§3. Products of one-sided ideals

The product of two left ideals needn't be a left ideal, and the expression for the left ideal generated by the product is slightly complicated.

3.1 (Product Lemma) If B is a left ideal and D a two-sided ideal, then DB is a left ideal. If B and C are left ideals, the left ideal generated by BC is

$$(B\hat{A})C = BC + [B,A,C] = \hat{A}(BC) .$$

Proof. For the first, $A(DB) \subset (AD + DA)B - D(AB) \subset DB$ since D absorbs A from either side and B absorbs A from the left. For the second, since $B\hat{A} = H(B)$ is two-sided we just saw $(B\hat{A})C$ is a left ideal containing BC; it is generated by BC since $(B\hat{A})C \subset [B,\hat{A},C] + B(\hat{A}C) \subset [A,B,C] + BC \subset A(BC) + BC = \hat{A}(BC)$ is contained in any left ideal in which BC is contained. In particular $(B\hat{A})C = \hat{A}(BC)$, and squeezed between is $BC + [B,A,C]$. ∎

3.2 Corollary. If BC are left ideals and $[A,B,C] = 0$, then their product BC is a left ideal. ∎

Thus the product BC of two left ideals will again be a left ideal in case B and C associate in the strong sense that any associator with factors from B and C vanishes.

We introduce the notations $I_L(S), I_R(S), I(S)$ for the left, right, (two-sided) ideal generated by a subset $S \subset A$. Although a product of one-sided ideals need not be one-sided, it still generates the correct two-sided ideals

3.3 (Proposition) If B,C are left ideals then the ideal generated by the product BC is the product

$$I(BC) = I(I_L(BC)) = I(B)I(C) .$$

Proof. Always $I(S) = I(I_L(S))$ for any set S ; the last equality is thus the formula

$$\{(B\hat{A})C\}\hat{A} = (B\hat{A})(C\hat{A})$$

by the Hull Lemma and the Product Lemma. Since $(B\hat{A})C \subset (B\hat{A})(C\hat{A})$ and the latter is an ideal (as the product of ideals), we have inclusion $\{(B\hat{A})C\}\hat{A} \subset (B\hat{A})(C\hat{A})$. [For any sets S_1, S_2, $I(S_1S_2) \subset I(S_1)I(S_2)$ since latter is an ideal containing S_1S_2] . For the opposite inclusion, note $(B\hat{A})(C\hat{A}) = (B\hat{A})(C \circ \hat{A}) - (B\hat{A})(\hat{A}C) \subset \{(B\hat{A})C\}\hat{A} + \{(B\hat{A})\hat{A}\}C - (B\hat{A})C \subset \{(B\hat{A})C\}\hat{A}$ using $(B\hat{A})\hat{A} \subset B\hat{A} = H(B)$. ∎

3.4 (Trivial Generation Corollary) A trivial one-sided ideal generates a trivial two-sided ideal.

Proof. We may assume B is left, $B^2 = 0$; then $I(B)^2 = I(B^2) = 0$. ∎

Exercises

3.1 Convince yourself the left ideal generated by BC (for B,C left ideals) is not $B(C\hat{A})$, and that if B is a left ideal and D is two-sided then BD is not a left ideal.

3.2 If B,C are left ideals such that BC is again a left ideal, show $(B\hat{A})(C\hat{A}) = (BC)\hat{A}$.

3.3 If B,C are left ideals show $I(BC) = \{\hat{A}(BC)\}\hat{A}$.

3.4 If B,C are disjoint left ideals, $B \cap C = 0$, show BC is a left ideal.

3.5 If B is a left ideal and C a right ideal, the product BC is a 2-sided ideal if A is associative. Is this true when A is alternative? Describe the ideal generated by BC.

3.6 If S is a subset of A, show $\hat{A}S$ is a left ideal iff $[A,S,A] \subset \hat{A}S$. Conclude $\hat{A}(S\hat{A}) = (\hat{A}S)\hat{A}$ is a 2-sided ideal for such S.