§2. Composition algebras

Alternativity of a degree 2 algebra has definite consequences for the norm form and involution. Conversely, the mere existence of a suitable norm form guarantees alternativity. The most important degree 2 algebras are the composition algebras, whose nondegenerate norm forms permit composition. They may be characterized in terms of having (i) degree 2, or (ii) a scalar involution, or (iii) a norm form permitting composition; here nondegeneracy of the norm form is equivalent to semisimplicity of the algebra.

Let us recall some terminology concerning quadratic and symmetric bilinear forms. If Q is a quadratic form, a vector $x \neq 0$ is **isotropic** if $Q(x) = 0$; a space is **isotropic** if it contains an isotropic vector and **anisotropic** if it doesn't ($Q(x) = 0 \Rightarrow x = 0$), and is **totally isotropic** if all its vectors are isotropic ($Q(x) \equiv 0$ for all x). The **radical** of the quadratic form is

(2.1a) $\quad \operatorname{Rad} Q = \{z \mid Q(z) = Q(z,x) = 0 \text{ for all } x\}.$

The radical is always a linear subspace, since $Q(\alpha z + \beta w, x) = 0$ if $Q(z,x) = Q(w,x) = 0$, and $Q(\alpha z + \beta w) = \alpha^2 Q(z) + \alpha\beta Q(z,w) + \beta^2 Q(w) = 0$ if $Q(z) = Q(w) = Q(z,\cdot) = 0$. If this radical is zero the quadratic form is **nondegenerate** ($\operatorname{Rad} Q = 0$), otherwise it is **degenerate** ($\operatorname{Rad} Q \neq 0$).

The **radical** of a symmetric bilinear form $B(x,y)$ is the subspace

(2.1b) $\quad \operatorname{Rad} B = X^{\perp} = \{z \mid B(z,x) = 0 \text{ for all } x\}.$

The form is **singular** if it has nonzero radical ($\mathrm{Rad}\, B \neq 0$), **nonsingular** if its radical is zero ($\mathrm{rad}\, B = 0$: $B(z,X) = 0 \Rightarrow z = 0$), and **totally singular** if its radical is the whole space X ($\mathrm{Rad}\, B = X$: $B(x,y) = 0$ for all x,y).

We have carefully reserved the words "isotropic" and "degeneracy" for quadratic forms, and "singularity" for bilinear forms. (This is not standard; one often speaks of **nondegenerate** instead of nonsingular bilinear forms, **nonisotropic** instead of nonsingular subspaces $Y \subset X$ with $\mathrm{Rad}\, Y = Y \cap Y^{\perp} = 0$, and **isotropic** instead of singular subspaces Y with $\mathrm{Rad}\, Y = Y \cap Y^{\perp} \neq 0$).

Comparing definitions (2.1a) and (2.1b) shows the radical of a quadratic form is always contained in the radical of the corresponding polarized bilinear form,

$$\mathrm{Rad}\, Q = \mathrm{Rad}\, Q(\cdot,\cdot) \cap \{z \mid Q(z) = 0\} \subset \mathrm{Rad}\, Q(\cdot,\cdot).$$

Put another way,

$$Q(\cdot,\cdot) \text{ nonsingular} \Rightarrow Q \text{ nondegenerate}.$$

Of course, the distinction disappears in characteristic $\neq 2$: if $z \in \mathrm{Rad}\, Q(\cdot,\cdot)$ then $2Q(z) = Q(z,z) = 0$ implies $Q(z) = 0$ and hence $z \in \mathrm{Rad}\, Q$,

$$\text{(2.2)} \qquad \begin{array}{l} \mathrm{Rad}\, Q = \mathrm{Rad}\, Q(\cdot,\cdot) \\ Q \text{ nondegenerate} \Leftrightarrow Q(\cdot,\cdot) \text{ nonsingular}. \end{array} \qquad \text{(characteristic } \neq 2)$$

WARNING: In characteristic 2 these concepts are not equivalent. Do not confuse nondegeneracy with nonsingularity, or the radical of the quadratic form with the radical of its polarized bilinear form. Nondegeneracy does not imply nonsingularity, that Rad $Q(\cdot,\cdot) = 0$, but only that all nonzero elements $z \in \text{Rad } Q(\cdot,\cdot)$ have nonzero norm $Q(z) \neq 0$. ∎

Degeneracy of the norm form corresponds to a certain degeneracy of the algebra. The next proposition gives us a glimpse of the interrelation of the various radicals considered in Chapter IV.

2.3 (Radical Proposition) If A is an alternative algebra of degree 2 then Rad A = Nil(A) = S(A) = P(A) = Triv (A) = Rad n is an ideal which consists entirely of elements z which are strictly trivial and generate trivial ideals $Z = Az = zA$.

Proof. Rad n is a left ideal since $n(xz) = n(x)n(z) = 0$ by (1.16) and $n(xz,y) = n(z,x^*y) = 0$ by (1.15), so that $xz \in \text{Rad } n$ when $z \in \text{Rad } n$. Similarly it is a right ideal. Its elements are trivial by the U-formula (1.19), $U_z y = n(z,y^*)z - n(z)y^* = 0$. We have $Az = zA$ since by (1.6) $t(z)1 = n(z,1)1 = 0$ implies $z^* = -z$, then $0 = n(z,x)1 = zx^* + xz^* = zx^* - xz$ implies

$$xz = zx^* .$$

$Z = Az = zA$ is (for example) a left ideal since $x(zy) = -x(z^*y) = z(x^*y) - n(x,z)y = z(x^*y) \in Z$; it is trivial since $Z^2 = (zA)(Az) = U_z A^2$ (middle Moufang) $= 0$ by triviality of z.

In general we know Rad (A) $\supset$ Nil(A) $\supset$ S(A) $\supset$ P(A) and S(A) $\supset$ Triv(A) by IV.0.0, and since A is algebraic (of degree 2) we know Rad(A) = Nil(A) by IV.0.0. By the above we know P(A) $\supset$ Rad n (since Z trivial implies $Z \subset P(A)$) and Triv(A) $\supset$ Rad n, so equality will be established if we show Nil(A) $\subset$ Rad n.

If B is a nil ideal in A we must have $n(B) = 0$ since if $b^n = 0$ then $n(b)^n = n(b^n) = 0$ (by 1.16) forces $n(b) = 0$. Then $b^2 = t(b)b$ implies $b^n = t(b)^{n-1}b$ for all n by induction; if $b^n = 0$ then $t(b)^{n-1} = 0$, again forcing $t(b) = 0$. Hence $n(b,a) = t(ba^*)$ (by (1.11) $\in t(B) = 0$. This shows $n(B) = n(B,A) = 0$, and all nil ideals B are contained in Rad n. Thus Nil(A) $\subset$ Rad n. ■

2.4 Corollary. The following are equivalent for an alternative algebra of degree 2:

(i) A is semisimple

(ii) A is semiprime

(iii) A is strongly semiprime

(iv) A has no nil ideals

(v) the norm form is nondegenerate. ■

If A is a unital alternative algebra of degree 2 then A has scalar involution $xx^* = n(x)1$ and $n(x)$ permits composition. Conversely, the presence of a well-behaved norm form permitting composition implies the algebra is alternative of degree 2. To see that just having any old norm form permitting composition is not enough, consider the following example: take any horrible algebra A_o and tack on a unit, $A = \Phi 1 \oplus A_o$; then the norm form $n(x) = \alpha^2$ $(x = \alpha 1 + a)$ permits composition, $n(xy) = \alpha^2\beta^2 = n(x)n(y)$, but tells us nothing about A_o (since there's nothing to tell).

Thus it is reasonable to require the quadratic form n to be nondegenerate, i.e. there is no element $z \neq 0$ with

$$n(z) = n(z,A) = 0 \ .$$

(We allow $n(z,A) = 0$ as long as $n(z) \neq 0$, so the bilinear form $n(\cdot,\cdot)$ can be singular)

2.5 (Composition Criterion) If a unital algebra A carries a nondegenerate quadratic form n permitting composition,

$$n(xy) = n(x)n(y) \ ,$$

then A is alternative of degree 2 over Φ with norm n.

Proof. We begin by noticing that

$$n(1) = 1 \ .$$

Indeed, $n(1) = n(1\cdot 1) = n(1)n(1)$ shows $n(1) = 1$ or 0 in the field Φ, and if $n(1) = 0$ then all $n(x) = n(1\cdot x) = n(1)n(x) = 0$, contrary to nondegeneracy of the quadratic form n.

Linearization $y \to y,z$ in the composition formula $n(xy) = n(x)n(y)$ yields

$$n(xy,xz) = n(x)n(y,z) \ ,$$

then linearization $x \to x,1$ yields

$$n(xy,z) + n(y,xz) = n(x,1)n(y,z) \ .$$

If we <u>define</u> $t(x) = n(x,1)$ and <u>define</u> $x^* = t(x)1 - x$ then we can rewrite the above relation as

$$n(xy,z) = n(y,x^*z) \ .$$

From this we see $n(x^*(xy),z) = n(xy,xz) = n(x)n(y,z)$ (by the above) $= n(n(x)y,z)$ and also $n(x^*(xy)) = n(x^*)n(x)n(y) = n(x)^2n(y) = n(n(x)y)$ (noting $n(x^*) = n(t(x)1-x) = t(x)^2n(1) - t(x)n(1,x) + n(x) = t(x)^2 - t(x)t(x) + n(x) = n(x)$). Now anytime $n(a) = n(b)$ and $n(a,z) = n(b,z)$ for all z we have $n(a-b,z) = 0$ for all z and $n(a-b) = n(a) - n(a,b) + n(b) = n(a) - n(a,a) + n(a) = 0$, therefore by nondegeracy $a-b = 0$.

In our case this says

$$x^*(xy) = n(x)y .$$

Setting $y = 1$ establishes the degree 2 nature of A

$$x^*x = n(x)1 \quad \text{or} \quad x^2 - t(x)x + n(x)1 = 0 .$$

Remember $n(1) = 1$, so $t(1) = n(1,1) = 2$:

$$n(1) = 1,\ t(1) = 2 .$$

Now $x^*(xy) = n(x)y = \{n(x)1\}y = (x^*x)y$, or $[x^*,x,y] = 0$. Thus $0 = [t(x)1 - x,x,y] = -[x,x,y]$ and A is left alternative. Similarly A is right alternative. ■

In view of this we will call a unital algebra over a field which carries a nondegenerate quadratic form permitting composition a **composition algebra**; these are necessarily alternative of degree 2, with scalar involution.

Summarizing our results,

2.6 (Equivalence Theorem for Composition Algebras) The following are equivalent for a unital algebra A over a field Φ:

(i) A is a composition algebra: it carries a nondegenerate quadratic form n which permits composition, $n(xy) = n(x)n(y)$

(ii) A is an alternative algebra of degree 2,

$x^2 - t(x)x + n(x)1 = 0$ where t is linear and n nondegenerate quadratic

(iii) A is a strongly semiprime (resp. semiprime, resp. semisimple) alternative algebra of degree 2 over Φ

(iv) A is a strongly semiprime (resp. semiprime, resp. semisimple) alternative algebra with scalar involution, $xx^* \in \Phi 1$. ∎

VII.2 Exercises

2.1 Find an expression for $n(U_x y)$ in a composition algebra. What can you say about $n(x \circ y)$? Prove $t(x^*) = t(x)$, $n(x^*) = n(x)$, $t(xy^*) = t(yx^*)$, $t(x,y) = t(xy)$ is a symmetric associative bilinear form.

2.2 Suppose $B \triangleleft A$ where A is alternative with scalar involution over a field Φ, and all elements $1-b$ for $b \in B$ are invertible (all b are quasi-invertible). Show directly B is a nil ideal, hence $B \subset \text{Rad } n$.

2.3 If $t(e) = 1$, $n(e) = 0$ show e is an idempotent $\neq 0, 1$. If A is of degree 2 over a field Φ, show any idempotent $e \neq 0, 1$ has $t(e) = 1$, $n(e) = 0$.

2.4 If $x_1(x_2(\cdots x_n)) = 1$ in a composition algebra, show each x_i is invertible. More generally, each one-sided inverse is two-sided.

2.5 In any composition algebra, establish the formulas

$$n(xy,zw) + n(xw,zy) = n(x,z)n(y,w)$$

$$x(yz) + \bar{y}(\bar{x}z) = t(x,y)z = (zy)x + (z\bar{x})\bar{y}$$

$$(xy)z - (\bar{y}\bar{z})x = t(x,y)z - t(x,\bar{z})\bar{y} = z(yx) - x(\bar{z}\bar{y}) .$$

2.6 Define the radical of a quadratic form over an arbitrary ring of scalars Φ. If n is nondegenerate show that $\alpha n(x) = 0$ for all x implies $\alpha = 0$. Conclude that if $n(x)n(1) = n(x)$ for all x then $n(1) = 1$.

2.7 Show directly that if z is trivial in a degree 2 algebra over a ring Φ without nilpotent elements, then z belongs to Rad n.

2.8 Show that if A is an alternative algebra with scalar involution over an arbitrary Φ then Rad n is a nil ideal which consists entirely of elements z which generate trivial ideals $Z = Az = zA$. If Φ contains no nilpotent elements show Rad n is the maximal nil ideal Nil(A).

2.9 Show that if a unital algebra A over an arbitrary ring of scalars Φ carries a nondegenerate quadratic form n permitting composition, $n(xy) = n(x)n(y)$ <u>and</u> $n(1) = 1$, then A is alternative of degree 2 over Φ with norm n.

VII. 2.1 Problem Set on Quasi-Composition Algebras

An algebra A carrying a nondegenerate quadratic form satisfying $n(xy) = n(x)n(y)$ is called a **quasi-composition algebra**; if A is unital it is an ordinary composition algebra. Throughout let A be quasi-composition over a field Φ.

1. Show that there is some $u \in A$ with $n(u) = 1$; in this case show L_u, R_u are injective. If A is finite-dimensional or a division algebra show L_u, R_u are bijective.
2. <u>Assume</u> L_u, R_u are bijective (not necessarily that A is finite dimensional). Define an isotope $\tilde{A}$ by $x \cdot y = (R_u^{-1}x)(L_u^{-1}y)$. Show $\tilde{A}$ is unital, that n still permits composition on $\tilde{A}$, and is still nondegenerate. Conclude $\tilde{A}$ is an alternative composition algebra.
3. Show how to recover A from the algebra $\tilde{A}$. Is A necessarily unital or alternative?