§9. Coincidence of radicals

In this section we want to see what our various radicals reduce to in the presence of suitable finiteness conditions. The most important result is that for algebras with d.c.c. on quadratic ideals, all the radicals coincide. Furthermore, in this situation semisimplicity, semiprimeness, and strong semiprimeness are all equivalent to regularity. We also establish Amitsur's result that for algebras over large fields the radical coincides with the nil radical.

Recall from (7.6) our general relations

(9.1) $$S(A) \subset T(A) \subset L(A) \subset Nil(A) \subset Rad(A).$$

In Chapter VIII we will develop a structure theory for alternative algebras with descending chain condition (d.c.c.) on inner ideals. It is important that for such algebras it doesn't matter what radical we use: all radicals coincide in the presence of the d.c.c.

9.2 (Radical Equality Theorem) If A is an alternative algebra with d.c.c. on inner ideals, then all radicals

$$S(A) = T(A) = L(A) = Nil(A) = Rad(A)$$

coincide with the maximal nilpotent ideal of A.

Proof. We first show Rad(A) is Jordan-solvable. The proof will be analogous to that used in the associative case with associative products replaced by Jordan products. Thus d.c.c. on right ideals becomes d.c.c. on inner ideals, nilpotent becomes Jordan-solvable, $(Rad\ A)^n$ becomes $J^n(Rad\ A)$, etc.

Recall that the associative proof proceeded in 5 stages: (1) the powers stabilize at $B = R^n = R^{n+1} = \cdots$ with $B^2 = B$; (2) if $B \neq 0$ there is a minimal right ideal $xB \neq 0$ (minimal $\text{Im } L_x$); (3) some $xbB \neq 0$ if $xB^2 = xB \neq 0$, so $\text{Im } L_{xb} = \text{Im } L_x$ by minimality; (4) $xb \in \text{Im } L_x = \text{Im } L_{xb}$ implies $xb = xbc$ for some $c \in B$; (5) $xb(1-c) = 0$ is impossible for $xb \neq 0$ and $c \in B \subset R$ q.i. [See Jacobson p. 38-39].

The alternative case is formally analogous. (1) If $R = \text{Rad } A$ then $R \supset J(R) \supset J^2(R) \supset \cdots$ terminates at $B = J^n(R) = J^{n+1}(R) = \ldots$ with $J(B) = U_B B = B$ by d.c.c. on ideals. If R is <u>not</u> Jordan solvable then $B \neq 0$; we show this leads to a contradiction.

(2) Invoke the d.c.c. to chose an inner ideal minimal among those of the form $\text{Im } L = L(B)$ for $L = L_{x_1} \cdots L_{x_n}$ ($n \geq 0$, $x_i \in B$; in the associative case $L = L_x$ for $x = x_1 \cdots x_n$) where $U(B) \neq 0$ for $U = U_{x_1} \cdots U_{x_n}$. (Such exist: take $n = 0$, $L = U = I$).

(3) $U(U_B B) = U(B) \neq 0$ implies $UU_b B \neq 0$ for some $b \in B$; then $L' = LL_b = L_{x_1} \cdots L_{x_n} L_b$ has $U' = UU_b \neq 0$ on B yet $\text{Im } L' \subset \text{Im } L$, so by minimality we must have $\text{Im } L' = \text{Im } L$.

(4) Thus $L(b) \in \text{Im } L = \text{Im } L' = \text{Im } LL_b$ implies $L(b) = L(bc)$ for some $c \in B$, and $L'(1-c) = L(b(1-c)) = 0$.

(5) Since $c \in B \subset R$ is quasi-invertible, $1-c$ is invertible and $(1-c)B = B$; therefore $U'(B) = U'((1-c)B) = L'(1-c)R'(B) = 0$ (using Middle Moufang $U_x(yz) = (L_x y)(R_y z)$ repeatedly to get $U_{x_1} \cdots U_{x_n}(yz) = (L_{x_1} \cdots L_{x_n} y)(R_{x_1} \cdots R_{x_n} z)$), contrary to our choice of $U' \neq 0$ on B.

Thus Rad A is Jordan solvable which implies $\text{Rad}(A) \subset S(A)$, so in view of (9.1) all the radicals coincide. By Zhevlakov-Slater Nilpotence 3.14 we know $S(A)$ is the maximal nilpotent ideal when A merely has d.c.c. on two-sided ideals. ■

An even more useful condition for semisimplicity in the presence of the d.c.c. is regularity.

9.3 (Regularity Theorem) The following are equivalent for an alternative algebra A with d.c.c. on principal inner ideals:

(i) A is semisimple

(ii) A is strongly semiprime

(iii) A is semiprime.

(iv) A is regular.

Proof. Always semisimple $\Rightarrow$ strongly semiprime $\Rightarrow$ semiprime by 7.6 and regular $\Rightarrow$ semisimple by 7.17, while semiprime $\Rightarrow$ semisimple in the presence of the d.c.c. by Radical Equality 9.2 Thus (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) and (iv) $\Rightarrow$ (i).

To show that A is regular if it is strongly semiprime with d.c.c., suppose there are irregular elements; then among the principal inner ideals $B = U_bA$ for b not regular we can choose a minimal one. For this b we must have all elements $c = U_ba$ of B regular, for if c were irregular then $U_cA = U_bU_aU_bA \subset U_bA$ would imply $U_cA = U_bA$ by minimality of U_bA, hence $c = U_ba \in U_bA = U_cA$, whereas we assumed c was not regular.

Now BY STRONG SEMIPRIMENESS $b \neq 0$ is not trivial: some $c = bab \neq 0$. We just saw c is regular, so by Regular Pairing III.3.12 c is regularly paired with some d: $c = cdc$, $d = dcd$. Then $e = cd$ is a NONZERO idempotent with $ec = c$, $de = d$; note $e = (bab)d = b\{a(bd)\}$ has the form $e = bz$, so in particular $e = e^2 = bzbz = (eb)z$ implies $eb \neq 0$. Since $eb = bzb \in U_bA$ but $b \notin U_bA$ by irregularity, the element $b' = (1-e)b = b-eb = b-bzb$ doesn't belong to U_bA either. But b' does belong to $Q(b) = \Phi b+U_bA$, so by III.3.11 $U_{b'}A \subset U_bA$ and therefore $b' \notin U_{b'}A$, so b' too is irregular. By minimality of U_bA we must have $U_bA = U_{b'}A = U_{(1-e)b}A \subset (1-e)U_bA$ by Left Fundamental. But then $eb = e(eb) \in e(U_bA) \subset$

$e\{(1-e)\cdot U_b A\} = \{e(1-e)\}U_b A = 0$, a contradiction. Thus there can be no irregular elements b. ∎

Since idempotents are regular, by 7.17 the radical contains no idempotents. Now if $z \in \text{Rad } A$ is algebraic over a field Φ it is well known that the associative subalgebra $\Phi[z] \subset \text{Rad } A$ contains an idempotent unless z is nilpotent; therefore an algebraic element of the radical must be nilpotent,

9.4 Proposition. If A is an algebra over a field, the elements of Rad(A) are either transcendental or nilpotent. ∎

9.5 Corollary. If A is algebraic over a field, Rad A = Nil A is nil. ∎

As in the associative case, if Φ is large enough the radical must be nil anyway.

9.6 (Amitsur's Theorem) If A is an algebra over a field Φ with enough elements, $|\Phi| > 2 + \dim_\Phi A$, then Rad A = Nil A is nil.

Proof. If $z \in \text{Rad } A$, let B be a maximal associative subalgebra containing z. By the Quasi-inverse Closure Theorem II.3.16 B is quasi-inverse closed. Clearly $|\Phi| > 2 + \dim_\Phi B$. Also, z belongs to the radical Rad B by the elementwise characterization: all bz for $b \in B$ are q.i. in B since they are q.i. in A (if z belongs to the radical) <u>and B is quasi-inverse closed</u> by construction.

First we show z is algebraic. For $\lambda \neq 0$ the element $\lambda^{-1}z \in \text{Rad } B$ is q.i., so $\lambda 1 - z = \lambda(1-\lambda^{-1}z)$ is invertible. There are $|\Phi|-1$ different elements $(\lambda 1 - z)^{-1}$ in $\hat{B}$ for $\lambda \neq 0$ in Φ, and $|\Phi|-1 > 1+\dim_\Phi B \geq \dim_\Phi \hat{B}$, so they cannot all be independent: we must have $\sum_1^n \alpha_i(\lambda_i - z)^{-1} = 0$ for some $\alpha_i \neq 0$ and distinct $\lambda_i \neq 0$.

Multiplying through by $\Pi_1^n(\lambda_i - z)$ gives $p(z) = \sum_1^n \alpha_i(\lambda_1 - z)\cdots\widehat{(\lambda_i - z)}\cdots(\lambda_n - z) = 0$ (where $\widehat{\ }$ denotes deletion). Here $p(\lambda) = \sum_1^n \alpha_i(\lambda_1 - \lambda)\cdots\widehat{(\lambda_i - \lambda)}\cdots(\lambda_n - \lambda)$ is a polynomial of degree n-1 which is nonzero since $p(\lambda_j) = \alpha_j(\lambda_1 - \lambda_j)\cdots\widehat{(\lambda_j - \lambda_j)}\cdots(\lambda_n - \lambda_j) \neq 0$ if $\alpha_j \neq 0$ and the λ_i are distinct.

Thus each $z \in$ Rad A is algebraic, so by 9.5 z is nilpotent. ∎

9.7 Corollary. If A is finitely (or even countably) generated over an uncountable field Φ then Rad A = Nil A.

Proof. If A is countably generated by $\{x_i\}$ it is countably spanned by the monomials in the x_i, hence has countable dimension over an uncountable field. ∎

Exercises IV.9

9.1 Use the Minimal Quadratic Ideal Theorem III.3.14 to show $T(A) = Nil(A) = Rad(A)$ whenever A has d.c.c. on quadratic ideals.

9.2 If A has a.c.c. on all subspaces of the form Ker L for $L = L_{x_1} \cdots L_{x_n} \neq 0$ on $N = Nil(A)$, show $N \cap Ann_R(N) \neq 0$ if $N \neq 0$. Note Ker L_x is an inner ideal, but $L_x^{-1}(B)$ does not seem to be for general inner B, so Ker $L_{x_1} \cdots L_{x_n}$ doesn't seem to be. If A has a.c.c. on all Ker L_x's for $x \in N$ show any maximal Ker L_x $(x \neq 0)$ has x trivial. Conclude that if A is strongly semiprime with a.c.c. on inner ideals then $T(A) = Nil(A)$.

9.3 If A has d.c.c. on principal inner ideals then any element x is either nilpotent or the principal inner ideal U_xA contains an idempotent. Conclude $Rad(A) = Nil(A)$ for such A.

IV.9.1 Problem Set on Chain Conditions

Consider the chain conditions

(I) d.c.c. on all chains $\operatorname{Im} L_x \supset \operatorname{Im} L_{x^2} \supset \cdots$

(II) a.c.c. on all chains $\operatorname{Ker} L_x \subset \operatorname{Ker} L_{x^2} \subset \cdots$

Note all $\operatorname{Im} L_y$ and $\operatorname{Ker} L_y$ are inner ideals.

1. Show any right ideal B or Peirce space eAe ($e^2 = e$) inherits I or II from A.

2. Show that if $xM = M$ where M is a subspace of an algebra A satisfying II, then $xm = 0 \Rightarrow m = 0$. Conclude $xm = x \Rightarrow m^2 = m$.

3. Show that if A satisfies I and II then A is a Zorn algebra.

IV.9.2 Problem Set on Weakly Artinian Algebras

Say A is weakly (resp. very weakly) Artinian if it has d.c.c. on principal inner ideals $U_{x_1}A \supset U_{x_2}A \supset U_{x_3}A \supset \cdots$ (resp. on $U_xA \supset U_{x^2}A \supset U_{x^3}A \supset \cdots$).

1. Show that an element x in a very weakly Artinian A has a regular power x^m. Conclude A is Zorn. Conclude Nil(A) = {z|z is p.n.} = {z|z is p.q.i.} = Rad(A).

2. If A is weakly or very weakly Artinian, so is any eAe or $\bar{A} = A/B$.

3. If A is strongly semiprime, very weakly Artinian, and all inner ideals contain minimal ones, show A is semisimple. Conclude such an A is strongly semiprime iff it is semisimple, and T(A) = Rad(A).

4. Show if A has d.c.c. on all chains $xA \supset x^2A \supset \cdots$ and on all $Ax \supset Ax^2 \supset \cdots$ then it is very weakly Artinian. Try the same for d.c.c. on all $x_1A \supset x_2A \supset \cdots$ and $Ax_1 \supset Ax_2 \supset \cdots$.

IV.9.3 Problem Set on Invariance of Radicals Under Derivations

1. Let D be a derivation of an arbitrary nonassociative algebra A. Given elements $x_1,\cdots,x_n$ A and some product $p(x_1,\cdots,x_n) = x_1\cdots x_n$ (with some distribution of parentheses) prove

$$D^k p(x_1,\cdots,x_n) = \sum_{i_1+\cdots+i_n=k} \binom{k}{i_1\cdots i_n} p(D^{k_1}x_1,\cdots,D^{k_n}x_n)$$

2. If B is an ideal show B+D(B) is too.

3. If R is the maximal nil ideal of A, and A is Z-torsion-free (characteristic zero) conclude $D(R) \subset R$ for every derivation D.

4. If R is a maximal nilpotent ideal of A and A is Z-torsion-free, show $D(R)^n \subset R$ if $R^n = 0$. If A/R is free of nilpotent ideals, conclude $D(R) \subset R$.

5. If A is again Z-torsion-free and $B \triangleleft A \Rightarrow B^2 \triangleleft A$, so we can define solvability, show $D(R) \subset R$ for any maximal solvable ideal R. Conclude $D(R) \subset R$ for the locally nilpotent radical R.