§8. Radicals of related algebras

The elementwise characterization of the radical allows us to quickly compute the radicals of related algebras, and establish that semisimplicity is inherited by ideals and Peirce subalgebras. The calculations leading up to the Characterization Theorem are illuminated by interpreting them in suitable homotopes. We characterize the radical by means of homotopes, then use this to re-compute the radicals of related algebras.

We begin by characterizing the radical of a principal inner ideal.

8.1 (Proposition) The radical of a principal inner ideal $B = bAb$ is

$$\mathrm{Rad}(B) = \{z \in B \mid bzb \in \mathrm{Rad}\ A\}.$$

Proof. The condition is <u>necessary</u> for z to belong to Rad(B), since if $z \in \mathrm{Rad}\ B$ all $(bzb)a = b\{z(ba)\} \in B\{zB\} \subset \mathrm{Rad}(B)$ (using left Moufang) are q.i., so bzb is p.q.i. in A and belongs to Rad(A). It is <u>sufficient</u>, for if $U_b z \in \mathrm{Rad}(A)$ then for any $c = bab$ in B we have $(zc)^2 = z \cdot U_c z$ (Artin) $= z \cdot U_b U_a U_b z$ (middle fundamental) $\in L_z U_b U_a (\mathrm{Rad}\ A) \subset \mathrm{Rad}(A)$ q.i., so all zc are q.i. and z is p.q.i. in B. ∎

The most important case is the Peirce space eAe.

8.2 (Corollary) For any idempotent e, $\mathrm{Rad}(eAe) = eAe \cap \mathrm{Rad}\ A$. ∎

8.3 (Semisimplicity Inheritance Theorem for Peirce Subalgebras) If A is semisimple, so is any Peirce subalgebra eAe. ∎

For left ideals we have

8.4 (Proposition) The radical of a left ideal B is

$$\mathrm{Rad}(B) = \{z \in B \mid Bz \subset \mathrm{Rad}(A)\}.$$

Proof. If all bz lie in Rad(A) all bz are q.i. in A, hence in B, so z is p.q.i. in B. Conversely, if $z \in$ Rad B then again all $(ab)z \in Bz \subset \mathrm{Rad}(B)$ are q.i., so a(bz) are q.i. by the x(yz), (xy)z - Lemma, and bz is p.q.i. in A. ∎

BEWARE: This result doesn't apply to Ae for an idempotent e since Ae needn't even be a subalgebra; we have seen in a split Cayley algebra $\mathfrak{C}$ that $\mathfrak{C}e = \Phi e + \mathfrak{C}_{01}$ is not a subalgebra since $\mathfrak{C}_{01}^2 = \mathfrak{C}_{10}$ (see I.1.15).

For ideals the criterion is more explicit.

8.5 (Radical Inheritance Theorem) The Jacobson-Smiley radical is hereditary: the radical of an ideal $B \lhd A$ is

$$\mathrm{Rad}(B) = B \cap \mathrm{Rad}(A).$$

Proof. If $z \in B \cap \mathrm{Rad}(A)$ then $Bz \subset \mathrm{Rad}(A)$, so by the criterion 8.4 for left ideals $z \in \mathrm{Rad}(B)$. (Or: $B \cap \mathrm{Rad}(A)$ is an ideal in B which is q.i. in A, hence in B, so lies in the radical). Conversely, if $z \in \mathrm{Rad}(B)$ then all $(aza)z \in Bz$ are q.i., so all $(az)^2$ are q.i., all az are too, and z is p.q.i. in A: $z \in \mathrm{Rad}(A)$. ∎

8.6 Corollary. $\mathrm{Rad}(A) = A \cap \mathrm{Rad}(\hat{A})$. ∎

8.7 (Semisimplicity Inheritance Theorem for Ideals). If A is semisimple so is any ideal $B \lhd A$. ∎

Remark. Since quasi-invertibility is a "strongly semiprime" radical property (A/Rad A is strongly semiprime) we could also use the argument of 6.4 to establish heredity without using an elementwise characterization.

Re-interpretation via homotopes

Both the left and right homotopes $A^{(u,L)}$ and $A^{(R,u)}$ have the same Jordan structure: $x^{2(u)} = U_x u$ and $U_x^{(u)} = U_x U_u$. We write $A^{(u)}$ indifferently to denote the left or right u-homotope. The derived y-operator is given by $V_x^{(u)} y = x \circ_u y = U_{x,y} u = V_{x,u} y$, so

$$(8.8) \qquad V_x^{(u)} = V_{x,u} \quad , \quad U_x^{(u)} = U_x U_u .$$

This makes clear what the transvection operators are: $T_{x,y} = I - V_{x,y} + U_x U_y = I - V_x^{(y)} + U_x^{(y)} = U^{(y)}_{1^{(y)}-x}$ is the U-operator of $1^{(y)}-x$ in the y-homotope

$$(8.9) \qquad T_{x,y} = U^{(y)}_{1^{(y)}-x} .$$

Thus x is q.i. in $A^{(y)}$ iff $T_{x,y}$ is bijective (or even contains $1^{(y)}$ in its image). Since $[1^{(y)}-x]^{2(y)} = 1^{(y)}-2x+x^{2(y)} = 1^{(y)}-2x+U_x y$ is always in the image of $T_{x,y}$, we see x is q.i. in $A^{(y)}$ iff $U_x y-2x$ is in the image. Comparing with 7.11,

8.10 (Lemma) xy is q.i. in A iff x is q.i. in $A^{(y)}$. ∎

Loosely speaking, x has a certain property in $A^{(y)}$ iff xy has that property in A.

The xy,yx-Lemma and x(yz),(xy)z-Lemma have elegant formulations in terms of homotopes.

8.11 (Symmetry Principle) x is q.i. in $A^{(y)}$ iff y is q.i. in $A^{(x)}$.

Proof. x is q.i. in $A^{(y)} \Leftrightarrow$ xy is q.i. $\Leftrightarrow$ yx is q.i. $\Leftrightarrow$ y is q.i. in $A^{(x)}$. ∎

8.12 (Shifting Principle) The following conditions are equivalent:

(i) x is q.i. in $A^{(yz)}$

(ii) xy is q.i. in $A^{(z)}$

(iii) z is q.i. in $A^{(xy)}$

(iv) zx is q.i. in $A^{(y)}$

(v) y is q.i. in $A^{(zx)}$

(vi) yz is q.i. in $A^{(x)}$.

Proof. By the Symmetry Principle (ii) $\Leftrightarrow$ (iii), (iv) $\Leftrightarrow$ (v), and (vi) $\Leftrightarrow$ (i). By the x(yz),(xy)z-Lemma and 8.10, (i) $\Leftrightarrow$ (ii), and similarly (iii) $\Leftrightarrow$ (iv) and (v) $\Leftrightarrow$ (vi). ∎

8.13 Remark. We could also add to the list of equivalences

(vii) x is q.i. in $A^{(y,z)}$

(viii) y is q.i. in $A^{(z,x)}$

(ix) z is q.i. in $A^{(x,y)}$.

Indeed, x is q.i. in $A^{(y,z)}$ iff it is q.i. in $A^{(yz)}$ since the Jordan structures of these two algebras coincide: $V_x^{(u,v)} = V_{x,uv} = V_x^{(uv)}$ and $U_x^{(u,v)} = U_x U_{uv} = U_x^{(uv)}$. ∎

8.14 Remark: If $1 \in A$ we can recover the Symmetry Principle from the Shifting Principle (setting z = 1 in (i) and (v)). ∎

Note that in shifting, an R becomes an L and vice versa: x in $A^{(R_z y)}$ becomes

$L_z x$ in $A^{(y)}$, x in $A^{(L_y z)}$ becomes $R_y x$ in $A^{(z)}$. This corresponds to the fact that "x q.i. in $A^{(yz)}$" is invariant under cyclic permutations.

The radical can also be characterized by means of homotopes.

8.15 (Homotope Characterization of the Radical) The following conditions on an element z are equivalent:

(i) $z \in \mathrm{Rad}(A)$

(ii) z is p.q.i.

(iii) all az are q.i.

(iv) all za are q.i.

(v) z is q.i. in all homotopes $A^{(a)}$

(vi) the homotope $A^{(z)}$ is a radical algebra (all its elements are q.i.)

Proof. We already know (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv); in view of 8.10 we have (iv) $\Leftrightarrow$ (v), and in view of Symmetry (v) $\Leftrightarrow$ (vi). ∎

We indicate the power of these tools by re-proving the results 8.1, 8.4, 8.5.

8.16 (Proposition) Let B be a subspace of an alternative algebra A. Then

(i) $\mathrm{Rad}(bAb) = \{z \in bAb \mid bzb \in \mathrm{Rad}(A)\}$ when $B = bAb$ is an inner ideal.

(ii) $\mathrm{Rad}(B) = \{z \in B \mid Bz \subset \mathrm{Rad}(A)\}$ when B is a left ideal.

(iii) $\mathrm{Rad}(B) = B \cap \mathrm{Rad}(A)$ when B is an ideal.

Proof. (i) bAb q.i. in $(bAb)^{(z)}$ $\Leftrightarrow$ bAb q.i. in $A^{(z)}$ (by (7.2)) $\Leftrightarrow$ A q.i. in $A^{(bzb)}$.

(ii) B q.i. in $B^{(z)} \Leftrightarrow$ AB q.i. in $B^{(z)}$ (for $\Leftarrow$ note $b^{2(z)} = (bz)b$ q.i. implies b q.i.) $\Leftrightarrow$ AB q.i. in $A^{(z)}$ (by (7.2)) $\Leftrightarrow$ A q.i. in $A^{(Bz)}$.

(iii) B q.i. in $B^{(z)} \Leftrightarrow$ B q.i. in $A^{(z)} \Leftrightarrow$ A q.i. in $A^{(z)}$ (for $\Rightarrow$, note $a^{2(z)} = aza \in B$ q.i. implies a q.i. in $A^{(z)}$). ■

We can also easily obtain a new result.

8.17 (Lawvere's Theorem) The radical of the u,v-homotope of A is

$$\mathrm{Rad}\ A^{(u,v)} = \{z \in A \mid (uv)z(uv) \in \mathrm{Rad}\ A\}$$

Proof. $z \in \mathrm{Rad}\ A^{(u,v)} \Leftrightarrow z \in \mathrm{Rad}\ A^{(uv)}$ (by 8.13) $\Leftrightarrow$ z q.i. in all $\{A^{(uv)}\}^{(x)} = A^{((uv)x(uv))}$ (as in I.5.4, ex.5.5.) $\Leftrightarrow$ (uv)z(uv) q.i. in all $A^{(x)}$ (by Shifting) $\Leftrightarrow (uv)z(uv) \in \mathrm{Rad}(A)$. ■

This sleight-of-hand with homotopes and shifting becomes important in Jordan algebras, because there we can no longer talk about the product xy but can still talk about homotopes $A^{(y)}$.

IV.8 Exercises

8.1 Prove that if B is an inner ideal in A which is also a subalgebra then

$$\mathrm{Rad}(B) = \{z \in B \mid bzb \in \mathrm{Rad}\,A \text{ for all } b \in B\}.$$

. Deduce Proposition 8.1 from this.

8.2 Prove that the radical of any left-principal inner ideal $B = Ab$ which happens to be a subalgebra is

$$\mathrm{Rad}\,B = \{z \in B \mid bz \in \mathrm{Rad}\,A\}.$$

8.3 Show $Z(A) = \{z \mid aza = 0 \text{ for all } a\}$ is a nil ideal in A. Show that if $B \lhd A$ then $\mathrm{Rad}(B) \lhd A$ by showing $\overline{A\,\mathrm{Rad}(B)} \subset Z(\bar{B})$ for $\bar{B} = B/\mathrm{Rad}(B)$.

8.4 Show $\mathrm{Rad}(B) \lhd A$ when $B \lhd A$ by showing all az ($a \in A$, $z \in \mathrm{Rad}(B)$) are p.q.i. in B.

8.5 Show the following are equivalent: (i) xy, wy are quasi-inverses in A, (ii) yx, yw are quasi-inverses in A, (iii) xy, xz are quasi-inverses in A ($z = ywy-y$), (iv) yx, zx are quasi-inverses in A, (v) x, w are quasi-inverses in $A^{(y)}$, (vi) y, z are quasi-inverses in $A^{(x)}$, (vii) u, y are quasi-inverses in $A^{(w)}$ ($u = yxy-y$), (viii) v, x are quasi-inverses in $A^{(z)}$ ($v = xyx-x$).

8.6 Prove $\mathrm{Rad}\,Ab = \{z \in Ab \mid bz \in \mathrm{Rad}(A)\}$ using $z \in \mathrm{Rad}(Ab) \Leftrightarrow z$ q.i. in all $(Ab)^{(ac)}$. Deduce $\mathrm{Rad}\,Ae = e(\mathrm{Rad}(A))e + (1-e)Ae$ when e is an idempotent for which Ae is a subalgebra.

8.7 Give alternate proofs that $Bz \subset \mathrm{Rad}(A) \Rightarrow z \in \mathrm{Rad}(B)$ and $z \in \mathrm{Rad}(B) \Rightarrow bz \in \mathrm{Rad}(A)$ (B a left ideal) and $z \in B \cap \mathrm{Rad}(A) \Rightarrow z \in \mathrm{Rad}(B)$ (B an ideal).

8.8 Extending (8.8), show $V^{(u)}_{x,y} = V_{x,U(u)y}$.

8.9 Reprove Exercises 8.1 and 8.2 using homotopes (as in Proposition 8.16).