## §2. Solvability

Solvability is a more tractable property than nilpotence since it is recoverable: if A/B and B are solvable so is A. Even more tractable is Jordan solvability, which is equivalent to solvability in characteristic $\neq 2$ situations. In general Jordan solvability doesn't imply solvability and solvability doesn't imply nilpotence.

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### Solvability

The **derived algebra** of a nonassociative algebra A is the subalgebra (actually ideal) spanned by all products,

$$(2.1) \qquad D(A) = A^2 = AA.$$

The **derived series** is the chain of subalgebras

$$A = D^\circ(A) \supset D^1(A) \supset \cdots \supset D^n(A) \supset \cdots$$

defined recursively by

$$(2.2) \qquad D^\circ(A) = A,\ D^1(A) = D(A),\ D^{k+1}(A) = D(D^k(A)) = D^k(A)^2.$$

(Although D(A) is always an ideal in A, the higher $D^k(A)$ need not be). We say A is **solvable** if $D^n(A) = 0$ for some n; the smallest such n is the **index** of A.

The derivation operators $D^k$ obey

$$(2.3) \qquad D^n(D^m(A)) = D^{n+m}(A)$$

since the recursive definition makes clear $D^k(A) = \overbrace{D(\cdots\cdots(DA))}^{k}$. Further, the derived series is preserved by any homomorphism: for any $A \overset{F}{\to} \tilde{A}$

$$(2.4) \qquad F(D^n(A\ )) = D^n(F(A\ ));$$

indeed, we need only observe $F(D(A)) = F(A^2) = F(A)^2 = D(F(A))$.

A subalgebra B is solvable if it is solvable as an algebra on its own, $D^n(B) = 0$. Notice that if B is an ideal and if products of ideals are ideals (even: if squares of ideals are ideals) then the recursive definition (2.2) shows all derived algebras $D^k(B)$ will also be ideals.

2.5 Proposition. If B is an ideal in an alternative algebra then so are all derived algebras $D^k(B)$. ■

2.6 (Weak Radical Property of Solvability). If a nonassociative algebra A is solvable so is any subalgebra or homomorphic image, and if B is a solvable ideal with solvable quotient A/B then A is solvable. Any finite sum of solvable ideals is solvable.

Proof. If B is a subalgebra of A it is clear inductively that $D^k(B) \subset D^k(A)$, so solvability $D^n(A) = 0$ of A forces solvability $D^n(B) = 0$ of B. Further, (2.4) shows that if $D^n(A) = 0$ then any homomorphic image $F(A)$ has $D^n(F(A)) = F(D^n(A)) = 0$.

Now suppose B and $\overline{A} = A/B$ are solvable of degrees n and m respectively, $D^n(B) = D^m(\overline{A}) = 0$. Then (applying (2.4) to the projection $A \xrightarrow{\pi} \overline{A}$) $\overline{D^m(A)} = D^m(\overline{A}) = 0$ implies $D^m(A) \subset B$. Therefore by (2.3) $D^{n+m}(A)$ $= D^n(D^m(A)) \subset D^n(B) = 0$, and A itself is solvable of degree n+m.

It suffices to prove a sum B+C of two solvable ideals is solvable. But $B+C/C \cong B/B \cap C$ is solvable as a homomorphic image of B, and C is solvable, so by recoverability B+C is solvable. ■

2.7 <u>Remark</u>. It is a general principle that to prove all finite sums $B_1 + \cdots + B_n$ of ideals $B_i$ having a given property inherit that property, it suffices to prove a sum of two ideals inherits the property and then induct.

For the induction step from n to n+1 summands, write $B_1+\cdots+B_{n+1}$ $= (B_1+\cdots+B_n) + B_{n+1}$ as the sum of two ideals $B_1+\cdots+B_n$ and $B_{n+1}$ having the property (by induction and hypothesis respectively).

As a consequence you will frequently see a theorem stated for all finite sums but proven only for sums of two objects; it is understood that the induction is left to the reader. ∎

It is this recoverability of solvability that makes it more manageable than nilpotence. As with nilpotence, solvability need not be retained by an infinite sum of ideals.

It is easy to see by induction that

$$D^k(A) \subset A^{2^k} \tag{2.8}$$

so that nilpotence of A implies solvability. In general it is easier to be solvable than to be nilpotent (recall the situation for Lie algebras), though we will see they are equivalent for alternative algebras in the presence of suitable finiteness conditions. Solvability has to do with vanishing of monomials having a particular association; for $D(A)$, $D^2(A)$, $D^3(A)$, etc. we are concerned only with

$$x_1x_2,\ (x_1x_2)(x_3x_4),\ \{(x_1x_2)(x_3x_4)\}\{(x_5x_6)(x_7x_8)\}, \quad \text{etc.}$$

and not such monomials as $x_1\{(x_2x_3)x_4\}$, $(x_1x_2)\{(x_3x_4)[(x_5x_6)(x_7x_8)]\}$, etc.

### Example of an alternative algebra which is solvable but not nilpotent

As we mentioned, it is more difficult to find an alternative example where solvability doesn't imply nilpotence. If E is solvable of index 1, $D(E) = E^2 = 0$, it is automatically nilpotent, so the first likely index is 2.

The easiest way to construct an algebra solvable of index 2 is to form a split null extension of a trivial algebra,

$$E = A \oplus M \qquad (A^2 = M^2 = 0,\ M \lhd E).$$

Indeed, $D(E) = A^2 + AM + MA + M^2 = AM + MA = N \subset M$ so $D^2(E) \subset D(M) = 0$. In order that E not be nilpotent we must choose M so A does not act nilpotently on M, i.e., the multiplication algebra $M(A|M)$ is not nilpotent. In terms of the birepresentation, the subalgebra D of End M generated by the $L_x, R_y$ must not be nilpotent.

2.9 Theorem. If $A = \oplus_{i=1}^{\infty} \Phi x_i$ is an infinite-dimensional trivial algebra and $M = m_1 \Lambda \oplus m_2 \Lambda$ a 2-dimensional free right module over the exterior algebra $\Lambda = \Lambda(A)$, then A does not act nilpotently on the A-bimodule M given by

$$a \cdot m_1\lambda = m_2(a \wedge \lambda) \qquad m_1\lambda \cdot a = -m_1(a \wedge \lambda)$$
$$a \cdot m_2\lambda = m_1(a \wedge \lambda) \qquad m_2\lambda \cdot a = -m_1(a \wedge \lambda) + m_2(a \wedge \lambda)$$

or equivalently by the birepresentation

$$\ell(a) = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \qquad r(a) = \begin{pmatrix} -a & -a \\ 0 & a \end{pmatrix} .$$

The split null extension $E = A \oplus M$ is a solvable alternative algebra of index 2 which is not nilpotent: $D^2(E) = 0$ but $E^n \neq 0$.

Proof. Since M is a free right module over $\Lambda$, the birepresentation $(\ell, r)$ is completely determined by the matrix of each $\ell(a)$ and $r(a)$. The birepresentation conditions I.7.3a,b $\ell(a^2) = \ell(a)^2$, $r(a^2) = r(a)^2$, $\ell(ab) - \ell(a)\ell(b) = [r(a), \ell(b)] = r(ba) - r(a)r(b)$ reduces for a trivial algebra to

$$\ell(a)^2 = 0, \quad -\ell(a)\ell(b) = [r(a), \ell(b)] = -r(a)r(b)$$

(note these imply $-r(a)^2 = -\ell(a)^2 = 0$). But

$$\ell(a)^2 = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} = \begin{pmatrix} a\wedge a & 0 \\ 0 & a\wedge a \end{pmatrix} = 0$$

$$\ell(a)\,\ell(b) = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} = \begin{pmatrix} a\wedge b & 0 \\ 0 & a\wedge b \end{pmatrix}$$

$$r(a)\,r(b) = \begin{pmatrix} -a & -a \\ 0 & a \end{pmatrix}\begin{pmatrix} -b & -b \\ 0 & b \end{pmatrix} = \begin{pmatrix} a\wedge b & 0 \\ 0 & a\wedge b \end{pmatrix}$$

$$r(a)\,\ell(b) - \ell(b)\,r(a) = \begin{pmatrix} -a & -a \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} - \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}\begin{pmatrix} -a & -a \\ 0 & a \end{pmatrix}$$

$$= \begin{pmatrix} -a\wedge b & -a\wedge b \\ a\wedge b & 0 \end{pmatrix} - \begin{pmatrix} 0 & b\wedge a \\ -b\wedge a & -b\wedge a \end{pmatrix}$$

$$= \begin{pmatrix} -a\wedge b & -a\wedge b - b\wedge a \\ a\wedge b + b\wedge a & b\wedge a \end{pmatrix} = \begin{pmatrix} -a\wedge b & 0 \\ 0 & -a\wedge b \end{pmatrix}$$

since in the exterior algebra $\bigwedge(A)$ we have $a\wedge a = 0$ and $a\wedge b + b\wedge a = 0$ for $a,b \in A$.

Thus $(\ell, r)$ is a birepresentation and M a bimodule. A does not act nilpotently on M since for the basis elements $x_1, \cdots, x_m$ of A we have $x_1\wedge \cdots \wedge x_m \neq 0$ in $\bigwedge(A)$ so

$$\ell(x_1)\,\ell(x_2)\cdots\ell(x_{2n-1})\,\ell(x_{2n}) = \begin{pmatrix} x_1\wedge x_2 & 0 \\ 0 & x_1\wedge x_2 \end{pmatrix}\cdots\begin{pmatrix} x_{2n-1}\wedge x_{2n} & 0 \\ 0 & x_{2n-1}\wedge x_{2n} \end{pmatrix}$$

$$= \begin{pmatrix} x_1\wedge\cdots\wedge x_{2n} & 0 \\ 0 & x_1\wedge\cdots\wedge x_{2n} \end{pmatrix} \neq 0$$

for any n. ∎

This example also shows nilpotence is not a recoverable property in general.

2.10 Theorem. If E is a solvable alternative algebra of index 2 which is not nilpotent then E/D(E) and D(E) are trivial algebras (in particular, nilpotent) but E is not nilpotent. ∎

It also shows a nilpotent algebra need not have a nilpotent multiplication algebra $M_E(A)$ (since $M_E(A|M)$ is not nilpotent).

2.11 Theorem. If A is an infinite-dimensional trivial algebra, then A has multiplication algebras $M_E(A)$ which are not nilpotent. ∎

Jordan Solvability

Solvability $D^n(A) = 0$, nilpotence $A^n = 0$, and left nilpotence $A^{n,L} = 0$ of an alternative algebra A have been defined in terms of the vanishing of various "powers" of the algebra. It is sometimes convenient to construct powers using the Jordan multiplications. The Jordan-derived algebra of a subalgebra $B \subset C$ is the Jordan cube

(2.12) $$J(B) = U_B B,$$

and the Penico-derived algebra of $B$ in $A$ is the Jordan square

(2.13) $$P(B) = U_B \hat{A} .$$

Note that J(B) depends only on the algebra B, whereas P(B) depends on the enveloping algebra A as well. (We could write $P_A(B)$ to indicate this dependence). If B is an ideal in A so are the Jordan and Penico derived algebras by III.1.5. We can iterate these constructions to obtain higher Jordan or Penico derived algebras $J^n(B)$ or $P^n(B)$, which again are ideals if B is. We say B is Jordan-solvable if some $J^n(B) = 0$, and Penico-solvable if some $P^m(B) = 0$.

Actually, these two notions of solvability using Jordan multiplications are equivalent to each other (and nearly equivalent to ordinary solvability).

2.14 (Jordan Solvability Criterion). An ideal B in an alternative algebra A is Jordan solvable, $J^n(B) = 0$ for some n, iff it is Penico-solvable, $P^m(B) = 0$ for some m. If 2 is injective or surjective on B then B is Jordan-solvable iff it is solvable in the ordinary sense. Moreover, for all n we have

$$4^n D^{4n}(B) \subset P^{2n}(B) \subset J^n(B) \subset P^n(B) \subset D^n(B).$$

Proof. In general, if $C_1$ and $C_2$ are methods of constructing new A-ideals out of old ones which are monotone in the sense that the bigger the ideal you start with the bigger the ideal you end up with ($B \subset \tilde{B} \Rightarrow C_1(B) \subset C_1(\tilde{B})$), and if $C_2$ builds bigger ideals than $C_1$ ($C_1(B) \subset C_2(B)$, which we write suggestively as $C_1 \subset C_2$), then the iterated construction $C_2^n$ also builds bigger ideals than $C_1^n$ ($C_1^n \subset C_2^n$). Thus by iteration we need only establish the first step

$$4D^4 \subset P^2 \subset J \subset P \subset D.$$

We have $J \subset P$ by comparing (2.12) and (2.13), and $P \subset D$ since $P(B) = U_B \hat{A} \subset B(\hat{A}B) \subset B^2 = D(B)$ if $B \triangleleft A$. By the Two-squares-and-a-cube Lemma III.1.6 $2D^2 \subset P$ (so $4D^4 \subset P^2$) and $P^2 \subset J$.

From this Jordan-solvability $J^n(B) = 0$ is equivalent to Penico solvability $P^m(B) = 0$ ($P^n(B) = 0 \Rightarrow J^n(B) = 0$ and $J^n(B) = 0 \Rightarrow P^{2n}(B) = 0$).

Solvable always implies Jordan-solvable; conversely, if B is Jordan-solvable then some $J^n(B) = 0$, so $4^n D^{4n}(B) = 0$. If 2 is injective we can cancel $4^n$ to get $D^{4n}(B) = 0$, while if 2 is surjective $D^{4n}(B) = D^{4n}(2B) = 2^{2^{4n}} D^{4n}(B) = 0$ (note $D(\lambda B) = \lambda^2 D(B)$ so $D^k(\lambda B) = (((\lambda^2)^2 \cdots)^2 D^k(B) = \lambda^{2^k} D^k(B))$, so in either case $D^{4n}(B) = 0$ and B is solvable in the ordinary sense. ∎

Thus Jordan-solvability can be defined in terms of the J or P. As usual, the existence of Jordan-solvable ideals is equivalent to the existence of Jordan-cube-trivial ideals B and to the existence of Jordan-square-trivial ideals C,

$$J(B) = U_B B = 0 \quad \text{and} \quad P(C) = U_C \hat{A} = 0,$$

since $J^n(B) = J(J^{n-1}(B)) = 0$ implies $J^{n-1}(B)$ is Jordan-cube-trivial and $P^n(C) = P(P^{n-1}(C)) = 0$ implies $P^{n-1}(C)$ is Jordan-square-trivial.

2.15 (Example of a Jordan-solvable algebra which is not solvable). Both notions of Jordan-solvability coincide with ordinary solvability in characteristic $\neq 2$ situations, but not in characteristic 2: if A is commutative associative over a field of characteristic 2 with $x^2 = 0$ for all x (e.g., any exterior algebra $\bigwedge(V)$ on a vector space V over a field of characteristic 2), then $J(A) = P(A) = 0$, but A need not be solvable (e.g., if V is infinite-dimensional with basis $x_1, x_2, \cdots$ then $x_1 \wedge x_2 \wedge \cdots \wedge x_{2n}$ is a nonzero element of $D^n(A)$). ∎

In characteristic 2 Jordan-solvability of B does not imply B itself is solvable, but <u>something</u> is solvable: an algebra contains Jordan solvable ideals iff it contains solvable ideals. In the next section we study the semiprime algebras, those free of solvable (equivalently, of Jordan solvable or of nilpotent) ideals.

## Exercises IV.2

2.1 If A is an infinite-dimensional trivial algebra, show Mu(A) is not nilpotent (see Problem Set I.7.1).

2.2 Let A be any nonassociative algebra in which the square of an ideal is again an ideal. Show A contains nilpotent ideals iff it contains solvable ideals iff it contains trivial ideals.

2.3 Improve on the Jordan Solvability Theorem to show $2^n \mathfrak{D}^{3n}(B) \subset J^n(A)$.

2.4 Show $F(J^n(B)) = J^n F(B))$ and $F(P^n_A(B)) = P^n_{F(A)}(F(B))$ for any ideal $B \triangleleft A$ and any homomorphism $A \xrightarrow{F} \tilde{A}$. Show that Jordan-solvability is a recoverable property, and conclude that a finite sum of Jordan-solvable ideals is Jordan-solvable.

2.5 If A is B-semiprime (in the sense that no trivial ideals of A are contained in B) show $U_B \hat{A} = 0$ implies the ideal $B = 0$.

2.6 Show directly that if $U_B B = 0$ for $B \triangleleft A$ then $U_{U(B)\hat{A}} \hat{A} = 0$.

2.7 Show $D(U_A \hat{A}) \subset J(A)$ directly.

2.8 Use induction and the fact that a trivial one-sided ideal generates a trivial two-sided ideal (III.2.9) to prove the <u>One-sidedness Theorem for Solvability</u>: If a one-sided ideal in an alternative algebra is solvable, so is the two-sided ideal it generates.

2.9 Prove One-sidedness by showing that if B is a solvable left ideal with kernel K(B) then K(B) and I(B)/K(B) are solvable. (When K(B) = 0 show all $D^n(B)$ are left ideals and $I(D^n(B)) = D^n(I(B))$).

2.10 Show that a semiprime alternative algebra contains no trivial or solvable one-sided ideals. Show that an alternative algebra without nilpotent ideals contains no nilpotent one-sided ideals.

## IV.2.1 Problem Set on Solvable-but-not-nilpotent algebras

We give another construction of a trivial algebra having a non-nilpotent action on a bimodule.

2.1 If A is trivial and M any A-bimodule, show $M(A|M)$ is spanned by operators $F(x_1,\cdots,x_n) = L_{x_1}\cdots L_{x_n}$ and $\tilde{F}(x_1,\cdots,x_{n-1},x_n) = L_{x_1}\cdots L_{x_{n-1}}R_{x_n}$ for $x_i \in A$. Show F is an alternating function of $x_1,\cdots,x_n$ and $\tilde{F}$ of $x_1,\cdots,x_{n-1}$. Show that on $N = AM+MA$ one has $U_x = 0$, so in $M(A|N)$ $L_xR_x = 0$ and $L_xR_y = -L_yR_x$ ; conclude on N that F is alternating in all its variables. Show in $M(A|N)$ we have the multiplicatinn rules

$$F(x_1,\cdots,x_n)F(y_1,\cdots,y_m) = F(x_1,\cdots,x_n,y_1,\cdots,y_m)$$

$$F(x_1,\cdots,x_n)\tilde{F}(y_1,\cdots,y_m) = \tilde{F}(x_1,\cdots,y_m)$$

$$\tilde{F}(x_1,\cdots,x_n)F(y_1,\cdots,y_m) = \varepsilon(m)F(x_1,\cdots,y_m)+\sigma(m)\tilde{F}(x_1,\cdots,y_m)$$

$$\tilde{F}(x_1,\cdots,x_n)\tilde{F}(y_1,\cdots,y_m) = -\sigma(m)F(x_1,\cdots,y_m)+\delta(m)\tilde{F}(x_1,\cdots,y_m)$$

for $\sigma(m) = (-1)^m$, $\varepsilon(m) = \sum_{i=1}^{m}(-1)^i$, $\delta(m) = \sum_{i=1}^{m-1}(-1)^i$. Show $\varepsilon = \frac{1}{2}(\sigma-1)$, $\delta = -\frac{1}{2}(\sigma+1)$.

2.2 What are the multiplication rules in $M(A|M)$? They are slightly more complicated than those in $M(A|N)$.

2.3 In 2.1 show $M(A|N)$ is also generated by the $F(x_1,\cdots,x_n)$ and $G(x_1,\cdots,x_n) = F(x_1,\cdots,x_n) + \tilde{F}(x_1,\cdots,x_n)$ (note $G(x) = V_x$). What are their multiplication rules?

2.4 Generalizing the multiplication algebra of 2.1, show that we can construct an associative algebra by the following recipe:

(Twisting Lemma) If $A = \oplus_{n=0}^{\infty} A_n$ is a graded associative algebra then $T(A) = A \oplus \tilde{A}$ with products

$$a \cdot b = ab \qquad \text{for} \qquad b \in A_n$$
$$a \cdot \tilde{b} = \widetilde{ab} \qquad \sigma(b) = (-1)^n$$
$$\tilde{a} \cdot b = \varepsilon(b)ab + \sigma(b)\widetilde{ab} \qquad \varepsilon(b) = \tfrac{1}{2}\{\sigma(b)-1\}$$
$$\tilde{a} \cdot \tilde{b} = -\sigma(b)ab + \delta(b)\widetilde{ab} \qquad \delta(b) = -\tfrac{1}{2}\{\sigma(b)+1\} = \varepsilon(b) - \sigma(b)$$

is an associative algebra containing A.

2.5 In terms of the decomposition $T(A) = A \oplus \tilde{A}$ find the matrix of $L_{a+\tilde{b}}$ (use S,E,D defined by $S(a) = \sigma(a)a$, $E(a) = \sigma(a)a$, $D(a) = \delta(a)a$). Prove $T(A)$ is associative by showing $L_x L_y = L_{xy}$.

2.6 Show that any multiplication algebra $M(A|N)$ as in 2.1 is a homomorphic image of $T(\Lambda(A)) = \Lambda(A) \oplus \widetilde{\Lambda(A)}$ ( $\Lambda(A)$ = exterior algebra of A) under $x_1 \wedge \cdots \wedge x_n + y_1 \wedge \cdots \wedge y_m \to F(x_1, \cdots, x_n) + \tilde{F}(y_1, \cdots, y_m)$.

2.7 If A is trivial show $\lambda(x) = x$, $\rho(x) = \tilde{x}$ defines a bispecialization $(\lambda, \rho)$ of A in $T(\Lambda(A))$, such that $\lambda(A)$ generates $\Lambda(A)$ and $\rho(A)$ generates $\widetilde{\Lambda(A)}$ modulo $\Lambda(A)$. Conclude that if $\Lambda^n(A) \neq 0$ for all n (e.g., if A is trivial on an infinite-dimensional free module $\oplus_{i=1}^{\infty} \Phi x_i$) then the bispecialization $\lambda$ is not nilpotent.

2.8 Deduce the <u>Theorem</u>. If $A = \oplus_{i=1}^{\infty} \Phi x_i$ is an infinite-dimensional trivial algebra and $M = T(\Lambda(A))$ the A-bimodule induced via the regular representation from the bispecialization $\lambda(x) = x$, $\rho(x) = \tilde{x}$ then the split null extension $E = A \oplus M$ is a solvable alternative algebra of index 2 which is not nilpotent: $D^2(E) = 0$ but $E^n \neq 0$.

2.9 If $A = \oplus_{n=0}^{\infty} A_n$ is a graded algebra show the twisted algebra $T(A) = A \oplus \tilde{A}$ of 2.4 is isomorphic to $A[t] = A \oplus At$ (with relations $ta_{2n} = a_{2n}t$, $ta_{2n+1} = a_{2n+1}\bar{t}$ for $\bar{t} = -1-t$, $t^2+t+1 = 0$) under $a+\tilde{b} \to a+bt$. Prove directly that $A[t]$ is associative.

2.10 If $A = \oplus_{n=0}^{\infty} A_n$ is graded, write $A = B \oplus C$ for $B = \oplus_n A_{2n}$ and $C = \oplus_n A_{2n+1}$. Show $b+c \to b1+cj = \begin{pmatrix} b & c \\ c & b \end{pmatrix}$ imbeds A in $B1+Cj \subset M_2(A)$. Show $t = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ satisfies $t^2+t+1 = 0$, $t(b1) = (b1)t$, $t(cj) = (cj)\bar{t}$. Conclude $T(A) = A \oplus \tilde{A} = B \oplus C \oplus \tilde{B} \oplus \tilde{C}$ is imbedded in $M_2(A)$ via $b'+c'+\tilde{b}+\tilde{c}$ $\to b'1+c'j+bt+cjt = \begin{pmatrix} b'-c & b+c'-c \\ c'-b & b'+c-b \end{pmatrix}$ (giving an alternate proof $T(A)$ is associative).

2.11 Show that the bispecialization $(\lambda,\rho)$ of a trivial A in $T(\Lambda(A))$ given by 2.7 is carried by the imbedding in 2.10 into the bispecialization $(\ell,r)$ of A in $M_2(A)$ given in Theorem 2.10.