§3. Peirce relations

The Peirce relations are the key to the structure of alternative algebras. Two Peirce spaces multiply just as in the associative case except that the product $A_{ij}A_{ij}$ need not be zero for $i \neq j$. There are a host of Peirce Associativity Relations measuring the non-associativity of Peirce spaces, leading to useful criteria in terms of the Peirce spaces for associativity, nuclearity, or triviality in the whole algebra.

The way Peirce spaces multiply in alternative algebras is more complicated than it was in associative algebras.

3.1 (Peirce Orthogonality Relations) The Peirce subspaces relative to idempotents $e_1, \cdots, e_n$ multiply according to the rules

(2-laws)

(3.2) $A_{ij}\, A_{jk} \subset A_{i\ell}$

(3.3) $A_{ij}\, A_{k\ell} = 0$ iff $j \neq k$ unless $i = k$, $j = \ell$

(3.4) $A_{ij}\, A_{ij} \subset A_{ji}$ with $x_{ij}^2 = x_{ij}y_{ij} + y_{ij}x_{ij} = 0$

for $x_{ij}, y_{ij} \in A_{ij}$.

3.5 Remark. These are the same as the Peirce relations for associative algebras EXCEPT FOR (3.4). Thus: A PRODUCT $x_{ij}y_{k\ell}$

FROM TWO PEIRCE SPACES A_{ij}, $A_{k\ell}$ IS ZERO UNLESS THE INDICES ARE LINKED ($j = k$) OR ELSE THE PEIRCE SPACES ARE THE SAME ($i = k$, $j = \ell$). ∎

Note that the diagonal Peirce spaces $A_{00}, A_{11}, \cdots, A_{nn}$ are subalgebras. These will be called Peirce subalgebras, and will play a large role in our work.

It is important to know just how close the Peirce spaces are to being associative.

3.6 (Peirce Associativity Relations) An associator $[x_{ij}, y_{k\ell}, z_{mn}]$ involving Peirce spaces vanishes unless at least two elements fall in the same Peirce space and at most two distinct indices occur. The only nonvanishing associators are (up to permutation) of the form

$$(3.7)\quad \begin{cases} [x_{ii}, y_{ii}, z_{ii}] \quad [x_{ij}, y_{ij}, z_{ij}] \\ [x_{ij}, y_{ij}, z_{ii}] \quad [x_{ij}, y_{ij}, z_{jj}] \quad \text{(5 unruly associators)} \\ [x_{ij}, y_{ij}, z_{ji}] \end{cases}$$

for $i \neq j$. Thus any three distinct Peirce spaces associate. We have further formulas involving 3 factors

(3-laws)

(3.8) (Slipping Formulas)

$$x_{ii}(y_{ji}z_{ji}) = (y_{ji}x_{ii})z_{ji} = y_{ji}(z_{ji}x_{ii})$$

$$(y_{ji}z_{ji})x_{jj} = (x_{jj}y_{ji})z_{ji} = y_{ji}(x_{jj}z_{ji})$$

(3.9) (Permuting Formulas)

$$x_{ij}(y_{ij}z_{ij}) = y_{ij}(z_{ij}x_{ij}) = z_{ij}(x_{ij}y_{ij})$$

$$(x_{ij}y_{ij})z_{ij} = (y_{ij}z_{ij})x_{ij} = (z_{ij}x_{ij})y_{ij}$$

(3.10)

$$[x_{ij}, z_{ji}, y_{ij}] = -(x_{ij}y_{ij})z_{ji} = z_{ji}(x_{ij}y_{ij})$$

and involving 4 factors

(4-laws)

(3.11) (Diagonal Commutativity)

$$[x_{ii}, y_{ii}](z_{ji}w_{ji}) = 0$$

$$(z_{ij}w_{ij})[x_{ii}, y_{ii}] = 0$$

(3.12) (Diagonal Associativity)

$$[x_{ii}, y_{ii}, z_{ij}w_{ji}] = 0$$

(3.13) (Diagonal Associativity)

$$[x_{ii}, y_{ii}, z_{ii}]w_{ij} = 0$$

$$w_{ji}[x_{ii}, y_{ii}, z_{ii}] = 0$$

3.14 Remark. The way to remember the Slipping Formulas (3.8) is that an x_{ii} or x_{jj} leaps in wherever it can attach itself (in the sense that x_{ii} can attach itself to the front of an a_{ij} or the back of an a_{ji}). These describe how a diagonal element x_{ii} or x_{jj} slips inside a product of off-diagonal elements.

The way to remember the Permuting Formula (3.9) is that a product of 3 factors from an A_{ij} is invariant under cyclic permutations.

The Diagonal Commutativity Formula (3.11) says that commutators of diagonal elements kill A_{ij}^2.

The Diagonal Associativity Formulas (3.13) and (3.12) say associators of diagonal elements kill A_{ij}, and associators of diagonal elements involving $A_{ij}A_{ji}$ are zero (i.e. that $A_{ij}A_{ji}$ falls in the nucleus of A_{ii}).

The associativity condition $[x_{ii}, y_{ii}, z_{ij}] = 0$ and the fact that $e_i z_{ij} = z_{ij}$ means the map $x_{ii} \to L_{x_{ii}}$ (restricted to A_{ij}) is a unital homomorphism (= associative specialization) of A_{ii} in End A_{ij}, called the *Peirce specialization* of A_{ii} on A_{ij}. (Similarly $x_{ii} \to R_{x_{ii}}$ leads to a *Peirce antispecialization* of A_{ii} on A_{ji}).

Insight into these relations is gained by seeing what they reduce to in a split Cayley algebra. For example, from the rules I.1.15 we see $\mathbb{C}_{ij}^2 = \mathbb{C}_{ji}$, $\mathbb{C}_{ij}\,\mathbb{C}_{ji} = \mathbb{C}_{ii}$, and $\mathbb{C}_{ii} = \Phi e_i$ is a commutative associative field (as hinted in (3.11)-(3.13)). (3.9) is reflected in the cyclic multiplication rules for $e_{12}^{(i)}(e_{12}^{(i+1)}e_{12}^{(i+2)})$.

Only the Peirce Orthogonality Relations appear frequently enough that you should memorize them; for the Peirce Associativity Relations just be aware of their existence and where to look them up. ∎

Proof of the Peirce Orthogonality and Associativity Relations.

Trivially $e_1, \cdots, e_n$ generate an associative subalgebra B ($B \cong \Phi \boxplus \cdots \boxplus \Phi$), and any $x \in A$ belongs to the nucleizer of B since $[e_i, e_j, x] = 0$ by orthogonality. Thus by the Throw-in-a-Nucleizer Lemma II.3.8, B and x and 1 generate an associative subalgebra C of A containing $e_0, e_1, \cdots, e_n$ and x (hence also the components $x_{ij} = e_i x e_j$ of x).

Suppose we take arbitrary x_{ij}, $x_{k\ell}$, x_{mn} from distinct Peirce spaces. Then the element $x = x_{ij} + x_{k\ell} + x_{mn}$ has Peirce components precisely x_{ij}, $x_{k\ell}$, x_{mn} so they belong to an associative subalgebra C. In particular, their associator vanishes: any associator $[x_{ij}, x_{k\ell}, x_{mn}]$ involving distinct Peirce components is zero.

From this we can immediately deduce the "2-laws": For (3.2) with $i \neq j$ or $j \neq k$ the elements x_{ij}, y_{jk} belong to distinct Peirce spaces, hence fall in an associative subalgebra C containing e_i, e_k, hence $x_{ij}y_{jk} \in C_{ij}C_{jk} \subset C_{jk} \subset A_{jk}$ by the associative Peirce relations (or directly $x_{ij}y_{jk} = e_i x_{ij} y_{jk} e_k \in e_i A e_k = A_{ik}$ by associativity). (If $i = j = k$ we argue differently: $x_{ii}y_{ii} = (e_i x_{ii})(y_{ii} e_i) = e_i(x_{ii}y_{ii})e_i \in A_{ii}$ by middle Moufang.) For (3.3), $x_{ij}\, y_{k\ell} = 0$ for $j \neq k$ by the Peirce relations in C except for the case $x_{ij}y_{ij}$ of a repeated Peirce space, since two different x_{ij}, y_{ij} can't be components of a single element x (note we can at least conclude $x_{ij}^2 = 0$ if $i \neq j$). (For (3.4) we argue differently: $e_j(x_{ij}y_{ij}) =$

$- [e_j, x_{ij}, y_{ij}] = [x_{ij}, e_j, y_{ij}] = x_{ij}y_{ij}$ since $e_jx_{ij} = e_jy_{ij}$ $= 0$, $x_{ij}e_j = x_{ij}$, and similarly $(x_{ij}y_{ij})e_i = x_{ij}y_{ij}$.)

We have already seen the only nonzero associators have the form $[x_{ii}, y_{ii}, z_{k\ell}]$ or $[x_{ij}, y_{ij}, z_{k\ell}]$ (involving a repeated Peirce space). Even these vanish if three of the indices i, j, k, ℓ are distinct: for example, if $k \neq i$ then $[x_{ii}, y_{ii}, z_{k\ell}]$ $= 0$ since $A_{ii}z_{k\ell} = 0$ by the 2-laws, and if $k \neq i, j$ $[x_{ij}, y_{ij}, z_{k\ell}]$ $= 0$ since $A_{ij}z_{k\ell} = A_{ji}z_{k\ell} = 0$, dually if $\ell \neq i$ use $[x_{ii}, y_{ii}, z_{k\ell}]$ $= [z_{k\ell}, x_{ii}, y_{ii}]$ and if $\ell \neq i, j$ use $[x_{ij}, y_{ij}, z_{k\ell}] = [z_{k\ell}, x_{ij}, y_{ij}]$. This leaves only the possibilities $[x_{ii}, y_{ii}, z_{ii}]$, $[x_{ij}, y_{ij}, z_{k\ell}]$ for $z_{k\ell}$ either $z_{ii}, z_{jj}, z_{ij}, z_{ji}$. These are the 5 Unruly Associators of (3.7).

When we turn to the 3-laws and 4-laws we cannot avail ourselves of associativity, since these laws all involve repeated Peirce spaces.

For (3.8) we have $x_{ii}(y_{ji}z_{ji}) = - y_{ji}(x_{ii}z_{ji}) + (x_{ii}\circ y_{ji})z_{ji}$ $= (y_{ji}x_{ii})z_{ji}$ by Peirce orthogonality, and by skew-symmetry in (3.4) $x_{ii}(y_{ji}z_{ji}) = - x_{ii}(z_{ji}y_{ji}) = - (z_{ji}x_{ii})y_{ji} =$ $+ y_{ji}(z_{ji}x_{ii})$. From this Diagonal Commutativity (3.11) follows via the HIDING TRICK: $(x_{ii}y_{ii})(z_{ji}w_{ji}) = x_{ii}\{y_{ii}(z_{ji}w_{ji})\} =$ $x_{ii}\{z_{ji}(w_{ji}y_{ii})\} = (z_{ji}x_{ii})(w_{ji}y_{ii}) = y_{ii}\{(z_{ji}x_{ii})w_{ji}\} =$ $y_{ii}\{x_{ii}(z_{ji}w_{ji})\} = (y_{ii}x_{ii})(z_{ji}w_{ji})$.

Similarly Diagonal Associativity (3.12) follows from a sort of HIDING: $(x_{ii}y_{ii})(z_{ij}w_{ji}) = \{(x_{ii}y_{ii})z_{ij}\}w_{ji} = \{x_{ii}(y_{ii}z_{ij})\}w_{ji}$ $= x_{ii}\{(y_{ii}z_{ij})w_{ji}\} = x_{ii}\{y_{ii}(z_{ij}w_{ji})\}$ by Peirce associativity.

Alternately: $[x_{ii},y_{ii},z_{ij}w_{ji}] = [x_{ii},y_{ii},z_{ij}\circ w_{ji}]$ (as $w_{ji}z_{ij} \in A_{jj}$) $= [x_{ii},y_{ii},z_{ij}]\circ w_{ji} + [x_{ii},y_{ii},w_{ji}]\circ z_{ij}$ (middle bumping) $= 0$.

For the other Diagonal Associativity formula (3.13) we can again use a HIDING TRICK: $\{(x_{ii}y_{ii})z_{ii}\}w_{ij} = (x_{ii}y_{ii})(z_{ii}w_{ij}) = x_{ii}\{y_{ii}(z_{ii}w_{ij})\} = x_{ii}\{(y_{ii}z_{ii})w_{ij}\} = \{x_{ii}(y_{ii}z_{ii})\}w_{ij}$. More elegantly, note $x_{ii} \to L_{x_{ii}}$ is a homomorphism of A_{ii} into the associative algebra End A_{ij}, $L_{x_{ii}y_{ii}} = L_{x_{ii}}L_{y_{ii}}$ on A_{ij}, so all associators lie in the kernel, $L_{[x_{ii},y_{ii},z_{ii}]}A_{ij} = 0$.

Instead of (3.9) we have the stronger

(3.15) $\qquad f(x_{ij},y_{ij},z_{ij}) = x_{ij}(y_{ij}z_{ij})$ and $g(x_{ij},y_{ij},z_{ij}) =$

Clearly $f\ (x_{ij}y_{ij})z_{ij}$ are alternating functions of their variables.

vanishes if $x_{ij} = y_{ij}$ or $y_{ij} = z_{ij}$ since $y_{ij}^2 = 0$, so by linearization also $f(x_{ij},y_{ij},x_{ij}) = -f(y_{ij},x_{ij},x_{ij}) = 0$. For (3.10) observe $[x_{ij},z_{ji},y_{ij}] = -[x_{ij},y_{ij},z_{ji}] = -(x_{ij}y_{ij})z_{ji}$ (since $x_{ij}(y_{ij}z_{ji}) \in x_{ij}A_{ii} = 0$), and dually. ■ ■

As an immediate consequence we obtain a useful associativity condition.

3.16 (Peirce Associativity Criterion) A is associative iff each A_{ii} is associative and $A_{ij}^2 = 0$ for $i \neq j$. Here A_{ii} will be associative if $A_{ii} = A_{ij}A_{ji}$ for some $j \neq i$, and

A_{ij}^2 will vanish if $A_{ij} = A_{ij}A_{kj}$ for some $k \neq i,j$.

Proof. Certainly the two conditions are necessary. They are sufficient since they guarantee the five unruly associators (3.7) vanish: $[x_{ii}, y_{ii}z_{ii}] = 0$ if A_{ii} is associative, and $[x_{ij}, y_{ij}, A] = 0$ because $A_{ij}^2 = 0$ means $A_{ij}A_{jk} \subset A_{ik}$ and $A_{ij}A_{k\ell} = 0$ if $j \neq k$ so that all nonzero products must have linked indices.

Since $A_{ij}A_{ji}$ associates with A_{ii} by (3.12), when $A_{ij}A_{ji} = A_{ii}$ the algebra A_{ii} is associative. When $A_{ij} = A_{ik}A_{kj}$ then $A_{ij}^2 = A_{ij}(A_{ik}A_{kj}) = (A_{ij}A_{ik})A_{kj} = 0$ since all three Peirce spaces are distinct if $i, j, k \neq$. ∎

We also have a useful condition for nuclearity.

3.17 (Peirce Nuclearity Condition) If $A = \bigoplus_{i,j=0}^{n-1} A_{ij}$ is the Peirce decomposition of A relative to $e_1, \cdots, e_{n-1}$ then for distinct indices $i,j,k \neq$

(3.18) $$A_{ij}\, A_{jk} \subset N(A)$$

(3.19) $$A_{ij}\, A_{ji} \cap A_{ik}\, A_{ki} \subset N(A)\ .$$

Proof. To prove $[A_{ij}A_{jk}, A, A] = 0$ it suffices to prove $[A_{ij}A_{jk}, A_{pq}, A_{rs}] = 0$ for Peirce spaces. The only unruly associators involving $A_{ij}A_{jk} \subset A_{ik}$ have all their indices $\{p,q,r,s\} \subset \{i,k\}$; but relative to $e = e_i + e_k$ (in $\hat{A}$ if e_i or e_k doesn't exist, i.e. if i or k is 0) the fact that $j \neq i, k$ means

$A_{ij}A_{jk} \subset A_{10}(e)\, A_{01}(e) \subset N(A_{11}(e)) = N(A_{ii} + A_{ik} + A_{ki} + A_{kk})$ by the associativity condition (3.12) and the Collection Formula (2.5). Thus $A_{ij}A_{jk}$ associates with Peirce spaces involving only the indices i and k.

Similarly, we need only prove $A_{ij}A_{ji} \cap A_{ik}A_{ki}$ associates with Peirce spaces A_{pq}, A_{rs} involving indices i and ℓ. If $i = \ell$ this follows from (3.12), $A_{ij}A_{ji} \subset N(A_{ii})$, while if $\ell \neq i$ then because i, j, k are distinct ℓ must differ from j or k, say $\ell \neq j$. But then relative to $e = e_i + e_\ell$ we have $A_{ij}A_{ji} \subset A_{10}(e)A_{01}(e) \subset N(A_{11}(e)) = N(A_{ii} + A_{i\ell} + A_{\ell i} + A_{\ell\ell})$ by (3.12) again, so $A_{ij}A_{ji}$ associates with Peirce spaces involving only indices i and ℓ for $\ell \neq j$. ∎

Less useful than the way the Peirce spaces multiply is the way they interact with U-operators.

3.20 (Peirce U-Relations) Each Peirce space A_{ij} is a quadratic ideal, and the quadratic multiplication is given by

U-laws

(3.21) $$U_{A_{ij}} A_{ji} \subset A_{ij}$$

(3.22) $$U_{A_{ij}} A_{k\ell} = 0 \text{ if } k \neq j \text{ or } \ell \neq i$$

Proof. Each $A_{ij} = e_iAe_j$ is a quadratic ideal since any $x(Ay)$ is. From the Fundamental Formulas $U_{x_{ij}} = U_{e_i(x_{ij}e_j)} = L_{e_i}U_{x_{ij}e_j}R_{e_i} = L_{e_i}\{R_{e_j}U_{x_{ij}}L_{e_j}\}R_{e_i} = E_{ij}U_{x_{ij}}E_{ji} = 0$ we see $U_{x_y} = 0$ on $A_{k\ell}$ if $(k,\ell) \neq (j,i)$, and $U_{x_{ij}}$ takes $A_{ji} = E_{ji}(A)$ into $A_{ij} = E_{ij}(A)$.

Alternately, we can use the 2-laws: $x_{ij}(y_{ji}x_{ij}) \in A_{ij}A_{jj} \subset A_{ij}$, while if $\ell \neq i$ then $x_{ij}(y_{k\ell}x_{ij}) = 0$ since $y_{k\ell}\, x_{ij} = 0$ unless $(k,\ell) = (i,j)$, in which case $x_{ij}(y_{ij}x_{ij}) = 0$ by alternating nature of (3.15). ∎

An observation which will prove useful later on is that if x_{ij} is nontrivial it must connect up with A_{ji}.

3.23 (Peirce Triviality Condition) If $x_{ij}A_{ji} = 0$ or $A_{ji}x_{ij} = 0$ then $x_{ij} \in A_{ij}$ is trivial. In particular, if $A_{jj} = 0$ then all elements of A_{ij} and A_{ji} are trivial.

Proof. $U_{x_{ij}}A = U_{x_{ij}}A_{ji} = x_{ij}A_{ji}x_{ij} = 0$ if $x_{ij}A_{ji} = 0$ or $A_{ji}x_{ij} = 0$. ∎

VI. 3 Exercises

3.1 In Chapter I we saw that if $Az = 0$ forces $z = 0$, then any right unit is also a left unit. Show how this also follows immediately from the Peirce relations.

3.2 In I.7.9 we saw that any bimodule for a unital algebra decomposes into a trivial module, a unital left module, a unital right module, and a unital bimodule. Show how this follows from the Peirce relations.

3.3. Though they are not as useful as the formulas for the U_x, prove the following formulas for the $U_{x,y}$:

(1) $U_{x_{ij},x_{ik}} A_{\ell m} = 0$ unless (ℓ,m) is (k,i) or (j,i),

$$U_{x_{ij},x_{ik}} a_{ji} = x_{ij}a_{ji}x_{ik} \in A_{ik} \quad (j \neq k)$$

(2) $U_{x_{ji},x_{ki}} A_{\ell m} = 0$ unless (ℓ,m) is (i,k) or (i,j),

$$U_{x_{ki},x_{ji}} a_{ij} = x_{ki}\, a_{ij}\, x_{ji} \in A_{ki} \quad (j \neq k)$$

(3) $U_{x_{ij},x_{k\ell}} A_{mn} = 0$ unless (m,n) is (j,k) or (ℓ,i),

$$U_{x_{ij},x_{k\ell}} a_{jk} = x_{ij}a_{jk}x_{k\ell} \in A_{i\ell} \quad (i \neq k, ; j \neq k,\ell)$$

(4) $U_{x_{ij},x_{ji}} A_{\ell m} = 0$ unless (ℓ,m) is (j,j), (j,i), (i,j) or (i,i),

$$U_{x_{ij},x_{ji}}\, a_{jj} = x_{ij}a_{jj}x_{ji} \in A_{ii},\ U_{x_{ij},x_{ji}}\, a_{ji} = x_{ij}(a_{ji}x_{ji}) \in A_{ji}.$$

3.4 If a_{ij} as usual denotes an element of the Peirce space A_{ij} $(i \neq j)$ show

(1) $$\{x_{ij}(y_{ij}z_{ij})\}w_{ij} - w_{ij}\{(y_{ij}z_{ij})x_{ij}\} = [x_{ij}\circ y_{ij}z_{ij}, w_{ij}] = -[x_{ij}, y_{ij}, z_{ij}] \circ w_{ij}$$

(2) $$\{(y_{ij}z_{ij})x_{ij}\}w_{ji} - w_{ji}\{x_{ij}(y_{ij}z_{ij})\} = [x_{ij}\circ y_{ij}z_{ij}, w_{ji}] = [x_{ij}, y_{ij}, z_{ij}] \circ w_{ji} = [x_{ij}, w_{ji}] \circ y_{ij}z_{ij} = [y_{ij}z_{ij}, x_{ij}\circ w_{ji}]\ .$$

Show (1) = (2) = 0 in associative or Cayley algebras, or whenever A has no 2-torsion but has a scalar involution with $e_i^* = e_j$. Do these always vanish?

3.5 Show $x_{ij}\circ y_{ij}z_{ij}$ lies in the center of $A = A_{11} + A_{10} + A_{01} + A_{00}$ iff (1) = (2) = 0 in #4.

3.6 Conclude from 3.16 that A is associative if $A_{ij}A_{ji} = A_{ii}$ and $A_{ij}^2 = 0$ for all $i \neq j$.

3.7 Show that $z_{ij} \in A_{ij}$ is nuclear iff $z_{ij}A_{ij} = A_{ij}z_{ij} = 0$ $(i \neq j)$, and $z_{ii} \in A_{ii}$ is nuclear iff $z_{ii} \in N(A_{ii})$ and $z_{ii}A_{ji}^2 = A_{ij}^2 z_{ii} = 0$ for all $i \neq j$.

3.8 Deduce from 3.7 that $A_{ij}A_{jk} \subset N(A)$ and $A_{ij}A_{ji} \cap A_{ik}A_{ki} \subset N(A)$ if $i,j,k \neq$.

3.9 Show A_{11} is nuclear iff it is associative and $A_{10}^2 = A_{01}^2 = 0$.

3.10 Show that if A_{11} is simple with unit e and $A_{10} \neq 0$ or $A_{01} \neq 0$ then A_{11} is associative. If also $A_{10}^2 \neq 0$ or $A_{01}^2 \neq 0$ show A_{11} is a (commutative, associative) field.

3.11 Prove the <u>Interconnected Associativity Criterion</u>: If the components of a Peirce decomposition $A = \bigoplus_{i,j=0}^{n}$ for $n \geq 3$ satisfy

(i) $$A_{ii} = A_{ij}A_{ji} \qquad (j \neq i)$$

(ii) $$A_{ik} = A_{ij}A_{jk} \qquad (i,j,k \neq)$$

then A is associative.

3.12 Establish the additional 4-laws

$$x_{ii}\{y_{ij}(z_{ij}w_{ij})\} = (x_{ii}y_{ij})\{z_{ij}w_{ij}\} = y_{ij}\{(x_{ii}z_{ij})w_{ij}\}$$
$$= y_{ij}\{z_{ij}(x_{ii}w_{ij})\} = y_{ij}\{(z_{ij}w_{ij})x_{ii}\} =$$
$$\{y_{ij}(z_{ij}w_{ij})\}x_{ii}$$

(Slipping Formulas)

$$x_{jj}\{(y_{ij}z_{ij})w_{ij}\} = \{x_{jj}(y_{ij}z_{ij})\}w_{ij} = \{(y_{ij}x_{jj})z_{ij}\}w_{ij}$$
$$= \{y_{ij}(z_{ij}x_{jj})\}w_{ij} = \{y_{ij}z_{ij}\}(w_{ij}x_{jj}) =$$
$$\{(y_{ij}z_{ij})w_{ij}\}x_{jj}$$

VI. 3.1 Problem Set on Peirce Relations

Establish the Peirce relations without recourse to any Artin-type theorems.

1. Show $A_{ij}A_{jk} \subset A_{ik}$ by verifying $e_i(x_{ij}y_{jk}) = x_{ij}y_{jk}$.
2. Show $A_{ij}A_{k\ell} = 0$ if $j \neq k$ unless $i = k$, $j = \ell$.
3. Show $A_{ij}A_{ij} \subset A_{ji}$ by verifying $e_j(x_{ij}y_{ij}) = x_{ij}y_{ij}$; show $x_{ij}^2 = 0$.
4. Show $[x_{ii}, y_{ii}, z_{jk}] = 0$ if $j \neq i$ or $k \neq i$. Deduce $[x_{ij}, y_{ij}, z_{k\ell}] = 0$ if $k \neq i, j$ or $\ell \neq i, j$. Conclude that the only possible nonzero associators involving two terms from the same Peirce space are our Unruly 5.
5. Show that when all terms are from distinct spaces, $[x_{ij}, y_{k\ell}, z_{mn}] = 0$ unless we have at least one link $j = k$ or $\ell = m$. Deduce it vanishes unless we have two links $j = k$, $\ell = m$. Conclude it vanishes unless $i = j = k = \ell = m = n$, contradicting distinctness.

We can also establish the Peirce relations from the ordinary Artin's Theorem (any two elements generate an associative subalgebra).

6. If $e_1, \cdots, e_n$ are orthogonal idempotents and $y_{ij} \in A_{ij}$ let $x = \Sigma\, \lambda_i e_i$, $y = \Sigma\, \lambda_{ij} y_{ij}$ in A_Ω, $\Omega = \Phi[\lambda_1, \cdots, \lambda_n, \lambda_{11}, \cdots, \lambda_{nn}]$ for λ_i, λ_{jk} $n + n^2$ indeterminates. From associativity of $\Phi[x,y]$ deduce that any associator involving elements y_{ij} from distinct Peirce spaces is associative. (Show

$\omega_i\omega_j\lambda_{ij}y_{ij} \in \Phi[x,y]$ for $\omega_i = \Pi_{j \neq i}(\lambda_i - \lambda_j))$. Also deduce $A_{ij}A_{jk} \subset A_{ij}$ (unless $i = j = k$) and $A_{ij}A_{k\ell} = 0$ if $j \neq k$ (unless $i = k$, $j = \ell$).

VI. 3.2 Problem Set on Proper Nilpotence with Peirce Decompositions

1. For any $x = x_{11} + x_{10} + x_{01} + x_{00}$ and $a_{ii} \in A_{ii}$ (relative to an idempotent e) prove

$$(a_{ii}x)^{m+1} = \{(a_{ii}x_{ii})^m a_{ii}\}x = (a_{ii}x_{ii})^m(a_{ii}x)$$

either by induction or using $(ax)^{2m}a = U_aU_x\cdots U_aU_xa$, $(ax)^{2m+1}a = U_aU_x\cdots U_ax$.

2. For any $a_{ij} \in A_{ij}$ $(i \neq j)$ prove

$$(a_{ij}x)^{m+1} = (a_{ij}x_{ij})(a_{ij}x_{ji})^m + (a_{ij}x_{ji})^m \Sigma_k a_{ij}x_{jk}.$$

3. Recall that $z \in A$ is **properly nilpotent** (abbreviated p.n.) if all zx are nilpotent (equivalently all xz are nilpotent, in view of the xy, yx-Lemma for Nilpotence). Show that if z is p.n. so are its Peirce components z_{ij}, using the xy, yx-Lemma and x(yz), (xy)z-Lemmas. Show that if all $z_{ii}x_{ii}$ are nilpotent then z_{ii} is p.n., and similarly for z_{ij} if all $z_{ij}x_{ji}$ are nilpotent. Conclude the set of p.n. elements is $N \subset \Sigma N_{ij}$ where $N_{ij} = \{z_{ij} \in A_{ij} | z_{ij}A_{ji}$ is nil$\}$.