§3 Structure of bimodules

Since the semisimple Artinian alternative algebras are just the associative and Cayley algebras, and we have separately classified the bimodules for each of these, we can put our results together to describe the bimodules for an arbitrary separable alternative algebra A over a field Φ .

There is one slight catch. The Cayley summands of A are Cayley algebras C over their centers Ω , which are separable extensions of Φ . In II.7.1 we classified only Ω-bimodules for C/Ω; a priori there might be some new Φ-bimodules for C which are not Ω-bimodules. To show this cannot happen we show that any Φ-bimodule carries a natural Ω-bimodule structure.

3.1 (Lemma) Let A be a unital alternative algebra with center Ω over a field Φ . Then all A-bimodules M in the category of Φ-algebras carry a natural left Ω-structure $\omega\cdot m = \omega m$ making them A-bimodules in the category of Ω-algebras, as soon as A (or some central extension $A \otimes_\Phi \Sigma$) is a direct sum of 2-interconnected algebras.

Proof. The scalar product $\omega\cdot m$ will make M a unital left Ω-module as soon as $(\omega_1\omega_2)\cdot m = \omega_i\cdot(\omega_2\cdot m)$, ie $[\Omega,\Omega,M] = 0$. The bimodule action of A on M will be Ω-bilinear, $\omega(am) = (\omega a)m = a(\omega m)$ and $(ma)\omega = m(a\omega) = (m\omega)a$, if $[\Omega,A,M] = 0$. (Then $[\omega,a,m] = 0$ implies $\omega(am) = (\omega a)m$, $\Omega =$ C(A) implies

$(\omega a)m = (a\omega)m$, and $[a,\omega,m] = -[\omega,a,m] = 0$ implies $(a\omega)m = a(\omega m)$. Dually on the right). Since $\Omega \subset A$ we are reduced to showing

$$[\Omega,A,M] = 0.$$

It suffices if $[\Omega,A,M]_\Sigma = [\Omega_\Sigma,A_\Sigma,M_\Sigma]$ vanishes in the bimodule $M_\Sigma = M \otimes_\Phi \Sigma$ for $A_\Sigma = A \otimes_\Phi \Sigma$. Replacing A/Φ by the direct sum of interconnected algebras A_Σ/Σ , and noting $\Omega_\Sigma \subset C(A_\Sigma)$, it suffices to assume from the start that A is a direct sum of 2-interconnected algebras.

Let $1 = \Sigma_{i=1}^n e_i$ where each e_i is interconnected with at least one other e_j . Since any $\omega \in \Omega = C(A)$ commutes with the e_i we have $\omega = \sum_{i=1}^n w_{ii}$ for $w_{ii} \in A_{ii}$. Thus the only unrully associators involving Ω are

$$[\Omega,\ A_{ii},\ M_{ii}],\ [\Omega,\ A_{ij},\ M_{ij}] \quad (0 \le i \ne j \le n)\ .$$

Now $[\Omega,\ A_{ii},\ M_{ii}] = [\Omega_{ii},\ A_{ii},\ M_{ii}] = 0$ since $A_{oo} = 0$ and if $1 \le i \le n$ by interconnectivily there exists $j \ne i$ with $A_{ii} = A_{ij}A_{ji} \subset E_{ij}E_{ji} \subset N(E_{ii})$ by (3.12) for $E = A \oplus M$ the split null extension. To see any $[\omega,\ a_{ij},\ m_{ij}]$ vanishes, compute $(\omega a_{ij})m_{ij} = (a_{ij}\ \omega)m_{ij}$ $(\omega \in C(A)) = (a_{ij}\ w_{jj})\ m_{ij}$ $= w_{jj}(a_{ij}\ m_{ij}) = \omega(a_{ij}\ m_{ij})$ by Slipping Formula 3.8.

Thus all unruly associators vanish, $[\Omega,\ A,\ M] = 0$, and M carries a natural Ω-bimodule structure. ∎

We determine the structure of bimodules for a separable A by decomposing them into bimodules for the simple summands A_i . In general, we have the following decomposition for a direct sum $A = \boxplus A_i$.

3.2 (Peirce Decomposition of Bimodules) Let $A = A_1 \boxplus \dots \boxplus A_n$ be a direct sum of alternative algebras A_i with units e_i. Then any alternative bimodule M for A has Peirce decomposition

$$M = \oplus_{i,j=0}^{n} M_{ij}$$

relative to the e_i, where for $i,j,0 \neq$

M_{oo} is a trivial bimodule

M_{ii} is a unital A_i - bimodule

M_{io}, M_{oi} are unital left, right A_i - modules

M_{ij} is a unital left A_i, right A_j module where the actions of A_i and A_j commute.

Proof. Since the Peirce decomposition of A is $A = \Sigma A_{ii}$ for $A_{ii} = A_i$, it is clear that M_{oo} is trivial, M_{io} is a unital left A_i - module (killed by all other A_j), M_{oi} is a unital right A_i - module, and M_{ii} a unital A_i - bimodule (killed by all other A_j). Since the Peirce specialization $x \to L_x$ of A_{ii} on M_{ij} $(i \neq j)$ and the Peirce antispecialization $y \to R_y$ of A_{jj} on M_{ij} commute, $(x_{ii}m_{ij})y_{jj} = x_{ii}(m_{ij}y_{jj})$, M_{ij} is a unital left A_i and right A_j module where the module actions commute. ∎

Note that if A_i, A_j are associative M_{ij} is simply a unital left module for $A_i \otimes A_j^{op}$ via $x \otimes y \to \ell_x r_y$.

Putting this together with our knowledge of bimodules for the basic building - blocks A_i leads to

3.3 (Bimodule Theorem) If A is a separable alternative algebra over a field Φ then all A-bimodules are completely reducible. If $A = A_1 \boxplus \cdots \boxplus A_n$ the irreducible A-bimodules are of the following types:

(i) a trivial bimodule $M = \Phi m$

(ii) an irreducible left or right A_i - module if A_i is associative

(iii) an irreducible unital left $A_i \otimes A_j^{op}$ module if A_i, A_j are associative $(i \neq j)$

(iv) a regular bimodule $M \cong reg(A_i)$ if A_i is associative

(v) an irreducible constituent of a Cayley bimodule $M \cong cay(A_i)$ if A_i is quaternion

(vi) a regular bimodule $M \cong reg(A_i)$ if A_i is Cayley.

Proof. Since a trivial bimodule is just a vector space, it is a direct sum of irreducible 1-dimensional bimodules $M = \Phi m$ as in (i) .

Now consider left or right A_i - modules M_{io} or M_{oi} . Since Cayley algebras admit no nonzero specializations (have no nonzero modules), such exist only when A_i is associative. Since A_i is semisimple artinian, from associative theory all its modules are completely reducible. This gives type (ii).

Consider next M_{ij} for $i \neq j$. Since again the Cayley algebras admit no specializations, $M_{ij} \neq 0$ only when A_i and A_j are associative. In this case it is just a unital left $A_i \otimes A_j^{op}$ module. Since A_i and A_j are separable over Φ, $A_i \otimes_\Phi A_j^{op}$ remains semisimple and all its modules are completely reducible. This gives (iii).

Finally we turn to the M_{ii}. When A_i is associative (a form of a matrix algebra $M_k(\Omega)$) its unital bimodules are associative if $k \neq 2$ (trivially if $k = 1$, $A_i = \Phi$, and by Bimodule Associativity VII.5.9 since A_i is strongly associative if $k \geq 3$ by VII.5.14). The associative bimodules for an associative A_i are again just $A_i \otimes A_i^{op}$ modules; since A_i is separable over Φ these are completely reducible. We know from associative theory that the irreducible associative A_i - bimodules are Φ-isomorphic to $reg(A_i)$. This gives (iv).

If A_i is quaternion (over its center Ω) its bimodules need not be associative. However, by 3.1 its Φ-bimodules are automatically Ω-bimodules, so by the Bimodule Theorem II.7.1 all its bimodules are completely reducible with irreducible bimodules either $reg(A_i)$ or irreducible constituents of $cay(A_i)$, as in (v).

If A_i is Cayley over its center Ω its bimodules are Ω-bimodules by 3.1 and so once more by the Bimodule Theorem II.7.1 are direct sums of regular bimodules and therefore completely reducible. ∎