§7. Lifting theorems

If N is a nil ideal we can lift families of orthogonal (resp. connected, interconnected, Cayley-connected) idempotents, families of associative matrix units, and families of Cayley matrix units from the algebra A/N into A. From this we deduce that an associative matrix algebra or Cayley matrix algebra can be lifted from A/N to A.

Because they are essentially exercises in Peirce relations, we will include at this point a series of lifting theorems which are the core of the Wedderburn Splitting Theorem discussed in Chapter VIII.

A structure or collection of elements in A/B can be *lifted* to A if there is a preimage in A having the same form. We say an ideal B in an alternative algebra A is a *lifting ideal* if idempotents mod B can be lifted: whenever $\bar{e} \in \bar{A} = A/B$ is an idempotent we can lift it back to an idempotent $e \in A$ with $\pi(e) = \bar{e}$ ($A \xrightarrow{\pi} \bar{A}$ the canonical projection). If we can lift $\bar{e}$ to an idempotent e which is a polynomial $e = p(x)$ for any preimage x of $\bar{e}$, we say B is a *strong lifting ideal*. The important example is

7.1 (Idempotent Lifting Theorem) Any nil ideal N in A is a strong lifting ideal: if $\pi(x) = \bar{e}$ is an idempotent in

$\bar{A} = A/N$ there is an idempotent $e = p(x)$ in A with $\pi(e) = \bar{e}$.

Proof. $B = \Phi[x]$ is an associative subalgebra of A, $M = B \cap N$ a nil ideal in B, and $\bar{B} = B/M = \pi(B)$ contains $\bar{e}$. By the associative Lifting Theorem (see Ex. 1, 2) there is a polynomials $p(x) = e$ with zero constant term which is an idempotent covering $\bar{e}$, $\pi(e) = \bar{e}$. ∎

Once we know how to lift one idempotent we can lift a whole string of them.

7.2 (Orthogonal Idempotent Lifting Theorem) If N is a strong lifting ideal then any countable family $\bar{e}_1, \bar{e}_2, \cdots$ of orthogonal idempotents in $\bar{A} = A/N$ may be lifted to an orthogonal family $e_1, e_2, \cdots$ in A. If N is quasi-invertible, when the $\bar{e}_i$ are connected (resp. interconnected or Cayley-connected) in $\bar{A}$ then so are the e_i in A, indeed strongly connecting (resp. Cayley-connecting) elements in $\bar{A}$ can be lifted to strongly connecting (resp. Cayley-connecting) elements in A.

Proof. We proceed by induction. The induction gets off the ground with the lifting of $\bar{e}_1$ to e_1, due to the hypothesis N is a lifting ideal. Suppose we have lifted $\bar{e}_1, \cdots, \bar{e}_n$ to $e_1, \cdots, e_n$ in A and set $e = e_1 + \cdots + e_n$. Since $\bar{e}_{n+1} =$

$(1-\bar{e})\bar{e}_{n+1}(1-\bar{e}) \in (\bar{1}-\bar{e})\bar{A}(\bar{1}-\bar{e}) = \overline{(1-e)A(1-e)}$ we can find a preimage x for $\bar{e}_{n+1}$ lying in $A_{00}(e) = (1-e)A(1-e)$. By the strong lifting hypothesis, there is a polynomial $e_{n+1} = p(x)$ without constant term which is an idempotent covering $\bar{e}_{n+1}$. Since $x \in A_{00}$ and the latter is a subalgebra, $e_{n+1} = p(x)$ also belongs to A_{00}. Since all e_i belong to $A_{11} = eAe$, Peirce orthogonality automatically gives orthogonality $e_i e_{n+1} = e_{n+1}e_i = 0$ $(1 \leq i \leq n)$. Thus $e_1,\cdots,e_{n+1}$ is an orthogonal family, and the induction is complete.

Now assume N is q.i. If $\bar{e}_i$, $\bar{e}_j$ are interconnected, so $\bar{e}_i$ lies in $\overline{A_{ij}}\ \overline{A_{ji}} = \overline{A_{ij}A_{ji}}$, then there is a preimage f_{ii} of $\bar{e}_i$ in $A_{ij}A_{ji}$. Since e_i is also a preimage of $\bar{e}_i$, the difference $z_{ii} = e_i - f_{ii}$ has $\bar{z}_{ii} = 0$ and therefore lies in N: $z_{ii} \in A_{ii} \cap N$. Then $f_{ii} = e_i - z_{ii} = e_i(1-z_{ii}) = (1-z_{ii})e_i$ is invertible in A_{ii} with inverse $f_{ii}^{-1} = e_i(1-z_{ii})^{-1} = (1-z_{ii})^{-1}e_i$ (remember $z_{ii} \in N$ is q.i.). Since $A_{ij}A_{ji}$ is an ideal in A_{ii} (by 4.1) containing the invertible element f_{ii}, it contains all of A_{ii}, and consequently e_i and e_j are interconnected.

If $\bar{e}_i$, $\bar{e}_j$ are (strongly) connected, $\bar{e}_{ij}\bar{e}_{ji} = \bar{e}_i$ for $\bar{e}_{ij} \in \bar{A}_{ij}$, $\bar{e}_{ji} \in \bar{A}_{ji}$, we can choose preimages $f_{ij} \in A_{ij}$, $f_{ji} \in A_{ji}$ of $\bar{e}_{ij}$, $\bar{e}_{ji}$. Then $f_{ii} = f_{ij}f_{ji}$ has $\pi(f_{ii}) = \bar{e}_{ij}\bar{e}_{ji} = \bar{e}_i$, $f_{ii} = e_i - z_{ii}$ for $z_{ii} \in A_{ii} \cap N$, so f_{ii} is invertible in A_{ii} if N is q.i., and e_i, e_j are connected in A. Once e_i, e_j are conneted by f_{ij}, f_{ji} we know they are strongly connected by

$e_{ij} = f_{ii}^{-1} f_{ij}$, $e_{ji} = f_{ji}$ by the Strong Connection Lemma. Here the connecting elements e_{ij} cover the $\bar{e}_{ij}$ since $\pi(e_{ji}) = \pi(f_{ji}) = \bar{e}_{ji}$, $\pi(e_{ij}) = \pi(f_{ii})^{-1}\pi(f_{ij}) = \bar{e}_i^{\,-1}\,\bar{e}_{ij} = \bar{e}_{ij}$.

Finally, suppose $\bar{e}_1$, $\bar{e}_2$ are (strongly) Cayley-connected: $\bar{e}_{12}^{(1)}(\bar{e}_{12}^{(2)}\bar{e}_{12}^{(3)}) = \bar{e}_1$, $(\bar{e}_{12}^{(2)}\bar{e}_{12}^{(3)})\bar{e}_{12}^{(1)} = \bar{e}_2$ for $\bar{e}_{12}^{(k)} \in \bar{A}_{12}$. Choose preimages $f_{12}^{(k)} \in A_{12}$ of the $\bar{e}_{12}^{(k)}$. Note $f_{12}^{(1)}\{f_{12}^{(2)}f_{12}^{(3)}\} = f_{11}$ covers $\bar{e}_{12}^{(1)}\{\bar{e}_{12}^{(2)}\bar{e}_{12}^{(3)}\} = \bar{e}_1$ and $\{f_{12}^{(2)}f_{12}^{(3)}\}\ f_{12}^{(1)} = f_{22}$ covers $\{\bar{e}_{12}^{(2)}\bar{e}_{12}^{(3)}\}\bar{e}_{12}^{(1)} = \bar{e}_2$; once more $f_{ii} = e_i - z_{ii}$ where z_{ii} lies in $N \cap A_{ii}$. Since N is q.i., the elements $f_{ii} = e_i - z_{ii}$ are again invertible in A_{ii}. By the strong Cayley-connection Lemma 6.2 the elements $e_{12}^{(1)} = f_{11}^{-1}\ f_{12}^{(1)}$, $e_{12}^{(2)} = f_{12}^{(2)}$, $e_{12}^{(3)} = f_{12}^{(3)}$ strongly Cayley-connect e_1 and e_2. Here $\pi(e_{12}^{(k)}) = \pi(f_{12}^{(k)}) = e_{12}^{(k)}$ for $k = 2, 3$, and $\pi(e_{12}^{(1)}) = \pi(f_{11})^{-1}\pi(f_{12}^{(1)}) = \bar{e}_1^{-1}\ \bar{e}_{12}^{(1)} = \bar{e}_{12}^{(1)}$, so the $e_{12}^{(k)}$ cover the $\bar{e}_{12}^{(k)}$. ∎

The theorem is actually true when N is only weak lifting (see Problem Set).

7.3 (Matrix Units Lifting Theorem) If N is a q.i. strong lifting ideal in an alternative algebra A, then any family of associative matrix units for A/N can be lifted to one for A. If A is unital and the original matrix units in A/N are supplementary, so automatically are those in A.

Proof. If $\{\bar{e}_{ij}\}$ are matrix units for $\bar{A}$ then $\bar{e}_{1i}$, $\bar{e}_{i1}$ strongly connect $\bar{e}_{11}$ to $\bar{e}_{ii}$, so by Orthogonal Idempotent Lifting 7.2 we can lift to orthogonal idempotents e_{11}, e_{ii} in A strongly connected by elements e_{1i}, e_{i1} in A_{1i}, A_{i1}. By the Corollary 5.3 to the Transitivity Lemma, the elements $e_{ij} = e_{i1}e_{1j}$, $e_{ji} = e_{j1}e_{1i}$ strongly connect e_{ii} to e_{jj} and satisfy $e_{ij}e_{jk} = e_{ik}$ for $i,j,k \neq$ (since $e_{ij}e_{jk} = (e_{i1}e_{1j})e_{jk} = e_{i1}(e_{1j}e_{jk})$ $= e_{i1}\{e_{1j}(e_{j1}e_{1k})\} = e_{i1}\{(e_{1j}e_{j1})e_{1k}\} = e_{i1}e_{1k} = e_{ik}$, where associativity follows from the fact that three e_{mn}, e_{pq}, e_{rs} either belong to distinct Peirce spaces or else two of them coincide). That is, the $\{e_{ij}\}$ form a family of matrix units, which cover the original $\bar{e}_{ij}$ since $\pi(e_{ij}) = \pi(e_{i1}e_{1j}) = \pi(e_{i1})\pi(e_{1j}) = \overline{e_{i1}}\ \overline{e_{1j}} = \overline{e_{ij}}$.

If A is unital and $\bar{1} = \Sigma\ \overline{e_{ii}}$, then $e = 1 - \Sigma\ e_{ii}$ has $\bar{e} = 0$ and consequently $e \in N$. Yet e is idempotent since $\Sigma\ e_{ii}$ is by orthogonality of the e_{ii}. The only idempotent which is q.i. is $e = 0$, so $1 = \Sigma\ e_{ii}$. ∎

We could have lifted the $\bar{e}_{ij}$ directly to elements e_{ij} strongly connecting e_{ii}, e_{jj}, but unless we are careful there is no reason why $e_{ij}e_{jk} = e_{ik}$ for $i,j,k \neq$.

Having a family of associative matrix units $\{e_{ij}\}_{i,j=1}^{n}$ is the same as having a matrix subalgebra $\Sigma\Phi e_{ij} \cong M_n(\Phi)$, so

7.4 (Matrix Algebra Lifting Theorem) If N is a nil ideal in an alternative algebra A then any matrix subalgebra $\bar{B} \cong M_n(\Phi)$ of $\bar{A} = A/N$ can be lifted to a matrix subalgebra $B \cong M_n(\Phi)$ of A. ∎

We can do the same thing for Cayley matrix units.

7.5 (Cayley Units Lifting Theorem) If N is a q.i. lifting ideal in an alternative algebra A, then any family of Cayley units for A/N can be lifted to one for A. If A is unital and the original family is supplementary in A/N, so automatically is the family in A.

Proof. Suppose $\bar{e}_{ii}$, $\bar{e}_{ij}^{(k)}$ $(1 \le i,j \le 2,\ 1 \le k \le 3)$ are Cayley units in $\bar{A}$. Then by the Orthogonal Idempotent Lifting Theorem 7.2 the strongly Cayley-connecting elements $\bar{e}_{12}^{(1)}$, $\bar{e}_{12}^{(2)}$, $\bar{e}_{12}^{(3)}$ can be lifted to strongly Cayley-connecting elements $e_{12}^{(1)}$, $e_{12}^{(2)}$, $e_{12}^{(3)}$. By the Cayley Units Construction 6.3 the elements $e_{12}^{(1)}$, $e_{12}^{(2)}$, $e_{12}^{(3)}$ and $e_{21}^{(1)} = e_{12}^{(2)}e_{12}^{(3)}$, $e_{21}^{(2)} = e_{12}^{(3)}e_{12}^{(1)}$, $e_{21}^{(3)} = e_{12}^{(1)}\ {}_{12}^{(2)}$ form a set of Cayley matrix units for A. Since the $e_{12}^{(k)}$ cover $\bar{e}_{12}^{(k)}$ we see $\pi(e_{21}^{(k)}) = \pi(e_{12}^{(k+1)})\pi(e_{12}^{(k+2)}) = \bar{e}_{12}^{(k+1)}\bar{e}_{12}^{(k+2)} = \bar{e}_{21}^{(k)}$ and all $e_{ij}^{(k)}$ cover $\bar{e}_{ij}^{(k)}$.

The supplementary case follows as in 7.3. ∎

Since having a family of Cayley units $\{e_{ij}^{(k)}\}$ is the same as having a subalgebra $\Sigma\Phi e_{ij}^{(k)} \cong \mathbb{C}(\Phi)$,

7.6 (Cayley Algebra Lifting Theorem). If N is a nil ideal in an alternative algebra A then any split Cayley subalgebra $\bar{C} \cong \mathbb{C}(\Phi)$ of $\bar{A} = A/N$ can be lifted to a split Cayley subalgebra $C \cong \mathbb{C}(\Phi)$ of A. ∎

VI. 7 Exercises

7.1 Show that if $x^2 - x = z$ is nilpotent then there is an idempotent $e = x + \sum_1^n \alpha_i z^i + x \sum_1^n \beta_i z^i$ where α_i, β_i are defined recursively by

$$\alpha_k = 2\beta_{k-1} + \Sigma_{i+j=k-1} \beta_i\beta_j + \Sigma_{i+j=k} \alpha_i\alpha_j$$

$$\beta_k = - 2\alpha_k - \Sigma_{i+j=k} (\alpha_i\beta_j + \alpha_j\beta_i + \beta_i\beta_j).$$

7.2 Alternately, if $x^2 - x = z$ is nilpotent show inductively there are x_n with $x_n^2 - x_n \in z^n\Phi[z]$ and $x_n \in x + z\Phi[z]$, so if $z \in N$ has $z^m = 0$ then $e = x_m$ is idempotent with $\bar{e} = \bar{x}$.

7.3 Show that any ideal N which is a direct summand of A is a lifting ideal, but give an example to see it need not be strongly lifting. Show N is a lifting ideal if it is merely a semidirect summand of A.

VI. 7.1 Problem Set on Lifting Ideals

1. If N is a strong lifting ideal in A and e is an idempotent, show eNe is a strong lifting ideal in eAe.

2. If N is a q.i. lifting ideal in an associative algebra A, show eNe is a lifting ideal in eAe. (It is important that N be q.i.).

3. If N is a q.i. lifting ideal in an alternative algebra A, show eNe is a lifting ideal in eAe.

4. Prove the Orthogonal Lifting Theorem. If N is a q.i. lifting ideal then any countable family $\bar{e}_1, \bar{e}_2, \cdots$ of orthogonal idempotents in $\bar{A} = A/N$ may be lifted to an orthogonal family $e_1, e_2, \cdots$ in A.