_i,m_i}$ with $3m + \Sigma[x_i,m_i] \in fMf$ can be written $\frac{1}{3} Ad_{3m+\Sigma[x_i,m_i]} - \frac{1}{3}\Sigma\{Ad_{[x_i,m_i]} - 3A_{x_i,m_i}\} = Ad_n - 1/3\, \Sigma D_{x_i,m_i}$ for $n = \frac{1}{3}[3m + \Sigma[x_i,m_i]] \in fMf$, and thus is f-strictly standard. In characteristic 0 we have $f(A) = 0$, so $n = 0$.

To prove f-strict innerness it suffices to pass to the algebraic closure A_Ω by the Linearity Lemma 4.4 (recalling $f(A_\Omega) = f(A)$), so <u>we may assume Φ is algebraically closed</u>. By the Algebra Sum Lemma 4.2 it suffices if each simple summand A_i is f_i-strictly inner for $f_i = f(A_i)$ (recalling $f = \boxplus f_i$), so <u>we may assume A is simple and M is unital</u>; by the Module Sum Lemma 4.3 <u>we may even assume M is irreducible</u> (since A is finite-dimensional and M completely reducible by the Bimodule Theorem 3.0).

As a simple finite-dimensional algebra over an algebraically closed field, we know A has the form $M_k(\Phi)$ or $\mathbb{C}(\Phi)$ by 2.0. First consider the Cayley case $A = \mathbb{C}(\Phi)$. Here every derivation is strictly inner by the Inner Derivation Theorem IV.5.1. Next consider $A = M_k(\Phi)$ for $k = 1$, ie. $A = \Phi$; now every unital derivation kills 1, so here $D = 0$ is indeed strictly inner.

When $A = M_k(\Phi)$ for $k \geq 3$ we know all unital bimodules M are associative, indeed the only irreducible unital bimodule is $M = reg(A)$. By associative theory we know $D = Ad_m$ is at least inner. If p/k then $f(A) = 1$ and certainly $m \in M = fMf$, so D is f-strictly inner (and since $k \geq 3$ f is strongly associative by VII.0.0). If $p \not| k$ then $f(A) = 0$; we must rewrite $D = Ad_m$ in the form of a strictly inner derivation. Now in $M = reg(A) = M_k(\Phi)$ the commutator subspace $[A,M]$ consists precisely of all matrices of trace zero. Since $tr(1) = k \neq 0$ in Φ if $p \not| k$, we have $\Phi 1 \notin [A,M]$ and $\Phi 1 + [A,M] = M$. Thus we can write $m = \omega 1 + \Sigma [x_i, m_i]$. But then $Ad_m = \Sigma\, Ad_{[x_i,m_i]}$ (1 is in the center of M) $= \Sigma\{Ad_{[x_i,m_i]} - 3A_{x_i,m_i}\}$ (by associativity) $= \Sigma\, D_{x_i,m_i}$. Thus when $p \not| k$ we can write $Ad_m = \Sigma\, D_{x_i,m_i}$ as a strictly inner derivation.

Finally consider the split quaternion case $A = M_2(\Phi)$. If $M \subset cay(A)$ is the Cayley bimodule, any derivation is strictly inner by the Split Inner Derivation Theorem VI.5.10. If $M = reg(A) = M_2(\Phi)$ any derivation is at least inner, $D = D_{e_1,m} - Ad_{m_{11}}$, and if the characteristic $\neq 2$ then $Ad_{m_{11}} = \frac{1}{2} D_{e_{12},m_{21}}$ is strictly inner by (5.11a'). However, if the characteristic = 2 and there is a quaternion summand $A = M_2(\Phi)$ then there exist derivations $Ad_{m_{11}}$ which are not strictly inner. (Of course, $Ad_{m_{11}}$ is f-strictly inner for $f = f(A)$, but since f is only 2-connected it is not strongly associative). ∎

4.6 <u>Remark</u>. It is worth recalling that the only time we really need derivation of the form Ad_m (ie. where we can't convert them into strictly inner $D_{x,n}$'s) is for summands $M_k(\Phi)$ of degree k divisible by the characteristic p. The reason we can't convert them is that we can't write m as a sum of central elements and commutators. For example, if $m = e_{11}$ there is no way to make $Ad_{e_{11}}$ strictly inner on $M_{pr}(\Phi)$. ∎

4.7 Corollary. If B is a separable subalgebra of A without quaternion summands of characteristic 2 then any derivation of B into A extends to an inner derivation of A.

Proof. A is a B-bimodule, so any D: B → A has the form $D = Ad_m + \Sigma A_{x_i, m_i}$ for $3m + \Sigma[x_i, m_i] = n \in N(M)$ (ie $[B,B,n] = 0$). This doesn't imply $n \in N(A)$, ($[A,A,n] = 0$) in general, but if B has no quaternion summands of characteristic 2 then D is actually f-strictly inner, so $n \in fAf \subset N(A)$ by strong associativity of $f = f(B)$, and D does extend to an inner derivation on all of A. ∎

Exercises

4.1 Show that if $A^2 = A$, the only derivation of A into a trivial module M is $D = 0$.

4.2 If $\Omega \supset \Phi$ is free as a Φ-module, to what extent is it true that $\mathrm{Der}(A_\Omega, M_\Omega) \cong \mathrm{Der}(A,M)_\Omega$, $\mathrm{Inder}(A_\Omega, M_\Omega) \cong \mathrm{Inder}(A,M)_\Omega$, $\mathrm{Outder}(A_\Omega, M_\Omega) \cong \mathrm{Outder}(A,M)_\Omega$?

4.3 Show $\mathrm{Der}(A, \prod M_i) \cong \prod \mathrm{Der}(A, M_i)$ and $\mathrm{Der}(A, \oplus M_i) \cong \prod^* \mathrm{Der}(A, M_i)$, where $\prod^*$ denotes the subspace of the direct product consisting of the locally finite elements $\prod d_i$ (ie those such that for each $a \in A$ only finitely many $d_i(a)$ are nonzero). Show $\mathrm{Der}(\boxplus_i A_i, M) \cong \prod_* \mathrm{Der}(A_i, M)$ where $\prod_*$ consists of those $\prod d_i$ such that $d_i(a_i)a_j + a_i d_j(a_j) = 0$ for $a_k \in A_k$, $i \neq j$. Give an example where $\mathrm{Der}(A_1 \boxplus A_2, M) < \mathrm{Der}(A_1, M) \oplus \mathrm{Der}(A_2, M)$.

4.4 If M is a unital left module for A show any derivation $A \xrightarrow{D} M$ has the form $D = \mathrm{Ad}_m = D_{1,m}$ for $m = -D(1)$. Thus non-unital bimodules admit only standard derivations.

4.5 If 4.1, show that if $D: A \to M$ is a derivation into a <u>unital</u> bimodule M then $\tilde{D} = -\Sigma_{i>j} D_{e_i, D(e_j)}$ kills all e_k . Deduce from this the case for non-unital M by tacking on an e_0 to A to make M a unital $\hat{A}$-bimodule, then applying the unital case to $\hat{D}: A \to M$.

Problem Set: Campbell-Casimir Operator

Let $\tau(x) = \operatorname{tr} L_x$ be the left trace form of the regular bimodule A over a field Φ. Assume τ is <u>nondegenerate</u>, so $\tau(x,y) = \tau(x \cdot y)$ is a nondegenerate associative symmetric bilinear form on A. If $\{x_i\}, \{x_i^*\}$ are dual bases for A relative to and (ℓ, r) any birepresentation, the (left) <u>Campbell-Casimir operator</u> is $C_\ell = \Sigma \, \ell_{x_i} \ell_{x_i^*}$.

1. Show a finite-dimensional algebra A has nondegenerate trace form iff A is separable, $A = \boxplus A_i$ for A_i simple with separable centers Γ_i/Φ, such that the degrees of the A_i/Φ (ie of A_i/Γ_i and Γ_i/Φ) are not divisible by the characteristic p.
2. In case $A = \mathbb{C}(\Phi)$ is split Cayley, choose natural dual bases $\{x_i\}$, $\{x_i^*\}$ and compute τ.
3. If $(\lambda_{ij}(x))$, $(\rho_{ij}(x))$ are the matrices of L_x, R_x relative to the bases $\{x_i\}$ and $(\lambda^*_{ij}(x))$, $(\rho^*_{ij}(x))$ those relative to $\{x_i^*\}$ show $(\rho^*_{ij}(x)) = (\lambda_{ij}(x))^t$, $(\lambda^*_{ij}(x)) = (\rho_{ij}(x))^t$.
4. Show for the regular representation $C = L_e$ for $e = \Sigma x_i x_i^*$.
5. Show $e = 1$ is a unit for A.
6. If $A \xrightarrow{D} M$ is a derivation into a unital bimodule show $D = \mathrm{Ad}_m + \Sigma A_{x_i, D(x_i^*)} + \Sigma A_{x_i^*, D(x_i)}$ for $m = \Sigma D(x_i)x_i^* = -\Sigma x_i D(x_i^*)$. Using #2, compare this with the Split Inner Derivation Theorem.
7. If $e_i^* = \Sigma \gamma_{ij} e_j$ show $\gamma_{ij} = \gamma_{ji}$. Conclude $C_\ell = \ell_e$ for any birepresentation (ℓ, r), so $C_\ell = I$ for a unital birepresentation.

This gives a uniform proof that derivations are inner, but only when none of the simple summands has degree divisible by the characteristic. Such summands contribute the "f-strictly inner" derivations of 4.5.