§6. Cayley Matrix units

Cayley matrix units are easily constructed from Cayley-connected idempotents. We establish the Zorn Coordinatization Theorem, which asserts that an alternative algebra with a family of Cayley matrix units is a Cayley matrix algebra. Ideals in a Cayley matrix algebra $\mathbb{C}(\Omega)$ are precisely all $\mathbb{C}(\Lambda)$ for ideals Λ in Ω.

A family of *Cayley matrix units* (or simply *Cayley units*) in an alternative algebra is a family of eight elements $e_{11}, e_{22}, e_{12}^{(k)}, e_{21}^{(k)}$ $(k = 1,2,3)$ which multiply like the matrix units for a Cayley matrix algebra in I. 1.15: e_{11}, e_{22} are orthogonal idempotents and $e_{ij}^{(k)}$ belong to the Peirce space A_{ij}, such that

(I) $e_{12}^{(k)}e_{21}^{(k)} = e_{11},\ e_{21}^{(k)}e_{12}^{(k)} = e_{22},\ e_{12}^{(k)}e_{21}^{(\ell)} = e_{21}^{(\ell)}e_{12}^{(k)} = 0\ (k \neq \ell)$

(II) $e_{12}^{(1)}\ e_{12}^{(2)} = e_{21}^{(3)},\ e_{12}^{(2)}\ e_{12}^{(3)} = e_{21}^{(1)},\ e_{12}^{(3)}e_{12}^{(1)} = e_{21}^{(2)}$

(III) $e_{21}^{(2)}e_{21}^{(1)} = e_{12}^{(3)},\ e_{21}^{(3)}\ e_{21}^{(2)} = e_{12}^{(1)},\ e_{21}^{(1)}\ e_{21}^{(3)} = e_{12}^{(2)}.$

Thus we have three sets of ordinary matrix units $e_{11}, e_{22}, e_{12}^{(k)}, e_{21}^{(k)}$ which are "orthogonal" to each other and generate each other cyclically:

$$\text{(I)}\quad e_{ij}^{(k)}\ e_{ji}^{(k)} = e_{ii},\ e_{ij}^{(k)}\ e_{ji}^{(\ell)} = 0 \quad (k \neq \ell)$$

$$(6.1) \qquad \text{(II)} \quad e_{12}^{(k)}\, e_{12}^{(k+1)} = e_{21}^{(k+2)} \qquad (k \bmod 3)$$

$$\text{(III)} \quad e_{21}^{(k+1)}\, e_{21}^{(k)} = e_{12}^{(k+2)} \qquad (k \bmod 3)\ .$$

WARNING: the product of two e_{12}'s generates the next e_{21} if the indices are a cyclic permutation of (123) (<u>increasing</u> from left to right), whereas the product of two e_{21}'s generates the next e_{12} if the indices are a cyclic permutation of (321) (<u>decreasing</u> from left to right). The situation is not symmetric in the subscripts 1 and 2: the indices go up for the e_{12}'s and down for the e_{21}'s.

Since $e_{21}^{(k)} e_{21}^{(k+1)} = - e_{21}^{(k+1)} e_{21}^{(k)} = - e_{12}^{(k+2)}$ (recall $x_{21}y_{21} + y_{21}x_{21} = 0$), we can summarize (II) and (III) in

$$(6.1) \qquad \text{(IV)} \quad e_{ij}^{(k)}\, e_{ij}^{(k+1)} = \varepsilon_{ij}\, e_{ji}^{(k+2)} \quad (\varepsilon_{12} = 1,\ \varepsilon_{21} = -1)\ .$$

Other notations for Cayley matrix units are $f_k = e_{21}^{(k)}$, $g_k = e_{21}^{(k)}$ with multiplication rules

$$f_i g_i = e_{11},\ g_i f_i = e_{22},\ f_i g_j = g_j f_i = 0 \quad (i \neq j)$$

$$f_i f_i = g_i g_i = 0,\ f_i f_{i+1} = g_{i+2},\ g_{i+1} g_i = f_{i+2}$$

or $e_{ij} = e_{ij}^{(1)}$, $f_{ij} = e_{ij}^{(2)}$, $g_{ij} = e_{ij}^{(3)}$ with rules

$$e_{ij}e_{ji} = f_{ij}f_{ji} = g_{ij}g_{ji} = e_{ii}\ ,$$

$$e_{ij}f_{ji} = e_{ij}g_{ji} = f_{ij}g_{ji} = f_{ij}e_{ji} = g_{ij}e_{ji} = g_{ij}f_{ji} = 0,$$

$e_{12}f_{12} = g_{21},\ f_{12}g_{12} = e_{21},\ g_{12}e_{12} = f_{21},$

$f_{21}e_{21} = g_{12},\ g_{21}f_{21} = e_{12},\ e_{21}g_{21} = f_{12},$

$e_{ij}{}^2 = f_{ij}{}^2 = g_{ij}{}^2 = 0.$

We say two orthogonal idempotents e_1, e_2 are **strongly Cayley-connected** if there are $x,y,z \in A_{12}$ with

$$x(yz) = e_1,\ (yz)x = e_2\ .$$

They are merely **Cayley-connected** if there exist $x,y,z \in A_{12}$ with

$$x(yz) = a_{11},\ (yz)x = a_{22}\quad (a_{ii} \text{ invertible in } A_{ii})\ .$$

Notice e_1, e_2 are strongly or weakly Cayley-connected iff they are strongly or weakly connected in the ordinary sense by elements $x \in A_{12}$ and $w = yz \in A_{12}^2 \subset A_{21}$, in other words where the second connecting element lies in A_{12}^2 rather than merely in A_{21}.

The Strong Connection Lemma 5.1 says that if $x \in A_{12}$, $w = yz \in A_{21}$ connect e_1 and e_2 then w and $a_{11}^{-1}x = xa_{22}^{-1}$ strongly connect e_1 and e_2 ($xw = a_{11}$, $wx = a_{22}$). This leads immediately to

6.2 (Strong Cayley-Connection Lemma) Two orthogonal idempotents e_1, e_2 in an alternative algebra A are Cayley-connected iff they are strongly Cayley-connected: if $x(yz) = a_{11}$, $(yz)x = a_{22}$ for a_{ii} invertible in A_{ii} then $x'(yz) = e_1$, $(yz)x' = e$ for $x' = a_{11}^{-1}x = xa_{22}^{-1}$.

Just as we can construct matrix units e_{ij} from the connecting elements e_{1i} and e_{i1} alone, so we can construct Cayley matrix units $e_{ij}^{(k)}$ from the Cayley-connecting elements x,y,z.

6.3 (Cayley Units Construction) If orthogonal idempotents e_1, e_2 are strongly Cayley-connected by elements $x,y,z \in A_{12}$ then

$$e_{11} = e_1, \; e_{22} = e_2 \; ,$$

$$e_{12}^{(1)} = x, \; e_{12}^{(2)} = y, \; e_{12}^{(3)} = z, \; e_{21}^{(1)} = yz, \; e_{21}^{(2)} = zx, \; e_{21}^{(3)} = xy$$

form a family of Cayley Matrix units for A.

Proof. By assumption the $e_{12}^{(k)}$ belong to A_{12}, and the $e_{21}^{(k)}$ belong to A_{21} since $A_{12}^2 \subset A_{21}$. The Permuting Formulas (3.9) and (3.15) say the products $x(yz)$ and $(yz)x$ are alternating, in particular invariant under cyclic permutations of the indices 1,2,3, so our hypotheses are cyclically invariant. Therefore the full set of defining relations (6.1) will follow if we can show

$$e_{ij}^{(1)} e_{ji}^{(1)} = e_{ii}, \; e_{ij}^{(1)} e_{ji}^{(2)} = e_{ji}^{(2)} e_{ij}^{(1)} = 0 \; ,$$

$$e_{12}^{(1)} e_{12}^{(2)} = e_{21}^{(3)} \; , \; e_{21}^{(2)} e_{21}^{(1)} = e_{12}^{(3)} \; .$$

We have $e_{12}^{(1)}e_{21}^{(1)} = x(yz) = e_1 = e_{11}$ and $e_{21}^{(1)}e_{12}^{(1)} = (yz)x$ $= e_2 = e_{22}$ by the Cayley-connection hypothesis; $e_{12}^{(1)}e_{21}^{(2)}$ $= x(zx) = -x(xz) = -x^2z = 0$, since $x^2 = 0$ in A_{12} (or directly from the alternating nature of products $x_{12}(y_{12}z_{12})$), similarly $e_{21}^{(2)}e_{12}^{(1)} = (zx)x = 0$, $e_{21}^{(1)}e_{12}^{(2)} = (yz)y = 0$, $e_{12}^{(2)}e_{21}^{(1)} = y(yz)$ $= 0$. By definition $e_{12}^{(1)}e_{12}^{(2)} = xy = e_{21}^{(3)}$, and $e_{21}^{(2)}e_{21}^{(1)} =$ $(zx)(yz) = \{z(xy)\}z$ (middle Moufang) $= \{x(yz)\}z$ (by (3.9)) $= e_1 z = z = e_{12}^{(3)}$. ∎

Having discussed how to construct Cayley matrix units, let's turn to what we can do with them once we have them. From now on we will be interested chiefly in Cayley matrix units which are **supplementary** in the sense that the idempotents e_{11}, e_{22} are supplementary:

$$e_{11} + e_{22} = 1 .$$

We can use Cayley units to refine the Peirce decomposition.

6.4 (Cayley-Peirce Decomposition) If $\{e_{ij}^{(k)}\}$ are a supplementary family of Cayley matrix units for A then we have a Cayley-Peirce decomposition

$$A = A_{11} \oplus \{A_{12}^{(1)} \oplus A_{12}^{(2)} \oplus A_{12}^{(3)}\} \oplus \{A_{21}^{(1)} \oplus A_{21}^{(2)} \oplus A_{21}^{(3)}\} \oplus A_{22}$$

where for $i \neq j$

$$A_{ij}^{(k)} = A_{ii}e_{ij}^{(k)} = e_{ij}^{(k)}A_{jj} \subset A_{ij} .$$

Proof. To show $A_{ii}e_{ij}^{(k)} = e_{ij}^{(k)}A_{jj}$ it suffices by symmetry to show $A_{ii}e_{ij}^{(k)} \subset e_{ij}^{(k)}A_{jj}$, and this follows from $A_{ii}e_{ij}^{(k)} = e_{ii}(A_{ii}e_{ij}^{(k)}) = (e_{ij}^{(k)}e_{ji}^{k)})(A_{ii}e_{ij}^{(k)}) = e_{ij}^{(k)}\{(e_{ji}^{(k)}A_{ii})e_{ij}^{(k)}\} \subset e_{ij}^{(k)}A_{jj}$ by Middle Moufang.

Once $A_{ij}^{(k)}$ is well-defined we established directness. Since $e_{ij}^{(k)}e_{ji}^{(\ell)} = \delta_{k\ell}\, e_{ii}$, we can recover the coefficients $a_{ii}^{(\ell)}$ from $x = \Sigma_k a_{ii}^{(k)}e_{ij}^{(k)}$ by multiplying by $e_{ji}^{(\ell)}$: $xe_{ji}^{(\ell)} = \Sigma_k(a_{ii}^{(k)}e_{ij}^{(k)})e_{ji}^{(\ell)} = \Sigma_k a_{ii}^{(k)}(e_{ij}^{(k)}e_{ji}^{(\ell)}) = a_{ii}^{(\ell)}$. Thus the coefficients are unique, and the sum $A_{ij}^{(1)} \oplus A_{ij}^{(2)} \oplus A_{ij}^{(3)}$ is direct.

To show the $A_{ij}^{(k)}$ span A_{ij}, note $A_{ij} = e_{ii}A_{ij} = (e_{ij}^{(1)}e_{ji}^{(1)})A_{ij} = -(e_{ij}^{(1)}A_{ij})e_{ji}^{(1)} + e_{ij}^{(1)}(e_{ji}^{(1)}A_{ij})$ (by (3.10) or right alternativity) $\subset A_{ji}e_{ji}^{(1)} + e_{ij}^{(1)}A_{jj}$, where $e_{ij}^{(1)}A_{jj} \subset A_{ij}^{(1)}$ and $\varepsilon_{ij}A_{ji}e_{ji}^{(1)} = A_{ji}(e_{ij}^{(2)}e_{ij}^{(3)}) = -e_{ij}^{(2)}(A_{ji}e_{ij}^{(3)}) + (e_{ij}^{(2)}A_{ji})e_{ij}^{(3)}$ (by (3.10) or right alternativity) $\subset e_{ij}^{(2)}A_{jj} + A_{ii}e_{ij}^{(3)} \subset A_{ij}^{(2)} + A_{ji}^{(3)}$. ∎

6.5 (Cayley-Peirce Orthogonality Relations) The Cayley-Peirce spaces multiply according to the rules

(6.6) $$A_{ij}^{(k)}\,A_{ji}^{(k)} \subset A_{ii}\,,\quad A_{ij}^{(k)}\,A_{ji}^{(\ell)} = 0 \quad (k \neq \ell)$$

(6.7) $$A_{ij}^{(k)}\,A_{ij}^{(k)} = 0\,,\quad A_{ij}^{(k)}\,A_{ij}^{(k+1)} \subset A_{ji}^{(k+2)}$$

or, in terms of elements,

(6.8) $$(a_{ii}e_{ij}^{(k)})(e_{ji}^{(\ell)}b_{ii}) = \delta_{jk}\, a_{ii}b_{ii}$$

(6.9) $$(a_{ii}e_{ij}^{(k)})(b_{ii}e_{ij}^{(k+\varepsilon)}) = \varepsilon\ \varepsilon_{ij}\ e_{ji}^{(k+2)}(a_{ii}b_{ii}) \quad (\varepsilon=0,1).$$

Proof. By (6.4) it suffices to prove (6.8) and (6.9). For the first, note $(a_{ii}e_{ij}^{(k)})(e_{ji}^{(\ell)}b_{ii}) = a_{ii}(e_{ij}^{(k)}e_{ji}^{(\ell)})b_{ii}$; if $k \neq \ell$ this vanishes since $e_{ij}^{(k)}e_{ji}^{(\ell)} = 0$, while if $k = \ell$ then $e_{ij}^{(k)}e_{ji}^{(k)} = e_{ii}$.

For the second, $(a_{ii}e_{ij}^{(k)})(b_{ii}e_{ij}^{(\ell)}) = \{(a_{ii}e_{ij}^{(k)})e_{ij}^{(\ell)}\}b_{ii}$ $= \{e_{ij}^{(k)}e_{ij}^{(\ell)}\}a_{ii}b_{ii}$ by the Slipping Formula (3.8); if $k = \ell$ this vanishes since $e_{ij}^{(k)}e_{ij}^{(k)} = 0$, while if $\ell = k+1$ then $e_{ij}^{(k)}e_{ij}^{(k+1)} = \varepsilon_{ij}e_{ji}^{(k+2)}$. This establishes the orthogonality relations. ∎

The Wedderburn Coordinatization Theorem says that an associative algebra with matrix units is a matrix algebra. In the same way, the Zorn Coordinatization Theorem says that an alternative algebra with Cayley matrix units is a Cayley matrix algebra (ie. split Cayley algebra).

6.10 (Zorn Coordinatization Theorem) An alternative algebra with a supplementary family of Cayley matrix units $\{e_{ij}^{(k)}\}$ is a Cayley matrix algebra: $A \cong \mathbb{C}(\Omega)$ for $\Omega = A_{11}$.

Proof. We know $\Omega = A_{11}$ is associative since associators $[x_{11}, y_{11}, z_{11}]$ kill $e_{12}^{(1)}$ by (3.13), hence kill $e_{12}^{(1)}e_{21}^{(1)} = e_{11}$, and so must vanish (or by (3.12) since $A_{12}A_{21} = A_{11}$), and it is commutative since by (3.11) commutators $[x_{11}, y_{11}]$ kill $e_{21}^{(3)} e_{21}^{(2)} = e_{12}^{(1)}$, hence kill $e_{12}^{(1)} e_{21}^{(1)} = e_{11}$ again and so vanish (or kill $A_{21}^2 = A_{12}$ and hence $A_{12}A_{21} = A_{11}$). Thus Ω is a commutative associative "ring of scalars". We define "coordinate functions" $\pi_{ij}^{(k)}: A \to \Omega$ by taking $\pi_{ij}^{(k)}$ to be zero on all Cayley-Peirce spaces except $A_{ij}^{(k)}$, where they are defined by

$$\pi_{11}(a_{11}) = a_{11}$$

$$\pi_{22}(a_{22}) = e_{12}^{(k)} a_{22} e_{21}^{(k)} = e_{12}^{(k)} \{e_{12}^{(k+1)} (e_{12}^{(k+2)} a_{22})\}$$

$$\pi_{12}^{(k)} \ (a_{12}^{(k)}) = a_{12}^{(k)} \ e_{21}^{(k)}$$

$$\pi_{21}^{(k)} \ (a_{21}^{(k)}) = e_{12}^{(k)} a_{21}^{(k)}.$$

Notice that π_{22} is well-defined: the right expression is invariant under cyclic permutations by Permuting (3.9) and Slipping (3.8) and hence independent of k, and agrees with the middle expression since $e_{12}^{(k+1)} (e_{12}^{(k+2)} a_{22}) = a_{22} (e_{12}^{(k+1)} e_{12}^{(k+2)}) = a_{22} \ e_{21}^{(k)}$ by the Slipping Formula (3.8).

In more concrete terms,

$$\pi_{12}^{(k)} \ (a_{11} e_{12}^{(k)}) = a_{11}, \quad \pi_{12}^{(k)} (e_{12}^{(k)} a_{22}) = \pi_{22}(a_{22}),$$

(6.11)

$$\pi_{21}^{(k)}(e_{21}^{(k)}a_{11}) = a_{11}, \qquad \pi_{21}^{(k)}(a_{22}e_{21}^{(k)}) = \pi_{22}(a_{22}) .$$

The relations for a_{11} all follow from $e_{12}^{(k)}e_{21}^{(k)} = e_{11}$. Those involving a_{22} result from $\pi_{12}^{(k)}(e_{12}^{(k)}a_{22}) = (e_{12}^{(k)}a_{22})e_{21}^{(k)} = \pi_{22}(a_{22}) = e_{12}^{(k)}(a_{22}e_{21}^{(k)}) = \pi_{21}^{(k)}(a_{22}e_{21}^{(k)})$.

All $\pi_{ij}^{(k)}$ are bijections: this is clear for π_{11}, $\pi_{12}^{(k)}$ is the inverse of $R_{e_{12}^{(k)}}: A_{11} \to A_{11}e_{12}^{(k)}$, $\pi_{21}^{(k)}$ is the inverse of $L_{e_{21}^{(k)}}$, while $\pi_{22} = L_{e_{12}^{(k)}} R_{e_{21}^{(k)}}$ has inverse $L_{e_{21}^{(k)}} R_{e_{12}^{(k)}}$. Thus by (6.4) we have a linear bijection $\pi: A \to \mathbb{C}(\Omega)$ by $\pi(a) = \Sigma\, \pi_{ii}(a)e_{ii} + \Sigma\pi_{ij}^{(k)}(a)e_{ij}^{(k)}$. It remains to show π is an algebra homomorphism,

$$\pi(ab) = \pi(a)\pi(b) .$$

We may restrict our attention to a, b in Cayley-Peirce spaces, leaving us the cases

(6.12)

$$\begin{array}{lll}
\text{(i)} & \pi_{ii}(ab) = \pi_{ii}(a)\pi_{ii}(b) & (a,b \in A_{ii}) \\
\text{(ii)} & \pi_{ij}^{(k)}(ab) = \pi_{ii}(a)\pi_{ij}^{(k)}(b) & (a \in A_{ii},\ b \in A_{ij}^{(k)}) \\
\text{(iii)} & \pi_{ij}^{(k)}(ab) = \pi_{ij}^{(k)}(a)\ \pi_{jj}(b) & (a \in A_{ij}^{(k)},\ b \in A_{jj}) \\
\text{(iv)} & \pi_{ii}(ab) = \pi_{ij}^{(k)}(a)\ \pi_{ji}^{(k)}(b) & (a \in A_{ij}^{(k)},\ b \in A_{ji}^{(k)}) \\
\text{(v)} & \pi_{ji}^{(k+2)}(ab) = \varepsilon_{ij}\pi_{ij}^{(k)}(a)\pi_{ij}^{(k+1)}(b) & (a \in A_{ij}^{(k)},\ b \in A_{ij}^{(k+1)})
\end{array}$$

in view of Cayley-Peirce orthogonality (6.5).

These are almost immediate from the Cayley-Peirce relations. Because we have chosen Ω to be A_{11}, the situation is not symmetric in i and j. Using (6.4) to express $A_{ij}^{(k)}$ and (6.11) to express $\pi_{ij}^{(k)}$, these become

$$\text{(ia)} \quad \pi_{11}(a_{11}b_{11}) = a_{11}b_{11}$$

$$\text{(iia)} \quad \pi_{12}^{(k)}(a_{11}[b_{11}e_{12}^{(k)}]) = a_{11}b_{11}$$

$$\text{(iiia)} \quad \pi_{21}^{(k)}([e_{21}^{(k)}\, a_{11}]b_{11}) = a_{11}b_{11}$$

$$\text{(iva)} \quad \pi_{11}([a_{11}e_{12}^{(k)}][e_{21}^{(k)}b_{11}]) = a_{11}b_{11}$$

$$\text{(va)} \quad \pi_{21}^{(k+2)}([a_{11}e_{12}^{(k)}][b_{11}e_{12}^{(k+1)}]) = a_{11}b_{11}$$

$$\text{(ib)} \quad \pi_{22}(a_{22}b_{22}) = \pi_{22}(a_{22})\pi_{22}(b_{22})$$

$$\text{(iib)} \quad \pi_{21}^{(k)}(a_{22}[b_{22}e_{21}^{(k)}]) = \pi_{22}(a_{22})\pi_{22}(b_{22})$$

$$\text{(iiib)} \quad \pi_{12}^{(k)}([e_{12}^{(k)}a_{22}]b_{22}) = \pi_{22}(a_{22})\pi_{22}(b_{22})$$

$$\text{(ivb)} \quad \pi_{22}([a_{22}e_{21}^{(k)}][e_{12}^{(k)}b_{22}]) = \pi_{22}(a_{22})\pi_{22}(b_{22})$$

$$\text{(vb)} \quad \pi_{12}^{(k+2)}([a_{22}e_{21}^{(k)}][b_{22}e_{21}^{(k+1)}]) = -\pi_{22}(a_{22})\pi_{22}(b_{22}).$$

Here (ia) - (va) follow immediately from (6.11), using Peirce associativity in (iia) and (iiia), (6.8) in (iva), and (6.9) in (va) to simplify the product ab. Similarly, simplifying ab and using (6.11) shows (ib) - (vb) all reduce to $\pi_{22}(a_{22}b_{22}) = \pi_{22}(a_{22})\pi_{22}(b_{22})$. This latter can be done directly, but a sneakier proof reduces π_{22} to $\pi_{12}^{(k)}$: $\pi_{22}(a_{22}b_{22}) = e_{12}^{(k)}a_{22}b_{22}e_{21}^{(k)}$ $\pi_{11}((e_{12}^{(k)}a_{22})(b_{22}e_{21}^{(k)})) = \pi_{12}^{(k)}(e_{12}^{(k)}a_{22})\cdot \pi_{21}^{(k)}(b_{22}e_{21}^{(k)})$ (by the established case (iva)) $= \pi_{22}(a_{22})\pi_{22}(b_{22})$ (by (6.11)). ∎

As in the case of associative matrix algebras, the ideals of a Cayley matrix algebra $\mathfrak{C}(\Omega)$ can be reduced to those of the coordinate algebra Ω.

6.13 (Theorem) Any one-sided ideal of a Cayley matrix algebra $\mathfrak{C}(\Omega)$ is two-sided, and the two-sided ideals B are precisely all $\mathfrak{C}(\Lambda)$ for Λ an ideal in Ω: $B = \mathfrak{C}(\Lambda)$, $\Lambda = B \cap \Omega 1$.

Proof. Let B be a left ideal, and set $\Lambda = B \cap \Omega 1$; Λ is a left ideal in Ω, hence by commutativity a two-sided ideal. It will be more convenient to note $\Lambda = \{\lambda \in \Omega | \lambda e_{ii} \in B\}$; certainly if $\lambda \in B \cap \Omega 1$ then $\lambda e_{ii} = e_{ii}\lambda \in e_{ii}B \subset B$ if B is a left ideal, and conversely if λe_{ii} belongs to B so does $\lambda e_{jj} = \varepsilon_{ji}e_{ji}^{(3)}\{e_{ji}^{(1)}(e_{ji}^{(2)}(\lambda e_{ii}))\}$ and so also $\lambda 1 = \lambda e_{ii} + \lambda e_{jj}$.

Clearly $\mathfrak{C}(\Lambda) = \mathfrak{C}(\Omega)\cdot\Lambda 1 \subset \mathfrak{C}\cdot B \subset B$. On the other hand if B contains $b = \Sigma\, \beta_{ii}e_{ii} + \Sigma\, \beta_{ij}^{(k)}e_{ij}^{(k)}$ it also contains

$$e_{ij}^{(k+1)}\{e_{ij}^{(k+2)}(e_{ii}b)\} = \varepsilon_{ij}\beta_{ij}^{(k)}e_{ii} \;,$$

$$e_{ij}^{(k)}[e_{ij}^{(k+1)}\{e_{ij}^{(k+2)}(e_{jj}b)\}] = \varepsilon_{ij}\beta_{jj}e_{ii}$$

since $e_{ij}^{(k+2)}(e_{ii}b) = e_{ij}^{(k+2)}(\beta_{ii}e_{ii} + \Sigma\beta_{ij}^{(\ell)}e_{ij}^{(\ell)}) = \beta_{ij}^{(k)}e_{ij}^{(k+2)}e_{ij}^{(k)} + \beta_{ij}^{(k+1)}e_{ij}^{(k+2)}e_{ij}^{(k+1)} = \varepsilon_{ij}\{\beta_{ij}^{(k)}e_{jj}^{(k+1)} - \beta_{ij}^{(k+1)}e_{jj}^{(k)}\} = c$ has $e_{ij}^{(k+1)}c = \varepsilon_{ij}\beta_{ij}^{(k)}e_{ii}$, and $e_{ij}^{(k+2)}(e_{jj}b) = e_{ij}^{(k+2)}(\beta_{jj}e_{jj} + \Sigma\beta_{ji}^{(\ell)}e_{ji}^{(\ell)}) = \beta_{jj}e_{ij}^{(k+2)} + \beta_{ji}^{(k+2)}e_{ii} = d$ has $e_{ij}^{(k)}[e_{ij}^{(k+1)}d] = \beta_{jj}e_{ij}^{(k)}[e_{ij}^{(k+1)}e_{ij}^{(k+2)}] = \varepsilon_{ij}\beta_{jj}e_{ii}$. Thus the coefficients $\beta_{ij}^{(k)}$ and β_{jj} of b all lie in Λ, and $B \subseteq \mathfrak{C}(\Lambda)$.

This shows $B = \mathfrak{C}(\Lambda)$ for some ideal $\Lambda \lhd \Omega$. Conversely, the multiplication rules show $\mathfrak{C}(\Lambda)$ is an ideal whenever Λ is. Thus $\mathfrak{C}(\Lambda) \longleftrightarrow \Lambda$ is a lattice isomorphism between the ideals of $\mathfrak{C}(\Omega)$ and those of Ω. ∎

VI, 6 Exercises

1. If $\{e_{ij}^{(k)}\}$ are Cayley matrix units, show $a_{ii}e_{ij}^{(k)} = e_{ij}^{(k)}a_{jj}$ iff $a_{jj} = e_{ji}^{(k)}a_{ii}e_{ij}^{(k)}$, $a_{ii} = e_{ij}^{(k)}a_{jj}e_{ji}^{(k)}$. If $x_{ij} \in A_{ij}$ $(i \neq j)$ show $(x_{ij}e_{ji}^{(k)})e_{ij}^{(k)} = e_{ij}^{(k)}(e_{ji}^{(k)}x_{ij})$ and $x_{ij} = \Sigma\, a_{ii}^{(k)}e_{ij}^{(k)}$ for $a_{ii}^{(k)} = x_{ij}e_{ji}^{(k)}$.
2. Show that the map $A_{ii} \overset{*}{\to} A_{jj}$ by $a_{ii}^{*} = e_{ji}^{(k)}a_{ii}e_{ij}^{(k)}$ is an isomorphism, independent of k, which is of period 2 in the sense $A_{ii} \overset{*}{\to} A_{jj} \overset{*}{\to} A_{ii}$ is the identity map. Show $a_{ii}x_{ij} = x_{ij}a_{ii}^{*}$ for $x_{ij} \in A_{ij}$ $(i \neq j$ always).
3. Prove the <u>Proposition</u>. If x', y', z' belong to the Peirce space A_{ij} $(i \neq j)$ relative to an idempotent e, such that some left multiple of $x'(y'z')$ is a nonzero idempotent f_{ii} in A_{ii}, then there are idempotents $f_{11} \in A_{11}$, $f_{00} \in A_{00}$ and $x,y,z \in f_{ii}Af_{jj}$ with

$$x(yz) = f_{ii}\ ,\ (yz)x = f_{jj}$$

so that the Peirce subalgebra $A_{11}(f_{11} + f_{00})$ has a family of Cayley matrix units. (Dually if $(y'z')x'$ has an idempotent right multiple).
4. Deduce as <u>Corollary</u>. If $1 = e_1 + e_2$ for completely primitive (= division) idempotents e_i with $A_{ij}A_{ij}^2 \neq 0$ (or $A_{ij}^2A_{ij} \neq 0$) then A contains a supplementary family of Cayley matrix units.

5. Succumb to brutish atavistic proclivities and verify (6.12 ia - vb) directly from the original definition of the $\pi_{ij}^{(k)}$.