§2. One-sided ideals

One sided ideals are messier than two-sided ideals, primarily because the product of two one-sided ideals need not be a one-sided ideal. The most useful positive fact about one-sided ideals is

2.1 (Associator Lemma). If B is a one-sided ideal of A then any associator with a factor from B falls back in B:

$$[A,B,A] \subset B \ .$$

Proof. To be definite, let us suppose B is a left ideal. Then by the alternating nature of associators we can move all multiplications by A to the left of B, whence they send B back into itself: $[A,B,A] = -[A,A,B] \subset A(AB) - (AA)B \subset AB \subset B$ if B is a left ideal. ∎

Using this we can quickly find an expression for the two-sided ideal generated by a one-sided ideal; just as in the associative case.

2.2 (Hull Lemma). The two-sided ideal generated by a left ideal B is the hull $H(B) = B\hat{A}$.

Proof. Any ideal containing B must also contain $B\hat{A}$; conversely, this space already constitutes an ideal since $A(B\hat{A}) = (AB)\hat{A} - [A,B,\hat{A}] \subset B\hat{A} + [A,B,A]$ and $(B\hat{A})A = [B,\hat{A},A] + B(\hat{A}A) \subset [B,A,A] + BA$, where we just saw $[A,B,A] \subset B \subset B\hat{A}$. ∎

Just as the <u>hull</u> $H(B)$ is the smallest ideal containing B, so the <u>kernel</u> $K(B)$ is the largest ideal contained in B. Again we have the expected characterization

2.3 (Kernel Lemma). If B is a left ideal, the largest ideal contained in B is the kernel $K(B) = \{b \in B \mid bA \subset B\}$;

Proof. Certainly if b belongs to an ideal K contained in B then $bA \subset K \subset B$. Conversely, the above set forms an ideal: it is a left ideal since $(ab)A = [a,b,A] + a(bA) \subset B + aB$ (Associator Lemma) $\subset B$ if B is left, and also $ab \in AB \subset B$; similarly it is a right ideal since $(ba)A = [b,a,A] + b(aA) \subset B + bA$ (Associator Lemma) $\subset B$ by choice of b. ∎

One general method of putting two one-sided ideals together to get a new one (besides sums and intersections) is transportation. If S is a set and B a subspace the <u>left transporter of S into B</u> is

$$L(S,B) = \{x \in A \mid L_x(S) \subset B\} .$$

There is always confusion with transporters as to which carries what into whom, and on what side; the notation is designed to indicate S is carried into B from the left. (One could also write $R_S^{-1}(B) = \{x \mid R_S(x) \subset B\}$, the set of elements which are carried by S into B). Similarly we have a <u>right transporter of S into B</u>

$$R(S,B) = \{x \in A \mid R_x(S) \subset B\} .$$

In case $B = 0$ these reduce to the left and right annihilators

$$\mathrm{Ann}_L(S) = L(S,0) = \{x \mid xS = 0\}$$
$$\mathrm{Ann}_R(S) = R(S,0) = \{x \mid Sx = 0\}$$

which kill S from the left and right respectively.

2.4 (Transportation Lemma). If B is a left ideal and D a two-sided ideal in A, then $R(B,D)$ is a right ideal and $R(D,B)$ is a left ideal. If B is a right and D a two-sided ideal then $L(B,D)$ is a left ideal and $L(D,B)$ is a right ideal.

Proof. We just prove the case when B is a left ideal. We observe

$$B(xa) = (Bx)a - [B,x,a]$$
$$D(ax) = (Da)x - [D,a,x].$$

If $x \in R(B,D)$ then in the first relation $(Bx)a \subset Da \subset D$ (D is right) and $[B,x,a] = [a,B,x] \subset (aB)x - a(Bx) \subset Bx - aD$ (B is left) $\subset D$ (D is left too). This shows xa lies in $R(B,D)$ if x does. If $x \in R(D,B)$ we use the second relation to see $D(ax) \subset B$: $(Da)x \subset Dx \subset B$ since D is right, $[D,a,x] = -[a,D,x]$ $= a(Dx) - (aD)x \subset aB - Dx$ (since D is left) $\subset B$ (since B is left). Thus ax lies in $R(B,D)$, and the latter is a left ideal. ∎

It is too confusing to remember how left and right get snarled up in transportation. Just note that for a left ideal B only right transporters $R(,)$ give rise to one-sided ideals, and that, like Hom (,), the functor $R(,)$ is "covariant" in the second variable but "contravariant" in the first (in the sense that a left ideal B gets converted into a right ideal R(B,D)).

Taking D = 0 , we see that for annihilators <u>everything</u> is backwards

2.5 (Annihilator Lemma). If B is a left ideal its right annihilator $Ann_R(B) = R(B,0)$ is a right ideal. If B is a right ideal its left annihilator $Ann_L(B) = L(B,0)$ is a left ideal. ∎

Thus annihilation is a contravariant process.

Exercise

2.1 Let B be an ideal in A, C = Be for $e \in N(B)$ a nuclear idempotent of B. Show C is a left ideal of A.

2.2 If B is a left ideal and C a right ideal, show $U_B C$ is a left ideal. In particular, $U_B \hat{A}$ is a left ideal if B is.

2.3 If C is a left ideal of A, show for any subset $S \subset A$ that $L(\hat{S},C) = C \cap L(S,C)$ is a left ideal of A contained in C. If $S \subset C$ show $L(S,C)$ is a left ideal of A containing C. If B, C are left ideals show $L(\hat{B},C)A \subset L(B,C)$.

2.4 Investigate why, if B is a left ideal and D a two-sided ideal, the <u>left</u> transporters L(B,D) and L(D,B) need't be one-sided ideals (as they would in the associative case).

2.5 If $n \in N(A)$ is nuclear, show Bn is a left ideal whenever B is.

2.6 The <u>middle annihilator</u> of a set S is $Ann_M S = \{a \in A \mid U_s a = 0$ for all $s \in S\}$. Show that if B is a left ideal then its middle annihilator $Ann_M(B)$ is a right ideal. Show that $Ann_M(B)$ is an ideal if B is.