§4. Ideal-building

An ideal in a Peirce space generates an ideal in the whole algebra which can be described explicitly. The connector ideal generated by A_{10} is a measure of how close $A_{10}A_{01}$ is to A_{11}, the alternator ideal generated by A_{10}^2 measures how close A_{10}^2 is to 0, and the ideal generated by an ideal B_{ii} of A_{ii} measures simplicity of A_{ii}. From such considerations we show a Peirce subalgebra A_{ii} inherits simplicity, primeness, or semiprimeness from A.

In this section we will use the Peirce relations to build ideals. We will be able to see that a Peirce subalgebra A_{ii} inherits many of the properties of A. Throughout we consider a Peirce decomposition relative to a single fixed idempotent e. All indices are either 0 or 1, and i and j are always assumed distinct.

We begin by investigating the ideal generated by a subspace of an off-diagonal Peirce space.

4.1 (Off-diagonal Ideal-Building Lemma) If B_{ij} is a subspace of A_{ij} such that $A_{ii}B_{ij} \subset B_{ij}$, $B_{ij}A_{jj} \subset B_{ij}$, $[A_{ij}, A_{ji}, B_{ij}] \subset B_{ij}$ then the ideal in A generated by B_{ij} is

$$I(B_{ij}) = B_{ii} + B_{ij} + B_{ji} + B_{jj}$$

for

$$B_{ii} = B_{ij}A_{ji},\ B_{jj} = A_{ji}B_{ij},\ B_{ji} = B_{jj}A_{ji} + A_{ji}B_{ii} + A_{ij}B_{ij}\ .$$

Proof. It suffices to prove the above sum B is an ideal, since it clearly is generated by B_{ij}. We check only that B is a left ideal.

We have $A_{ii}B = A_{ii}B_{ii} + A_{ii}B_{ij}$ where $A_{ii}B_{ii} = A_{ii}(B_{ij}A_{ji}) = (A_{ii}B_{ij})A_{ji} \subset B_{ij}A_{ji} = B_{ii}$ and $A_{ii}B_{ij} \subset B_{ij}$ by Peirce associativity and our hypothesis. We have $A_{jj}B = A_{jj}B_{jj} + A_{jj}B_{ji}$ where $A_{jj}B_{jj} = A_{jj}(A_{ji}B_{ij}) = (A_{jj}A_{ji})B_{ij} \subset A_{ji}B_{ij} = B_{jj}$ and $A_{jj}B_{ji} = A_{jj}\{B_{jj}A_{ji} + A_{ji}B_{ii} + A_{ij}B_{ij}\} \subset (A_{jj}B_{jj})A_{ji} + (A_{jj}A_{ji})B_{ii} + (A_{ij}A_{jj})B_{ij}$ (Peirce associativity and Slipping Formula) $\subset$ $B_{jj}A_{ji} + A_{ji}B_{ii} + A_{ij}B_{ij} = B_{ji}$.

We have $A_{ij}B = A_{ij}B_{jj} + A_{ij}B_{ji} + A_{ij}B_{ij}$ where $A_{ij}B_{ij}\ \ B_{ji}$ by definition, $A_{ij}B_{jj} = A_{ij}(A_{ji}B_{ij}) = (A_{ij}A_{ji})B_{ij} - [A_{ij}, A_{ji}, B_{ij}]$ $\ \ A_{ii}B_{ij} + B_{ij}\ \ B_{ij}$ by our stronger invariance hypotheses about B_{ij}, and $A_{ij}B_{ji} = A_{ij}\ \ B_{jj}A_{ji} + A_{ji}B_{ii} + A_{ij}B_{ij}\ \ =$ $(A_{ij}B_{jj})A_{ji} + (A_{ij}A_{ji})B_{ii} + B_{ij}\{A_{ij}A_{ij}\}$ (Peirce associativity and Permuting Formula) $\ \ B_{ij}A_{ji} + A_{ii}B_{ii} + B_{ij}A_{ji}$ $(A_{ij}B_{jj}\ \ \ B_{ij}$ above) $\ \ B_{ii}$.

Similarly $A_{ji}B = A_{ji}B_{ii} + A_{ji}B_{ij} + A_{ji}B_{ji}$ where $A_{ji}B_{ij} = B_{jj}$, $A_{ji}B_{ii} \subset B_{ji}$ by definition, and $A_{ji}B_{ji} = A_{ji}\{B_{jj}A_{ji} + A_{ji}B_{ii} + A_{ij}B_{ij}\}$ $\subset \{A_{ji}A_{ji}\}B_{jj} + B_{ii}\{A_{ji}A_{ji}\} + (A_{ji}A_{ij})B_{ij} - [A_{ji}, A_{ij}, B_{ij}]$ (Slipping Formula) $\subset A_{ij}B_{jj} + B_{ii}A_{ij} + 0 - B_{ij}$ (hypothesis) where we just saw $A_{ij}B_{jj}$ B_{ij} above and $B_{ii}A_{ij} = (B_{ij}A_{ji})A_{ij}$ $= [B_{ij}, A_{ji}, A_{ij}] + B_{ij}(A_{ji}A_{ij}) \subset B_{ij} + B_{ij}A_{jj}$ $B_{ij} \subset$ (by hypothesis).

Thus $AB \subset B$ and B is a left ideal. ■

<u>Important</u>: note the ij-component of $I(B_{ij})$ is just the B_{ij} we started with.

Certainly $B_{ij} = A_{ij}$ is as invariant as can be. Adding $I(A_{10})$ and $I(A_{01})$ gives

4.2 (Connector Lemma) If e is an idempotent in A then the ideal generated by A_{10} and A_{01} is the connector ideal

$$C = A_{10}A_{01} + A_{10} + A_{01} + A_{01}A_{10}\ . \quad ■$$

This has the very important result that idempotents in a simple algebra are "connected" (in a sense to be made clear in the next section.)

4.3 (Connector Corollary) If $e \neq 1, 0$ is a proper idempotent in a simple alternative algebra then $A_{10}A_{01} = A_{11}$, $A_{01}A_{10} = A_{00}$.

Proof. If $C = 0$ then $A_{10} = A_{01} = 0$, $A = A_{11} + A_{00}$ is by Peirce orthogonality a direct sum of ideals, so by simplicity either $A = A_{11}$ (whence $e = 1$) or $A = A_{00}$ (whence $e = 0$), contradiction. Therefore $C = A$ and $A_{ij}A_{ji} = C_{ii} = A_{ii}$. ■

Since $A_{ij}A_{ji}$ is always in the nucleus of A_{ii} by (3.12), we see that a proper Peirce subalgebra A_{ii} of a simple alternative algebra is always associative: the diagonal pieces are associative, and any non-associativity creeps in from the off-diagonal pieces. We proceed to estimate this non-associativity.

4.4 (Alternator Lemma) If e is an idempotent in A then the ideal generated by A_{ji}^2 is the alternator ideal

$$B = B_{ii} + B_{ij} + B_{ji} + B_{jj}$$

for $B_{ii} = A_{ji}^2 A_{ji}$, $B_{jj} = A_{ji} A_{ji}^2$, $B_{ij} = A_{ji}^2$, $B_{ji} = B_{jj} A_{ji} + A_{ji} B_{ii} + A_{ij} B_{ij}$.

Proof. This follows immediately from the Ideal-Building Lemma once we verify invariance of $B_{ij} = A_{ji}^2$. We have $A_{ii} A_{ji}^2 \subset A_{ji}^2$ and $A_{ji}^2 A_{jj} \subset A_{ji}^2$ by Slipping Formulas (3.8), and $[A_{ij}, A_{ji}, B_{ij}] = - [A_{ji}, A_{ij}, B_{ij}] = - (A_{ji} A_{ij}) B_{ij} + A_{ji}(A_{ij} B_{ij}) \subset 0 + A_{ji} A_{ij}^2 \subset A_{ji} A_{ji} = B_{ij}$. ∎

This too has important consequences for simple algebras.

4.5 (Alternator Corollary) If $e \neq 1, 0$ is a proper idempotent in a simple alternative algebra A then either $A_{10}^2 = A_{01}^2 = 0$ and A is associative, or else $A_{10}^2 = A_{01}$ and $A_{01}^2 = A_{10}$.

Proof. If $A_{10}^2 = A_{01}^2 = 0$ then (since we already know $A_{10} A_{01} = A_{11}$, $A_{01} A_{10} = A_{00}$ by the Connector Corollary 4.3) we can conclude A is associative by the Peirce Associativity Criterion 3.16.

On the other hand, if (say) $A_{01}^2 \neq 0$ then the alternator ideal B determined by $B_{10} = A_{01}^2$ is nonzero, hence by simplicity must be all of A; in particular, $A_{10} = B_{10} = A_{01}^2$ and $A_{11} = B_{11} = B_{10}A_{01} = A_{01}^2 A_{01}$. If $A_{10}^2 \neq 0$ we would have $A_{01} = A_{10}^2$ similarly.

What if $A_{01}^2 \neq 0$ but $A_{10}^2 = 0$? It seems strangely messy to rule this out directly (see Ex. 4.4), but with a little help from some radicals we can dispose of it. Since A contains an idempotent it is not nil; if its nil radical is not everything, by simplicity it must be nothing, and in particular A is strongly semiprime (no trivial elements; see V.4). But if $A_{10}^2 = 0$, $A_{10}^2 = A_{10}$ the elements of A_{01} are trivial: $U_{x_{01}}A = U_{x_{01}}A_{10} = U_{x_{01}}A_{01}^2 = (x_{01}A_{01})(A_{01}x_{01}) \subset A_{10}A_{10} = 0$ using middle Moufang. Then $A_{01} = 0$ contradicts $A_{01}^2 \neq 0$. ∎

4.6 Remark. The Peirce relations $A_{10} = A_{01}^2$ and $A_{01}A_{10} = A_{00}$ or a simple not-associative alternative algebra provide another demonstration that a Cayley algebra (split or not) contains no proper one-sided ideals. If B is proper then $n(B) = 0$, if $t(B) = 0$ as well then $n(\mathbb{C}, B) = t(\mathbb{C}^* B) \subset t(B)$ (left ideal!) $= 0$ would contradict nondegeneracy of $n(x,y)$, so we must have $t(e) = 1$ (and of course $n(e) = 0$) for some $e \in B$. Then e is idempotent; in the Peirce decompositon $\mathbb{C} = \mathbb{C}_{11} \oplus \mathbb{C}_{10} \oplus \mathbb{C}_{01} \oplus \mathbb{C}_{00}$

we have $\mathbb{C}_{11} \oplus \mathbb{C}_{01} = \mathbb{C}e \subset B$, hence also $\mathbb{C}_{10} = \mathbb{C}_{10}^2 \subset B$ (the trick of getting $\mathbb{C}_{10}$ from $\mathbb{C}_{01}$ is impossible for associative algebras) and so $\mathbb{C}_{00} = \mathbb{C}_{01}\ \mathbb{C}_{10} \subset B$. Thus $B = \mathbb{C}$, which is most improper. ∎

4.7 (Diagonal Ideal-Building Lemma) If B_{ii} is an ideal in the Peirce space A_{ii} then the ideal generated by B_{ii} is

$$I(B_{ii}) = B_{ii} + B_{ij} + B_{ji} + B_{jj} = B_{ii} + B_{ii}A_{ij} + A_{ji}B_{ii} + A_{ji}B_{ii}A_{ij}.$$

Proof. Clearly B_{ii} generates the above subspace B, so it is enough to verify it is an ideal, and as usual we only check it is a left ideal.

We have $A_{ii}B = A_{ii}B_{ii} + A_{ii}B_{ij}$ where $A_{ii}B_{ii} \subset B_{ii}$ since B_{ii} is an ideal in A_{ii}, and $A_{ii}B_{ij} = A_{ii}(B_{ii}A_{ij}) = (A_{ii}B_{ii})A_{ij} \subset B_{ii}A_{ij} = B_{ij}$ by Peirce associativity. Similarly $A_{jj}B = A_{jj}B_{jj} + A_{jj}B_{ji}$ where $A_{jj}B_{ji} = A_{jj}(A_{ji}B_{ii}) \subset (A_{jj}A_{ji})B_{ii} \subset A_{ji}B_{ii} = B_{ji}$ by Peirce associativity, so $A_{jj}B_{jj} = A_{jj}(B_{ji}A_{ij}) = (A_{jj}B_{ji})A_{ij} \subset B_{ji}A_{ij} = B_{jj}$.

We have $A_{ji}B = A_{ji}B_{ii} + A_{ji}B_{ij} + A_{ji}B_{ji}$ where $A_{ji}B_{ii} = B_{ji}$, $A_{ji}B_{ij} = B_{jj}$, and $A_{ji}B_{ji} = A_{ji}(A_{ji}B_{ii}) = B_{ii}(A_{ji}A_{ji}) \subset B_{ii}A_{ij} = B_{ij}$ by Slipping Formula (3.8).

We have $A_{ij}B = A_{ij}B_{jj} + A_{ij}B_{ji} + A_{ij}B_{ij}$ where $A_{ij}B_{ji} = A_{ij}(A_{ji}B_{ii}) = (A_{ij}A_{ji})B_{ii} \subset A_{ii}B_{ii} \subset B_{ii}$ as usual, $A_{ij}B_{ij} = A_{ij}(B_{ii}A_{ij}) = (A_{ij}A_{ij})B_{ii} \subset A_{ji}B_{ii} = B_{ji}$ by the Slipping Formula (3.8), and $A_{ij}B_{jj} = A_{ij}(A_{ji}B_{ij}) = (A_{ij}A_{ji})B_{ij} - [A_{ij}, A_{ji}, B_{ij}] \subset A_{ii}B_{ij} - A_{ji}(A_{ij}B_{ij})$ (by (3.10)) $\subset B_{ij} - A_{ji}B_{ji}$ (by the above) $\subset B_{ij}$ (by the previous A_{ji} case). ∎

Again, it is very important that the ii-component of $I(B_{ii})$ is just the original space B_{ii} we began with. Thus $I(B_{ii})$ is a proper ideal in A if B_{ii} is a proper ideal in A_{ii}. As an immediate consequence,

4.8 (Simple Inheritance Theorem) If A is a simple alternative algebra, so is any Peirce subalgebra A_{ii}. ∎

To see that A_{ii} inherits properties other than simplicity from A we need a

4.9 Lemma. If B_{ii}, C_{ii} are ideals in A_{ii} then the ideal generated by their product is the product of the ideals they generate,

$$I(B_{ii}C_{ii}) = I(B_{ii})I(C_{ii}) .$$

Proof. Containment one direction is easy: as the product of ideals, $I(B_{ii})I(C_{ii})$ is itself an ideal, containing $B_{ii}C_{ii}$ and therefore $I(B_{ii}\ C_{ii})$. In the other direction, by the explicit expression for $I(C_{ii})$ we have $B_{ii}I(C_{ii}) = B_{ii}\{C_{ii} + C_{ii}A_{ij} + A_{ji}C_{ii} + A_{ji}C_{ii}A_{ij}\} = B_{ii}C_{ii} + (B_{ii}C_{ii})A_{ij} \subset I(B_{ii}C_{ii})$, so B_{ii} belongs to the left transporter of $I(C_{ii})$ into $I(B_{ii}C_{ii})$; since this transporter is an ideal by the Transportation Lemma IV.2.4, $I(B_{ii})$ also belongs to the transporter, and $I(B_{ii})I(C_{ii}) \subset I(B_{ii}C_{ii})$. ∎

From this we see that if B_{ii} is trivial, solvable, or nilpotent so is $I(B_{ii})$. Therefore

4.10 (Inheritance Corollary) If A is simple, prime, or semiprime so is any Peirce subalgebra A_{ii}. ∎

We can sum up by saying that the correspondence $B_{ii} \to I(B_{ii})$ is an injective map from the ideals in A_{ii} to the ideals in A, which preserves sums and products. If B_{ii} has a property definable in terms of inclusion, sums, and products then $I(B_{ii})$ will inherit this property. If A is free of ideals with this property, the same will be true of A_{ii}.

VI.4 Exercises

4.1 Show that if B_{ij}, C_{ji} are subspaces of A_{ij}, A_{ji} satisfying the invariance conditions $A_{ii}B_{ij} \subset B_{ij}$, $C_{ji}A_{ii} \subset C_{ji}$ then $B_{ij}C_{ji}$ is an ideal in A_{ii}. Conclude $A_{ij}A_{ji}$, $A_{ij}A_{ij}^2$, $A_{i}^2A_{ji}$ are ideals in A_{ii}.

4.2 Prove directly that $A_{10}A_{01} + A_{10} + A_{01} + A_{00}$ is an ideal in A.

4.3 Use the remarks after the Connection Corollary 4.3 to show that a simple alternative algebra with two nonzero non-supplementary orthogonal idempotents e_1, e_2 ($e_1, e_2 \neq 0$ and $e_1 + e_2 \neq 1$) is associative.

4.4. If $A_{10}^2 = 0$ but $A_{01}^2 = A_{10}$ and $A_{10}A_{01} = A_{11}$, show $[A_{10}, A_{10}, A_{01}] = [A_{01}, A_{01}, A_{10}] = 0$. Show $B_{01} = \hat{A}_{00}x_{01}\hat{A}_{11}$ has the necessary invariance properties for the Ideal-Building Lemma 4.1 whenever $x_{01} \in A_{01}$, and $B_{01}^2 = 0$. Conclude $I(B_{01}) \neq A$. Deduce that $A_{10}^2 = 0$, $A_{01}^2 \neq 0$ is impossible in a simple algebra A. (There must be a better way !)

4.5 Show $Z = Z_{10} + Z_{01}$ ($Z_{ij} = \{z_{ij} \in A_{ij} \mid z_{ij}A_{ji} = A_{ji}z_{ij} = 0\}$ is a nilpotent ideal in A, $Z^3 = 0$, consisting entirely of trivial elements z, $U_zA = 0$. Conclude that if A is semiprime then $Z = 0$, so $A_{ij}A_{ij}^2 = A_{ij}^2A_{ij} = 0$ implies $A_{ij}^2 = 0$.

4.6 If B_{ii} is an <u>arbitrary</u> subspace of A_{ii}, find an expression for the ideal generated by B_{ii}. Repeat for an arbitrary subspace B_{ij} of A_{ij}.

4.7 Find an expression for the left ideal in A generated by a left ideal B_{ii} of the Peirce subalgebra A_{ii}.

4.8 Show Rad(eAe) = eAe $\cap$ Rad A without availing yourself of the elementwise characterization of the radical, by showing B_{ii} = Rad(eAe) generates a q.i. ideal in A.

4.9 Prove computationally that $I(B_{ii})I(C_{ii}) \subset I(B_{ii}C_{ii})$ as in Lemma 4.9.

4.10 Although it is not in general true that $I(B_{ii} \cap C_{ii}) = I(B_{ii}) \cap I(C_{ii})$ when $B_{ii}, C_{ii} \triangleleft A_{ii}$, show it comes close: $I(B_{ii} \cap C_{ii}) \subset I(B_{ii}) \cap I(C_{ii})$ and $\{I(B_{ii}) \cap I(C_{ii})\}^2 \subset I(B_{ii})I(C_{ii}) \subset I(B_{ii} \cap C_{ii})$.

VI. 4.1 Problem Set on Prime Algebras

1. If A is prime show $zC = 0$ (C a nonzero ideal) implies $z = 0$.

2. If $z_{ii}A_{ij} = 0$ show z_{ii} kills the connector ideal.

3. Show that if A is prime and $e \neq 0, 1$ a proper idempotent then $z_{ii}A_{ij} = 0$ or $A_{ji}z_{ii} = 0$ implies $z_{ii} = 0$. Conclude that A_{11} and A_{00} are associative.

4. Show that if A is prime and $z_{jj}A_{ij}^2 = 0$ then either $z_{jj} = 0$ or $A_{ij}^2 = 0$.

5. Prove the <u>Theorem</u>. If A is a prime alternative algebra and $e \neq 0, 1$ a proper idempotent, then either A is associative or else $A_{10}^2 + A_{01}^2 \neq 0$ and A_{11}, A_{00} are commutative, associative integral domains acting faithfully on $A_{10}^2 + A_{01}^2$.

This generalizes the results for simple algebras (where the A_{ii} are fields).