§5. Quadratic ideals

In this section we want to describe the structure of minimal quadratic ideals (when they exist). We will see that a minimal quadratic ideal B almost has the form $B = eAe$ for some division idempotent e, so that a rich supply of minimal ideals affords a rich supply of division idempotents. In turn, these idempotents provide the key which unlocks the structure of semisimple alternative algebras with d.c.c. on quadratic ideals.

A more general concept than that of invertibility is regularity, where an element x is (von Neumann) <u>regular</u> if there exists an element y such that

$$xyx = x .$$

Notice in this case that the elements

$$e = xy \qquad\qquad f = yx$$

are idempotents: $e^2 = (xyx)y = xy = e$ and $f^2 = y(xyx) = yx = f$ by Artin. Also $xf = x = ex$ since $xyx = x$. In particular, e and f are nonzero if x is nonzero, so regular elements give rise to idempotents.

We say x,y are <u>regularly paired</u> if

$$xyx = x, \qquad yxy = y .$$

For example, if x and y are inverses (even one-sided inverses) they are regularly paired. Any regular element x can be regularly paired with some y because

5.1 Lemma. If x is regular, $xy'x = x$, then x is regularly paired with $y = y'xy'$.

Proof. $U_x y = U_x U_{y'} x = U_x U_{y'}(U_x y') = U_{xy'x} y' = U_x y' = x$ so $U_y x = U_{y'} U_x U_{y'} x = U_{y'} U_x y = U_{y'} x = y$. ∎

If two elements are paired, the quadratic ideals they generate are also paired

5.2 (Pairing Lemma) If b,d are regularly paired with corresponding idempotents $e = bd$, $f = db$ then the principal quadratic ideals U_bA, U_dA, U_eA, U_fA are isomorphic under linear bijections which preserve subquadratic ideals

$$U_bA \underset{R_b}{\overset{R_d}{\rightleftarrows}} U_eA \underset{L_b}{\overset{L_d}{\rightleftarrows}} U_dA$$

and similarly

$$U_bA \underset{L_b}{\overset{L_d}{\rightleftarrows}} U_fA \underset{R_b}{\overset{R_d}{\rightleftarrows}} U_dA \, .$$

Proof. We first establish the operator identities

$$R_bR_dU_b = U_b \qquad R_dR_bU_e = U_e$$

$$R_bU_e = U_bL_d \qquad R_dU_b = U_eL_b \, .$$

The first follows from $R_bR_d(R_bL_b) = R_{bdb}L_b = R_bL_b = U_b$ and similarly $R_dR_bU_e = U_e$

since $U_e = U_{bd} = R_dU_bL_d$ where the first factor absorbs an R_dR_b. Then $R_bU_e = R_b(R_dU_bL_d) = U_bL_d$, $U_eL_b = (R_dU_bL_d)L_b = R_dU_b$.

From this we get

$$R_d(U_bA) = U_eL_bA \subset U_eA$$

$$R_b(U_eA) = U_bL_dA \subset U_eA$$

so we have maps $U_bA \xrightarrow{R_d} U_eA \xrightarrow{R_b} U_bA$ which are inverses because $R_bR_d = I$ on U_bA and $R_dR_b = I$ on U_eA: $R_bR_dU_b = U_b$ and $R_dR_bU_e = U_e$.

A dual argument (interchanging L's and R's) shows $U_bA \underset{L_b}{\overset{L_d}{\rightleftarrows}} U_{db}A = U_fA$, while a symmetric argument (interchanging b and d) establishes $U_dA \underset{R_d}{\overset{R_b}{\rightleftarrows}} U_{db}A = U_fA$, and dualizing this gives $U_dA \underset{L_d}{\overset{L_b}{\rightleftarrows}} U_{bd}A = U_eA$.

Since R_d and its inverse R_b take quadratic ideals into quadratic ideals (Cx is quadratic if C is), the bijections preserve quadratic ideals. ∎

This will be useful in classifying minimal quadratic ideals; note that if U_bA contains no proper quadratic ideals then neither will U_eA or U_fA .

5.3 (Minimal Quadratic Ideal Theorem) If B is a minimal quadratic ideal in an alternative algebra A, it has one of the following forms

(I) $B = \Phi z$ for z trivial

(II) $B = eAe$ a division algebra for e a division idempotent

(III) B is a trivial algebra, $B^2 = 0$, but $B = bAb$ for any $b \neq 0$ in B .

In the last case, for each $b \neq 0$ in B there is $d \in A$ and nonzero orthogonal division idempotents $e = bd$, $f = db$ with

$$B = (eAe)b = b(fAf) \quad eAe = Bd \quad fAf = dB .$$

Proof. If B contains a trivial element z then Φz is already a nonzero quadratic ideal, so $B = \Phi z$ is of type (I). [We are not saying conversely that z is minimal if z is trivial; this is true if Φ is a field, but not in general].

From now on assume B has no trivial elements. Thus for $b \neq 0$ we have $bAb \neq 0$; since the latter is a quadratic ideal, contained in B, it must be all of B by minimality:

$$bAb = B \quad \text{if} \quad b \neq 0 .$$

This forces B to be a subalgebra, $BB \subset B$: if $b \neq 0$ then any b' has the form $b' = bab$ for suitable $a \in A$, so $bb' = b\{(ba)b\}$ $= b(ba)b \in U_b A \subset B$.

If $bB = 0$ for some $b \neq 0$ then $BB = (bAb)B = b\{A(bB)\} = 0$, so $B^2 = 0$ and B is of type (III). Similarly if $Bb = 0$.

Both bB and Bb are quadratic ideals, and since B is a subalgebra they are contained in B; if they are nonzero they must be all of B by minimality: $bB = Bb = B$. Then L_b, R_b are surjective for all $b \neq 0$, and therefore by the all L-Test for Units I. 4.6 B has unit e and is a division algebra (implying e is a division idempotent, so we have case II).

We return to case (III). Since $B = U_b A$ for $b \neq 0$ in B we must have $b = U_b a$ for some $a \in A$. Then we know b is regularly paired with $d = U_a b$: $U_b d = b$, $U_d b = d$. By the Pairing Lemma $B = U_b A$ is isomorphic to $U_e A$ and $U_f A$ for $e = bd$, $f = db$

$$B = (eAe)b = b(fAf) \quad Bd = eAe \quad dB = fAf .$$

Since these linear isomorphisms preserve quadratic ideals, B is minimal (= has no proper quadratic ideals) iff eAe is minimal. Since eAe has unit e, it is of type II, so e is division (and similarly for f).

If we are willing to work a little harder we can even arrange it so that e,f are orthogonal. If we have

$$b^2 = d^2 = 0$$

then $ef = (bd)(db) = bd^2b = 0$, $fe = (db)(bd) = db^2d = 0$ and e,f are orthogonal. Automatically $b^2 = 0$ in case III. If $d^2 \neq 0$ we replace d by $c = d - bd^2 = d - ed$; then $ec = ed - e^2d = ed - ed = 0$ and $ce = de - ede = d - ed = c$ (note $de = dbd = d$) and $be = bd - b^2d^2 = bd$, so we still have $bcb = (bd)b = b$ and $cbc = c(bd) = ce = c$, but now also $c^2 = (ce)c = c(ec) = 0$. Replacing d by c we obtain new $e' = bc = bd = e$ and $f' = cb = db - db^2d = f - fe$ which are now orthogonal. ∎

5.4 Remark. There is another way of interpreting case III. When $b^2 = d^2 = 0$ we have $(b + d)^2 = bd + db = e + f$ an idempotent and $u = (1 - e - f) + b + d$ in A' has $u^2 = (1 - e - f)^2 + (b + d)^2 = (1 - e - f) + (e + f) = 1$ (as $(e + f)b = b = b(e + f)$, $(e + f)d = d = d(e + f)$ from $eb = b = bf$, $fb = 0 = be$, $ed = df = 0$, $fd = de = d$). Thus u is invertible, and in the isotope $A^{(1,u)}$ we still have B a minimal quadratic ideal, but

Now the element $b \in B$ satisfies $b^{2(u)} = bub = b(b + d)b$ $= bdb = b$. In other words, b becomes an idempotent in $A^{(L,u)}$, so B becomes type II in the isotope. ∎

Exercises

5.1 Show Φz for an absolute zero divisor is a minimal quadratic ideal iff $z^{\perp} = \{\alpha \in \Phi \mid \alpha z = 0\}$ is a maximal ideal in Φ.

5.2 Use a Schur's Lemma type argument to show a minimal quadratic ideal B is either trivial (as in I or III) or is a division algebra.

5.3 Rather than use the quadratic-ideal-preserving bijection $B \leftrightarrow eAe$ $(e = bd)$, prove directly eAe is a division algebra in case III by showing $U_{eae}(eAe) = eAe$ for all $eae \neq 0$.

5.4 Consider the associative algebra $A = M_n(\Phi)$ (Φ a field). Decide which of the following quadratic ideals are minimal, and classify them according to type: (i) Φe_{12} , (ii) Φe_{11} , (iii) $\Phi e_{22} + \Phi e_{23} + \Phi e_{32} + \Phi e_{33}$, (iv) $e_{12} A e_{12}$, (v) $(e_{11} + e_{12})A(e_{11} + e_{12})$, (vi) $(e_{11} + e_{22})A(e_{11} + e_{22})$, (vii) $e_{11} A e_{11}$.

5.5 Describe all minimal quadratic ideals in a Cayley division algebra (over a field Φ) ; in a split Cayley algebra.

5.6 Describe all quadratic ideals in $A = \mathbb{Z}$ and $A = \mathbb{Z}_4$; which are minimal?