§4. Jordan ideals

A subspace J of an alternative algebra A is called a <u>Jordan ideal</u> if every Jordan product having one factor in J falls back into J. Explicitly

$$U_J \hat{A} \subset J \quad U_{\hat{A}} J \subset J .$$

This implies that for $x \in J$, $a,b \in A$ the products

$$x^2 = U_x 1 , \ xax = U_x a , \ x \circ a = U_{a,1} x , \ axa = U_a x \text{ lie in } J.$$

Another important inclusion is

$$U_{A,J} A \subset J$$

since $U_{a,x} b = a \circ (b \circ x) - U_{a,b} x \in A \circ J - U_A J \subset J$.

Notice that every ordinary ideal is a Jordan ideal, but not conversely (nor is an ordinary one-sided ideal a Jordan ideal).

The basic result relating Jordan-ideals to ordinary ideals is

4.1 (Hull-Kernel Theorem for Jordan Ideals). If J is a Jordan ideal of A then the largest ideal of A contained in J is the kernel

$$K(J) = \{x \in J \mid xA \subset J\} = \{x \in J \mid Ax \subset J\}$$

and the smallest ideal containing J is the hull

$$H(J) = J\hat{A} = \hat{A}J .$$

Furthermore,

$$P^+(J) = U_J \hat{A} \subset K(J)$$

Proof. Note $J\hat{A} = \hat{A}J$ since $\hat{A} \circ J \subset J$, similarly $xA \subset J$ iff $Ax \subset J$ since $A \circ x \subset J$ for x in the Jordan ideal J.

Clearly any ideal B contained in J must have $BA \subset B \subset J$, i.e. $B \subset K(J)$, and any ideal containing J also contains $J\hat{A} = H(J)$. Thus we need only show $K(J), H(J)$ are ideals, and by symmetry merely that they are left ideals.

For $H(J)$, $AH(J)$ is spanned by elements $a(bx)$ for $a,b \in \hat{A}$, $x \in J$; here $a(bx) = U_{a,x} b - x(ba) \in U_{A,J} A + JA \subset H(J)$.

To show $AK(J) \subset K(J)$ we want $A(AK(J)) \subset J$, i.e. $a(bx)$ lies in J if $a,b \in A$ and $x \in K(J)$. But again $a(bx) = U_{a,x} b - x(ba)$ $\in U_{A,J} A + K(J)A \subset J$.

Furthermore, to see $P^+(J) \subset K(J)$ we show $AP^+(J) \subset J$: if $x \in J$ then $a(U_x b) = \{(ax)b\}x$ (right Moufang) $= U_{ax,x} b$ $- (xb)(ax) \in U_{A,J} A - U_x(ba)$ (Middle Moufang) $\subset J$ by definition of Jordan ideal. ∎

4.2 Lemma. If J is a Jordan ideal in A, so is any

$$J + J^2 + \quad + J^n$$

as well as the subalgebra generated by J.

Proof. Since the subalgebra generated by J is $\sum_1^\infty J^k$, and since the union of a chain of Jordan ideals is again a Jordan ideal, it suffices to consider the partial sums $\sum_1^n J^k$. Since $\sum_1^{n+1} J^k = (\sum_1^n J) + (\sum_1^n J^k)^2$, using induction it in fact suffices to consider the case $n = 2$: $J + J^2$ is a Jordan ideal if J is.

To see $U_a(xy) \in J + J^2$ for $a \in A$, $x,y \in J$, note $U_a R_y x = \{U_{a,ya} - L_y U_a\}x \in U_{A,A} J - J(U_A J) \subset J + JJ$. To see

$U_z a \in J + JJ^2$, note that for a spanning set of x's or xy's we have $U_x a \in J$, $U_{xy} = L_x U_y R_x a \in L_x U_y A \subseteq JJ$. ∎

Exercises

4.1 If J,K, L are Jordan ideals in A show $U_J K$ and $U_{J,K} L$ are also Jordan ideals.

4.2 If $1/2 \in \Phi$ show that $U_{\hat{A}} J \subset J$ implies $U_J \hat{A} \subset J$ and therefore that J is a Jordan ideal. The converse is false; $U_J \hat{A} \subset J$ is just the condition that J be a strict quadratic ideal.

4.3 Show directly from the definitions that if J is a Jordan ideal and $x \in J$ then $x^2A + Ax^2 \subset J$.

#00. Problem Set on Jordan Simplicity

We wish to show that if A is simple as alternative algebra then A^+ is simple as a "Jordan algebra," in the sense that A has no proper Jordan ideals.

1. Show that if A contains no trivial elements then any nonzero Jordan ideal J contains a nonzero alternative ideal K(J).

We will see later (Kleinfeld's Strong Semiprimeness Theorem) that if A is simple of characteristic $\neq 3$ it contains no trivial elements; this shows immediately in characteristic $\neq 3$ that alternative simplicity implies Jordan simplicity. Rather than use the high-powered Strong Semiprimeness Theorem, and to cover characteristic 3 as well, we give an alternate proof.

2. If J is any linear subspace satisfying $J \circ A \subseteq J$, and an element $z \in J$ has $U_{z,J}\hat{A} = 0$, show $U_{z,A}J = 0$ and $zH = J(zA)$ is a left A-ideal ($H = H(J)$). Dually Hz is a right ideal. Show $B = \hat{H}z + z\hat{H} \triangleleft H$; if $U_z = 0$ show $U_B \hat{A} = 0$.
3. Conclude that if J is a Jordan ideal in a semiprime algebra A with $U_J \hat{A} = 0$, then $J = 0$ (using 1.9 and the fact that any ideal H in A is also semiprime as an algebra, by Semiprime Inheritance Theorem Ch. VI).
4. (Herstein Construction) If A is semiprime, all Jordan ideals $J \neq 0$ contain an alternative ideal $K(J) \neq 0$.

5. (Jordan Simplicity Theorem) If A is simple as alternative algebra it contains no proper Jordan ideals, so is simple as Jordan algebra.