## §2. Peirce decompositions

We can decompose an algebra relative to a family of orthogonal idempotents $\{e_i\}$ into a direct sum of Peirce subspaces $e_iAe_j$. An important special case of such a Peirce decomposition is the Peirce decomposition relative to a single idempotent e. If e is a sum of orthogonal idempotents $e_i$, the Peirce spaces relative to e can be expressed in terms of those relative to the $e_i$ by the Collection Formula.

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As with associative algebras, a decomposition of the unit of an alternative algebra into supplementary orthogonal idempotents leads to a decomposition of the algebra into Peirce spaces. The way these spaces multiply (treated in the next section) gives very specific information about the structure of an alternative algebra.

Idempotence of the element e implies idempotence of the operator $L_e$, $L_e^2 = L_{e^2} = L_e$ (and conversely, in a unital algebra, but not in general; see ex. 2.1). Orthogonality $ef = fe = 0$ of two idempotents implies orthogonality of the operators $L_e$, $L_f$ because $L_eL_f + L_fL_e = L_{ef+fe} = 0$ (by linearized left alternativity) and $L_eL_fL_e = L_{efe}$ (left Moufang), so orthogonality is a consequence of the general

2.1 (Jordan Criterion for Orthogonality) If $E$, $F$ are idempotent operators with $EF + FE = EFE = 0$, then $E$ and $F$ are orthogonal.

Proof. $EF = E^2F = E(EF + FE) - EFE = 0$, hence $FE = (EF + FE) - EF = 0$. ∎

The converse holds even in non-unital algebras: if $L_e$, $L_f$ are orthogonal so are $e$, $f$ since $ef = ef^2 = L_eL_f(f) = 0$ and $fe = L_fL_e(e) = 0$.

Since $L_1 = I$ in the unital case, supplementary orthogonal idempotents $1 = \Sigma e_i$ give rise to supplementary orthogonal projections $I = \Sigma L_{e_i}$, where we have just seen the $L_{e_i}$ are orthogonal idempotent operators. We have exactly the same situation for the right multiplications $R_{e_i}$: $I = \Sigma R_{e_i}$ is a decomposition of $I$ into supplementary orthogonal projections. Then $I = (\sum_i L_{e_i})(\sum_j R_{e_j}) = \sum_{i,j} L_{e_i} R_{e_j}$. If we knew the $L_{e_i}$ and $R_{e_j}$ commuted then the $E_{ij} = L_{e_i}R_{e_j} = R_{e_j}L_{e_i}$ would be supplementary orthogonal projections: $E_{ij}^2 = (L_{e_i}R_{e_j})(L_{e_i}R_{e_j}) = L_{e_i}^2 R_{e_j}^2 = L_{e_i}R_{e_j} = E_{ij}$ and $E_{ij}E_{k\ell} = L_{e_i}L_{e_k}R_{e_j}R_{e_\ell} = 0$ if $i \neq k$ or $j \neq \ell$. In the associative case $L_xR_y = R_yL_x$ for any $x$, $y$ (this is just a restatement of the associative law); in the alternative case $L_xR_y = R_yL_x$ doesn't hold for all $x,y$, but it does in the particular cases when $x = y = e$ (by flexibility) or when $x = e$,

$y = f$ are orthogonal (by $[R_f L_e]z = [e,z,f] = -[e,f,z] = (L_e L_f - L_{ef})z = 0$ from orthogonality $ef = L_e L_f = 0$).

Any time the identity operator $I = \Sigma E_\alpha$ decomposes into supplementary orthogonal projections $E_\alpha$ we get a decomposition $A = \oplus A_\alpha$ of the space on which the operators act into a direct sum of subspaces $A_\alpha = E_\alpha(A)$.

2.2 (Unital Peirce Decomposition) If $1 = \sum_{i=1}^{n} e_i$ is a decomposition of 1 into supplementary orthogonal idempotents then $I = \sum_{i,j=1}^{n} E_{ij}$ is a decomposition of 1 into supplementary orthogonal projections $E_{ij} = L_{e_i} R_{e_j}$, giving rise to a (two-sided) unital Peirce decomposition

$$A = \bigoplus_{i,j=1}^{n} A_{ij} \qquad (A_{ij} = E_{ij}A = e_i A e_j)$$

of the alternative algebra A into Peirce spaces $A_{ij}$, where

$$A_{ij} = \{x \in A \mid e_i x = x e_j = x,\ e_k x = x e_\ell = 0 \text{ for } k \neq i,\ \ell \neq j\}.$$

Thus $e_k x_{ij} = \delta_{ki} x_{ij}$, $x_{ij} e_\ell = \delta_{j\ell} x_{ij}$ for $x_{ij} \in A_{ij}$. ∎

These results generalize to the non-supplementary case (in particular, to the case when A is not unital). Let $e_i$ be orthogonal idempotents in A. They remain so in the unital hull $\hat{A}$; the orthogonal sum $e = e_1 + \cdots + e_n$ is also an idempotent in A and $\hat{A}$ so $e_0 = 1-e$ is an idempotent in $\hat{A}$. Further, $e_0$ is orthogonal to all the original $e_i$ : $ee_i = e_i = e_i e$ so

$e_0 e_i = 0 = e_i e_0$. Thus $1 = (1-e) + e = e_0 + e_1 + \cdots + e_n$ is a decomposition of 1 into supplementary orthogonal idempotents in $\hat{A}$, giving rise to a decomposition $\hat{I} = \Sigma_{i,j=0}^{n} \hat{L}_{e_i} \hat{R}_{e_j} = \Sigma \hat{E}_{ij}$ of the identity operator on $\hat{A}$; since each $\hat{E}_{ij}$ is composed of multiplications, it leaves the ideal $A \lhd \hat{A}$ invariant, and by restriction we obtain a decomposition $I = \Sigma E_{ij}$ of the identity operator on A. Here the restriction $E_{ij}$ of $\hat{E}_{ij}$ retains the form $E_{ij} = L_{e_i} R_{e_j}$ in terms of multiplications in A, where for i = 0 or j = 0 the idempotent $e_0 = 1-e$ is a "virtual" idempotent: it exists only in $\hat{A}$, not in the original A. In turn, the decomposition of I leads to one of A:

2.3 (Peirce Decomposition) Orthogonal idempotents $e_1, \cdots, e_n$ in an alternative algebra give rise to a (two-sided) Peirce decomposition

$$A = \oplus_{i,j=0}^{n} A_{ij}$$

where for $1 \le i, j \le n$

$$A_{ij} = e_i A e_j = \{x \in A \mid e_i x = x e_j = x,\ e_k x = x e_\ell = 0 \text{ for } k \neq i,\ \ell \neq j\}$$

$$A_{i0} = e_i A(1-e) = \{x \in A \mid e_i x = x,\ e_k x = x e_\ell = 0 \text{ for } k \neq i, \text{ all } \ell\}$$

$$A_{0j} = (1-e)Ae_j = \{x \in A \mid x e_j = x,\ x e_\ell = e_k x = 0 \text{ for } \ell \neq j, \text{ all } k\}$$

$$A_{00} = (1-e)A(1-e) = \{x \in A \mid e_i x = x e_j = 0 \text{ for all } i,j\}.$$

Thus $e_k x_{ij} = \delta_{ki} x_{ij}$, $x_{ij} e_\ell = \delta_{j\ell} x_{ij}$ for $x_{ij} \in A_{ij}$ $(k, \ell = 1, \cdots, n)$. ∎

It is important that you remember these multiplication rules, since in the future we will use them without comment. The indices serve as a mnemonic device: for $A_{ij}$ with left index i and right index j, on the left it is the idempotent $e_i$ that acts as the identity while all other $e_k$ act as zero, on the right $e_j$ is the identity and all other $e_\ell$ are zero.

The most important case is that of a single idempotent $e_1 = e$.

2.4 (Single Peirce Decomposition) If e is an idempotent in an alternative algebra A we have a Peirce decomposition

$$A = A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00}$$

for $A_{11} = eAe$, $A_{10} = eA(1-e)$, $A_{01} = (1-e)Ae$, $A_{00} = (1-e)A(1-e)$. Here $ex_{11} = x_{11}e = x_{11}$; $ex_{00} = x_{00}e = 0$; $ex_{10} = x_{10}$, $x_{10}e = 0$; $ex_{01} = 0$, $x_{01}e = x_{01}$. ∎

On occasion we will be concerned with Peirce decompositions relative to several idempotents at once; in this case we denote the Peirce i, j space relative to e by

$$A_{ij}\ (e)$$

to indicate its dependence on e. Also, we will constantly write $x_{ij}$ to denote an element of the Peirce space $A_{ij}$,

$$x_{ij} \in A_{ij} .$$

It will also be useful to have an expression for the Peirce spaces of an idempotent $e = \Sigma e_i$ formed by collecting together several idempotents $e_i$.

2.5 (Collection Formula) If $e_1, \cdots, e_n$ are orthogonal idempotents then for any subset $J \subset \{1, \cdots, n\}$ the element $e = e_J = \Sigma_{j \in J} e_j$ is an idempotent with Peirce spaces

$$A_{11}(e) = \Sigma_{i,j \in J} A_{ij}$$

$$A_{10}(e) = \Sigma_{i \in J, \ell \in K} A_{i\ell}$$

$$A_{01}(e) = \Sigma_{k \in K, j \in J} A_{kj}$$

$$A_{00}(e) = \Sigma_{k, \ell \in K} A_{k\ell} .$$

where K consists of the index 0 as well as all indices between 1 and n not in J. The Peirce spaces $A_{ij}$ can be recovered from the Peirce spaces for the individual idempotents $e_k$ by

$$A_{ii} = A_{11}(e_i)$$

$$A_{ij} = A_{10}(e_i) \cap A_{01}(e_j) \quad (i \neq j) .$$

Proof. The projection operators for e are $E_{\alpha\beta} = L_{(\alpha)}R_{(\beta)}$ where $L_{(1)} = L_e = \Sigma_J L_{e_j}$, $L_{(0)} = L_{1-e} = \Sigma_K L_{e_k}$, $R_{(1)} = R_e = \Sigma_J R_{e_j}$, $R_{(0)} = R_{1-e} = \Sigma_K R_{e_k}$ so $E_{11} = \Sigma_{J\times J} E_{ij}$, $E_{10} = \Sigma_{J\times K} E_{i\ell}$, $E_{01} = \Sigma_{K\times J} E_{kj}$, $E_{00} = \Sigma_{K\times K} E_{k\ell}$.

If $J = \{e_i\}$ then $A_{11}(e_i) = \Sigma_{\{i\}} A_{ij} = A_{ii}$ and $A_{10}(e_i) = \Sigma_{\ell\neq i} A_{i\ell}$, similarly $A_{01}(e_j) = \Sigma_{k\neq j} A_{kj}$, so that $A_{10}(e_i) \cap A_{01}(e_j) = (\Sigma_{\ell\neq i} A_{i\ell}) \cap (\Sigma_{k\neq j} A_{kj}) = A_{ij}$ when $i \neq j$. ∎

Again, these are not hard to remember: for example, $A_{10}(e) = A_{10}(e_J)$ is built up from those spaces $A_{i\ell}$ whose left index i is part of J ($e_J$ acts like 1 on $e_i$) and whose right index $\ell$ is not ($e_J$ acts like 0 on $e_\ell$).

## Exercises

2.1 If $zA = 0$ and $e$ is an idempotent with $ez = 0$, show $x = e+z$ satisfies $L_x^2 = L_x$ but $x^2 \neq x$. Show conversely that any $x$ with $L_x^2 = L_x$ but $x^2 \neq x$ has the form $x = e+z$ where $L_z = 0$, $ez = 0$.

2.2 If $1 = \Sigma e_i$ is a decomposition of 1 into supplementary orthogonal idempotents $e_i$, then $I = \Sigma L_{e_i}$ gives rise to the unital left Peirce decomposition $A = A_1 \oplus \cdots \oplus A_n$ for $A_i = e_i A = \{x \in A | e_i x = x,\ e_j x = 0 \text{ for } j \neq i\}$. (This is usually called the right Peirce decomposition, since in the associative case at least the $A_i$ are right ideals.) State the analogous unital right Peirce decomposition. State and prove the analogous non-unital decompositions. What do these amount to for a single idempotent $e$?

2.3 If $e$, $f$ are orthogonal idempotents, show $e+f$ is again idempotent. Is the converse true? Generalize to arbitrary finite sums of orthogonal idempotents.

2.4 Show $A_{ij}(1-e) = A_{ji}(e)$ for $0 \leq i, j \leq 1$. If $1 = e_1 + e_2 + e_3$ find $A_{11}(e_1+e_3)$, $A_{10}(e_1+e_3)$, $A_{01}(e_1+e_3)$, $A_{00}(e_1+e_3)$. Prove $A_{ij} = A_{10}(e_i) \cap A_{01}(e_j)$ in general using $A_{ij} = \{x | e_k x = \delta_{ki} x,\ xe_\ell = \delta_{j\ell} x\}$.

2.5 Relative to $e_1, \cdots, e_n$ show $E_{ii} = U_{e_i}$ and $E_{ij} + E_{ji} = U_{e_i, e_j}$.