Chapter VIII

Structure Theory

In this chapter we will develope a structure theory for alternative algebras with d.c.c. on quadratic ideals analogous to the Artin-Wedderburn theory for associative algebras. We show semisimple algebras break into a direct sum of simple algebras, and simple algebras are associative or Cayley algebras.

§1. The First Structure Theorem

In this section we want to decompose semisimple Artinian alternative algebras into simple pieces; the next section will investigate the nature of these pieces.

An alternative algebra is Artinian if it has the d.c.c. on quadratic ideals. Since quadratic ideals play in alternative algebras the role that one-sided ideals play in associative algebras, this definition is formally analogous to the definition of Artinian associative algebras.

By the Radical Equivalence Theorem VI.0.0, in the presence of the d.c.c. semisimplicity is equivalent to strong semiprimeness. It will be more convenient to work with strong semiprimeness, so although we state our theorems in terms of semisimplicity we prove them in terms of strong semiprimeness.

We digress momentarily to note a useful consequence of the d.c.c., namely the maximum condition on idempotents. Roughly, the bigger the idempotent the smaller its Peirce zero-space (which is a quadratic ideal).

1.1 Lemma. If A has d.c.c. on quadratic ideals of the form $A_{00}(e) = U_{1-e}A$ for idempotents $e \in A$, then it has the a.c.c. on idempotents $e \in A$.

Proof. Let $e_1 < e_2 < \ldots$ be an increasing chain of idempotents. Then $A_{00}(e_1) \supset A_{00}(e_2) \supset \ldots$ is a decreasing chain of quadratic ideals since

$$e < f \Rightarrow A_{00}(e) \supset A_{00}(f) .$$

Indeed, $x \in A_{00}(f) \Rightarrow fx = xf = 0 \Rightarrow ex = \{fef\}x = f[e(fx)] = 0 = [(xf)e]f = x\{fef\} = xe \Rightarrow x \in A_{00}(e)$. If this chain terminates, $A_{00}(e_n) = A_{00}(e_{n+1}) = \ldots$, so must the original chain in view of

$$e < f,\ A_{00}(e) = A_{00}(f) \Rightarrow e = f .$$

Indeed, $f = e + g$ for e, g orthogonal idempotents in fAf so g belongs to $A_{11}(f)$ and also to $A_{00}(e) = A_{00}(f)$; but since $A_{11}(f) \cap A_{00}(f) = 0$ this forces $g = 0$ and $e = f$. ∎

In building up and tearing down algebras we need to know d.c.c. and strong semiprimeness are preserved.

1.2 (Inheritance Lemma). $A = A_1 \boxplus \ldots \boxplus A_n$ is Artinian or strongly semiprime iff each factor A_i is. If A is Artinian or strongly semiprime so is any Peirce subalgebra eAe (for $e \in \hat{A}$).

Proof. Since multiplication in $A_1 \boxplus \ldots \boxplus A_n$ is componentwise, B is a quadratic ideal in A iff $B = B_1 \boxplus \ldots \boxplus B_n$ for B_i quadratic ideals in A_i. From this it is clear A has d.c.c. iff each A_i has d.c.c. We also see (VI.0.0) $z = z_1 + \ldots + z_n$ is trivial in A iff each z_i is trivial in A_i, so A is strongly semiprime iff each A_i is.

If A has dcc so does any Peirce subalgebra A_{ii} $(i = 0,1)$ because any quadratic ideal B_{ii} in A_{ii} is already a quadratic ideal in A: $U_{B_{ii}} A = U_{B_{ii}} A_{ii}$ (by Peirce U-Relations VII.3.19) $\subset B_{ii}$. We also noted (VI.4.0) that if A is strongly semiprime so is any A_{ii}. ∎

It is not hard to show that strong semiprimeness is inherited by ideals (see VI.4.0), but it is not clear that ideals inherit the d.c.c. Our next result will say ideals in Artinian algebras have units; it would be conceptually simpler to prove algebras have units (though the proof would be almost exactly the same), then deduce ideals have units because they inherit the d.c.c. from the parent algebra.

In lieu of any such inheritance result, the requisite lemma becomes slightly technical.

1.3 (Unit-building Lemma). Suppose B is an ideal in a strongly semiprime Artinian algebra A containing an idempotent f. If e is maximal among all idempotents of B having the form $e = f + \sum_{i=1}^{n} e_i$ for e_i orthogonal division idempotents in B, then e is actually a unit for B.

Proof. Such maximal idempotents exist by the a.c.c. on idempotents in A. In the Peirce decompositions $A = A_{11} + A_{10} + A_{01} + A_{00}$ and $B = B_{11} + B_{10} + B_{01} + B_{00}$ relative to e, we have $A_{11} = B_{11}$, $A_{10} = B_{10}$, $A_{01} = B_{01}$ since $A_{11} + A_{10} + A_{01} = eA + Ae \subset BA + AB \subset B$. We hope to show $B_{00} = 0$; then as in the Peirce Triviality Condition VII.3.2 the elements of B_{10} and B_{01} will be trivial in A, $(U_{b_{10}} A = U_{b_{10}} A_{01} = b_{10}(A_{01} b_{10}) \subset b_{10} B_{00} = 0$ and $U_{b_{10}} A \subset B_{00} b_{10} = 0)$ so by strong semiprimeness $B_{10} = B_{01}$

$= 0$ and $B = B_{11}$ has unit e.

Now $B_{00} = B \cap A_{00}$ is a quadratic ideal in A; if it were nonzero it would contain a minimal quadratic ideal C. Case I ($C = \Phi z$, z trivial) of the Minimal Quadratic Ideal Theorem V.5.3 is ruled out by strong semi-primeness, Case II ($C = eAe$, e division) furnishes a division idempotent, as does Case III ($C = bAb$ where $e = bd$ is division; here $e \in BA \subset B$ and also in A_{00}, since if $b = bab$ for $h \in B_{00}$ then also $b = ba_{00}b$ and we can take $d = a_{00}ba_{00} \in A_{00}$). Thus if B_{00} were nonzero it would contain a division idempotent $e_{n+1} \neq 0$, in which case $e + e_{n+1} > e$ would be a bigger idempotent of the form $f + \sum_{i=1}^{n+1} e_i$, contrary to maximality of e. We must therefore have $B_{00} = 0$, and e is the desired unit. ∎

Thus we can enlarge any idempotent f in B to a unit e for B. We can always start with $f = 0$, so

1.4 (Ideal Unit Theorem). Every ideal in a semisimple Artinian algebra has a unit. ∎

This will guarantee we can split the algebra into simple pieces.

Another conclusion we can draw is that the whole algebra has a unit, even a nice one:

1.5 (Theorem of the Unit). Every semisimple Artinian algebra has a unit $1 = \sum_{i=1}^{n} e_i$ which is a sum of orthogonal division idempotents. ∎

The reason we want ideals to have units is the

1.6 Proposition. If an ideal $B \lhd A$ in an alternative algebra has a unit element e, it is a direct summand

$$A = B \boxplus B' \quad (B, B' \lhd A) .$$

Proof. In the Peirce decomposition of A relative to e we have $A_{11} + A_{10} + A_{01} = eA + Ae \subset B$ if $e \in B$ and $B \subset A_{11}$ if e acts as unit for B. Therefore $B = A_{11}$ and $A_{10} = A_{01} = 0$. Then $A = A_{11} \boxplus A_{00} = B \boxplus B'$ where in the absence of A_{10} and A_{01} the Peirce subalgebra A_{00} becomes an ideal orthogonal to A_{11}. ∎

Once we can split off ideals we can break the algebra into simple pieces. This is our first structure theorem.

1.7 (First Artin-Zorn Structure Theorem) An alternative algebra A is semisimple Artinian iff it is a finite direct sum $A = A_1 \boxplus \ldots \boxplus A_n$ of ideals which are simple unital Artinian algebras.

Proof. If A is strongly semiprime we have just seen that all ideals are direct summands. Since two-sided ideals are just the subspaces invariant under the multiplication algebra M(A), the fact that all M(A)-submodules have complements means A is completely reducible as M(A)-modules, so A is a direct sum of minimal ideals (= irreducible M(A)-modules). [More prosaically, we can split off one minimal ideal at a time until the process stops with a decomposition of A as a direct sum of minimal ideals. In one fell swoop: let B be minimal among those ideals which are <u>not</u> direct sums of minimal A-ideals. Then B is not itself minimal, so by d.c.c. on ideals it contains a minimal C; from $A = C \boxplus C'$ we have $B = C \boxplus B'$ for $B' = B \cap C'$ by Dedekind's Modular Law, where C and B' are smaller A-ideals than B and consequently direct sums of minimal ideals, so $B = C \boxplus B'$ is also such a direct sum, contradiction.]

Thus $A = A_1 \boxplus \ldots \boxplus A_n$ for (unital) minimal ideals A_i. As direct summands the A_i inherit the d.c.c. (by Inheritance 1.2) and also simplicity because by directness any A_i-ideal $B_i \triangleleft A_i$ is automatically invariant under the other A_j and hence an A-ideal $B_i \triangleleft A$; thus as algebras the A_i contain no proper ideals, and $A_i^2 \neq 0$ by semiprimeness of A or unitality of A_i, so the A_i are simple.

Conversely, if $A = A_1 \boxplus \ldots \boxplus A_n$ for simple unital Artinian A_i then A is Artinian and strongly semiprime by Lemma 1.2 (note A_i are strongly semiprime: $S(A_i)$ is a nil ideal by VI.4.0, so $S(A_i) \neq A_i$ since A_i is unital, therefore $S(A_i) = 0$). ∎

As in the associative case, such a decomposition is strongly unique. The following argument works in any linear algebra.

1.8 (First Uniqueness Theorem) If A is a direct sum $A = A_1 \boxplus A_2 \boxplus \ldots \boxplus A_n$ of simple unital ideals then this decomposition is unique: the A_i are precisely all the minimal ideals of A. Moreover, any ideal B of A is composed of certain of these simple pieces: $B = A_{i_1} \boxplus \ldots \boxplus A_{i_n}$.

Proof. We only have to prove the last part, for it implies that any minimal ideal must have the form $B = A_i$ for some i, so the A_i are uniquely determined as the minimal ideals.

The projection of A onto the summand A_i is a multiplication operator $L_{e_i} = R_{e_i}$, e_i the unit of A_i: if $x = x_1 + \ldots + x_n$ for $x_j \in A_j$ then $e_i x = x_i = x e_i$ since $e_i x_i = x_i = x_i e_i$ (e_i is unit on A_i) and $e_j x_j = 0 = x_j e_i$ (A_i and A_j are orthogonal for $i \neq j$). If B is an ideal it is invariant under these multiplications, so $B = B_1 \boxplus \ldots \boxplus B_n$ is the direct sum of its components $B_i = e_iB = B \cap A_i$. But $B_i = B \cap A_i$ is an ideal of A contained in A_i, so by minimality B_i is either A_i or 0. Thus B is the sum of those A_i for which $B_i = A_i$. ∎

1.9 (Isomorphism Criterion) Two semisimple Artinian algebras $A = A_1 \boxplus \ldots \boxplus A_n$ and $\tilde{A} = \tilde{A}_1 \boxplus \ldots \boxplus \tilde{A}_m$ are isomorphic iff $n = m$ and (after re-ordering) $A_i \cong \tilde{A}_i$ for each i.

Proof. If $A \xrightarrow{F} \tilde{A}$ is an isomorphism then $F(A_1),\ldots,F(A_n)$ are all the minimal ideals of $\tilde{A}$, hence up to order are $\tilde{A}_1,\ldots,\tilde{A}_m$. ∎

Exercises

1. Show $e < f$ iff $A_{11}(e) \prec A_{11}(f)$. Show $e < f$ in A iff $1-e > 1-f$ in $\hat{A}$, with $A_{00}(e) = A \cap \hat{A}_{11}(1-e)$, $A_{00}(f) = A \cap \hat{A}_{11}(1-f)$. Deduce $e < f$ implies $A_{00}(e) \supset A_{00}(f)$.
2. If A is unital, show A has a.c.c. on idempotents iff it has d.c.c. on idempotents. In general, show A has a.c.c. or d.c.c. on idempotents iff it has a.c.c. or d.c.c. on quadratic ideals of the form $eAe = A_{11}(e)$.
3. If $c \in A$ is regular and $B \triangleleft A$ show $bBb \subset cBc$ iff $bAb \subset cAc$. If $b_i \in B \triangleleft A$ are regular conclude the chain of quadratic ideals b_iBb_i is decreasing iff the b_iAb_i are, so if A is regular and has d.c.c. on principal quadratic ideals so does B. Repeat for left and right principal quadratic ideals.
4. Show that if $B = bAb$ contains no trivial elements and is minimal among principal quadratic ideals of A, it is actually minimal among all quadratic ideals.
5. If A is strongly semiprime with d.c.c. on principal quadratic ideals show every quadratic ideal contains a minimal quadratic ideal.

   Thus in the strongly semiprime case we can get by with d.c.c. only on principal quadratic ideals. However, this does not suffice in the general case.
6. If A is a trivial algebra show A has d.c.c. on principal quadratic ideals xAx; show A has dcc on all xAx for $x \in \hat{A}$ iff A has d.c.c. on submodules αA for $\alpha \in \Phi$; show A has d.c.c. on all quadratic ideals iff A has d.c.c. on submodules.